Stability of derivations in fuzzy normed algebras

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Abstract

In this paper, we find a fuzzy approximation of derivation for an \(m\)-variable additive functional equation. In fact, using the fixed point method, we prove the Hyers-Ulam stability of derivations on fuzzy Lie \(C^\ast\)-algebras for the following additive functional equation

\[
\sum_{i=1}^{m} f(mx_i + \sum_{j=1, j \neq i}^{m} x_j) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right)
\]

for a given integer \(m\) with \(m \geq 2\). ©2015 All rights reserved.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam \[9\] concerning the stability of group homomorphisms:

\textit{Let} \((G_1, \ast)\) \textit{be a group and let} \((G_2, \circ, d)\) \textit{be a metric group with the metric} \(d(\cdot, \cdot)\). \textit{Given} \(\epsilon > 0\), \textit{does there exist a} \(\delta(\epsilon) > 0\) \textit{such that if a mapping} \(h : G_1 \to G_2\) \textit{satisfies the inequality} \(d(h(x \ast y), h(x) \circ h(y)) < \delta\) \textit{for all} \(x, y \in G_1\), \textit{then there is a homomorphism} \(H : G_1 \to G_2\) \textit{with} \(d(h(x), H(x)) < \epsilon\) \textit{for all} \(x \in G_1\) ?

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Let \( m \in \mathbb{N} \) be a set. A function \( d : \Omega \times \Omega \to [0, \infty] \) is called a \textit{generalized metric} on \( \Omega \) if \( d \) satisfies the following:

1. \( d(x, y) = 0 \) if and only if \( x = y \) for all \( x, y \in \Omega \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in \Omega \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in \Omega \).

\textbf{Theorem 1.1.} \( [6] \) Let \( (\Omega, d) \) be a complete generalized metric space and let \( J : \Omega \to \Omega \) be a contractive mapping with Lipschitz constant \( L < 1 \). Then for each given element \( x \in \Omega \), either \( d(J^n x, J^{n+1} x) = 0 \) for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that

1. \( d(J^n x, J^{n+1} x) < \infty \) for all \( n \geq n_0 \);
2. the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);
3. \( y^* \) is the unique fixed point of \( J \) in the set \( \Gamma = \{ y \in \Omega : d(J^n x, y) < \infty \} \);
4. \( d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \) for all \( y \in \Gamma \).

In this paper, using the fixed point method, we prove the Hyers-Ulam stability of homomorphisms and derivations in fuzzy Lie \( C^* \)-algebras for the following additive functional equation (see \[10\])

\[
\sum_{i=1}^{m} f(mx_i + \sum_{j=1, j \neq i}^{m} x_j) + f\left(\sum_{i=1}^{m} x_i\right) = 2f\left(\sum_{i=1}^{m} mx_i\right)
\]

(1.1)

for all \( m \in \mathbb{N} \) with \( m \geq 2 \).

We use the definition of fuzzy normed spaces given in \[1 \] \[4 \] \[6 \] \[7 \] \[8 \] to investigate a fuzzy version of the Hyers-Ulam stability for the Cauchy-Jensen functional equation in the fuzzy normed algebra setting.

\textbf{Definition 1.2.} \( [6] \) Let \( X \) be a vector space. A function \( N : X \times \mathbb{R} \to [0, 1] \) is called a \textit{fuzzy norm} on \( X \) if

1. \( N(x, t) = 0 \) for all \( x \in X \) and \( t \in \mathbb{R} \) with \( t \leq 0 \);
2. \( x = 0 \) if and only if \( N(x, t) = 1 \) for all \( x \in X \) and \( t > 0 \);
3. \( N(cx, t) = N(x, \frac{t}{|c|}) \) for all \( x \in X \) and \( c \neq 0 \);
4. \( N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\} \) for all \( x, y \in X \) and \( s, t \in \mathbb{R} \);
5. \( N(x, \cdot) \) is a non-decreasing function of \( \mathbb{R} \) and \( \lim_{t \to \infty} N(x, t) = 1 \) for all \( x \in X \) \( t \in \mathbb{R} \);
6. for all \( x \in X \) with \( x \neq 0 \), \( N(x, \cdot) \) is continuous on \( \mathbb{R} \).

The pair \((X, N)\) is called a \textit{fuzzy normed vector space}.

\textbf{Definition 1.3.} \( [6] \) (1) Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is said to be \textit{convergent} to a point \( x \in X \) or \textit{converges} if there exists \( x \in X \) such that

\[
\lim_{n \to \infty} N(x_n - x, t) = 1
\]

for all \( t > 0 \). In this case, \( x \) is called the \textit{limit} of the sequence \( \{x_n\} \) and we denote it by \( N\text{-lim}_{n \to \infty} x_n = x \).

(2) Let \((X, N)\) be a fuzzy normed vector space. A sequence \( \{x_n\} \) in \( X \) is called \textit{Cauchy} if, for each \( \varepsilon > 0 \) and \( t > 0 \), there exists an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) and all \( p > 0 \), we have \( N(x_{n+p} - x_n, t) > 1 - \varepsilon \).

It is well-known that every convergent sequence in a fuzzy normed vector space is a Cauchy sequence. If each Cauchy sequence is convergent, then the fuzzy normed vector space is said to be \textit{complete} and the complete fuzzy normed vector space is called a \textit{fuzzy Banach space}.

We say that a mapping \( f : X \to Y \) between fuzzy normed vector spaces \( X \) and \( Y \) is \textit{continuous} at a point \( x_0 \in X \) if, for each sequence \( \{x_n\} \) converging to \( x_0 \) in \( X \), the sequence \( \{f(x_n)\} \) converges to \( f(x_0) \).

If \( f : X \to Y \) is continuous at each \( x \in X \), then \( f : X \to Y \) is said to be \textit{continuous} on \( X \) (see \[6\]).
Definition 1.4. A fuzzy normed algebra \((X, N)\) is a fuzzy normed space \((X, N)\) with the algebraic structure such that

\[(N7) \quad N(xy, ts) \geq N(x, t)N(y, s) \quad \text{for all} \quad x, y \in X \quad \text{and} \quad t, s > 0.\]

Every normed algebra \((X, \| \cdot \|)\) defines a fuzzy normed algebra \((X, N)\), where \(N\) is defined by

\[N(x, t) = \frac{t}{t + \|x\|}\]

for all \(t > 0\). This space is called the induced fuzzy normed algebra.

Definition 1.5. Let \((X, N)\) and \((Y, N)\) be fuzzy normed algebras. (1) An \(A\)-linear mapping \(f : X \rightarrow Y\) is called a homomorphism if

\[f(xy) = f(x)f(y)\]

for all \(x, y \in X\).

(2) An \(A\)-linear mapping \(f : X \rightarrow X\) is called a derivation if

\[f(xy) = f(x)y + xf(y)\]

for all \(x, y \in X\).

Definition 1.6. Let \((U, N)\) be a fuzzy Banach algebra. Then an involution on \(U\) is a mapping \(u \mapsto u^*\) from \(U\) into \(U\) which satisfies the following:

(a) \(u^{**} = u\) for any \(u \in U\);
(b) \((\alpha u + \beta v)^* = \overline{\alpha}u^* + \overline{\beta}v^*\);
(c) \((uv)^* = v^*u^*\) for any \(u, v \in U\).

If, in addition, \(N(u^*u, ts) = N(u, t)N(u, s)\) and \(N(u^*, t) = N(u, t)\) for all \(u \in U\) and \(t, s > 0\), then \(U\) is a fuzzy \(C^*\)-algebra.

2. Stability of derivations on fuzzy \(C^*\)-algebras

Throughout this section, assume that \(A\) is a fuzzy \(C^*\)-algebra with the norm \(N_A\).

For any mapping \(f : A \rightarrow A\), we define

\[D_\mu f(x_1, \cdots, x_m) := \sum_{i=1}^{m} \mu f(mx_i + \sum_{j=1, j \neq i}^{m} x_j) + f\left(\mu \sum_{i=1}^{m} x_i\right) - 2f\left(\mu \sum_{i=1}^{m} mx_i\right)\]

for all \(\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}\) and \(x_1, \cdots, x_m \in A\).

Note that a \(C\)-linear mapping \(\delta : A \rightarrow A\) is called a fuzzy \(C^*\)-algebra derivation on fuzzy \(C^*\)-algebras if \(\delta\) satisfies the following:

\[\delta(xy) = y\delta(x) + x\delta(y)\]

and

\[\delta(x^*) = \delta(x)^*\]

for all \(x, y \in A\).

Now, we prove the Hyers-Ulam stability of fuzzy \(C^*\)-algebra derivations on fuzzy \(C^*\)-algebras for the functional equation

\[D_\mu f(x_1, \cdots, x_m) = 0.\]
Theorem 2.1. Let \( f : A \to A \) be a mapping for which there are functions \( \varphi : A^m \times (0, \infty) \to [0, 1] \), \( \psi : A^2 \times (0, \infty) \to [0, 1] \) and \( \eta : A \times (0, \infty) \to [0, 1] \) such that

\[
N_A(D_{\mu}f(x_1, \cdots, x_m), t) \geq \varphi(x_1, \cdots, x_m, t),
\]

\[
\lim_{j \to \infty} \varphi(m^j x_1, \cdots, m^j x_m, m^j t) = 1,
\]

\[
N_A(f(xy) - xf(y) - xf(y), t) \geq \psi(x, y, t),
\]

\[
\lim_{j \to \infty} \psi(m^j x, m^j y, m^j t) = 1,
\]

\[
N_A(f(x^*) - f(x)^*, t) \geq \eta(x, t),
\]

\[
\lim_{j \to \infty} \eta(m^j x, m^j t) = 1
\]

for all \( \mu \in \mathbb{T}^1, x_1, \cdots, x_m, x, y \in A \) and \( t > 0 \). If there exists an \( L < 1 \) such that

\[
\varphi(mx_0, \cdots, 0, mLt) \geq \varphi(x_0, \cdots, 0, t)
\]

for all \( x \in A \) and \( t > 0 \), then there exists a unique fuzzy \( C^* \)-algebra derivation \( \delta : A \to A \) such that

\[
N_A(f(x) - \delta(x), t) \geq \varphi(x_0, \cdots, 0, (m - mL)t)
\]

for all \( x \in A \) and \( t > 0 \).

Proof. Consider the set \( X := \{ g : A \to A \} \) and introduce the generalized metric on \( X \):

\[ d(g, h) = \inf \{C \in \mathbb{R}^+ : N_A(g(x) - h(x), Ct) \geq \varphi(x, 0, \cdots, 0, t), \forall x \in A, t > 0 \} \]

It is easy to show that \((X, d)\) is complete. Now, we consider the linear mapping \( J : X \to X \) such that \( Jg(x) := \frac{1}{m}g(mx) \) for all \( x \in A \). By \([2, \text{Theorem 3.1}]\), we have

\[ d(Jg, Jh) \leq Ld(g, h) \]

for all \( g, h \in X \). Letting \( \mu = 1, x = x_1 \) and \( x_2 = \cdots = x_m = 0 \) in \( 2.1 \), we get

\[
N_A(f(mx) - mf(x), t) \geq \varphi(x, 0, \cdots, 0, t)
\]

for all \( x \in A \) and \( t > 0 \). So

\[
N_A(f(x) - \frac{1}{m}f(mx), t) \geq \varphi(x, 0, \cdots, 0, mt)
\]

for all \( x \in A \) and \( t > 0 \). Hence \( d(f, Jf) \leq \frac{1}{m} \). By Theorem 1.1 there exists a mapping \( \delta : A \to A \) such that

\( \delta(mx) = m\delta(x) \)

for all \( x \in A \). The mapping \( \delta \) is a unique fixed point of \( J \) in the set

\[ Y = \{ g \in X : d(f, g) < \infty \} \]

This implies that \( \delta \) is a unique mapping satisfying \( 2.10 \) such that there exists \( C \in (0, \infty) \) satisfying

\[
N_A(\delta(x) - f(x), Ct) \geq \varphi(x, 0, \cdots, 0, t)
\]

for all \( x \in A \) and \( t > 0 \).
(2) \(d(J^n f, \delta) \to 0\) as \(n \to \infty\). This implies the equality
\[
\lim_{n \to \infty} \frac{f(m^n x)}{m^n} = \delta(x)
\]
for all \(x \in A\).
(3) \(d(f, \delta) \leq \frac{1}{1-L}d(f, Jf)\), which implies the inequality \(d(f, \delta) \leq \frac{1}{m-mL}\). This implies that the inequality (2.8) holds.

It follows from (2.1), (2.2) and (2.11) that
\[
N_A\left( \sum_{i=1}^{m} \delta(\sum_{j=1, j \neq i}^{m} x_j) + \delta(\sum_{i=1}^{m} x_i) - 2\delta(\sum_{i=1}^{m} mx_i), t \right)
= \lim_{n \to \infty} N_A\left( \sum_{i=1}^{m} f(m^{n+1} x_i) + \sum_{j=1, j \neq i}^{m} m^n x_j + f(\sum_{i=1}^{m} m^n x_i) - 2f(\sum_{i=1}^{m} m^{n+1} x_i), m^n t \right)
\leq \lim_{n \to \infty} \phi(m^n x_1, \ldots, m^n x_m, m^n t)
= 1
\]
for all \(x_1, \ldots, x_m \in A\) and \(t > 0\) and so
\[
\sum_{i=1}^{m} \delta(\sum_{j=1, j \neq i}^{m} x_j) + \delta(\sum_{i=1}^{m} x_i) = 2\delta(\sum_{i=1}^{m} mx_i)
\]
for all \(x_1, \ldots, x_m \in A\).

By the similar method to above, we get \(\mu \delta(mx) = \delta(m \mu x)\) for all \(\mu \in \mathbb{T} \) and all \(x \in A\). Thus one can show that the mapping \(\delta : A \to A\) is \(\mathbb{C}\)-linear.

It follows from (2.3), (2.4) and (2.11) that
\[
N_A(\delta(xy) - y\delta(x) - x\delta(y), t)
= \lim_{n \to \infty} N_A(f(m^n xy) - m^n yf(m^n x) - m^n xf(m^n y), m^n t)
\leq \lim_{n \to \infty} \psi(m^n x, m^n y, m^{2n} t)
= 1
\]
for all \(x, y \in A\). So \(\delta(xy) = y\delta(x) + x\delta(y)\) for all \(x, y \in A\). Thus \(\delta : A \to A\) is a derivation satisfying (2.7).

Also, by (2.5), (2.6), (2.11) and a similar method, we have \(\delta(x^*) = \delta(x)^*\). \qed

3. Stability of derivations on fuzzy Lie \(C^*\)-algebras

A fuzzy \(C^*\)-algebra \(C\), endowed with the Lie product
\[
[x, y] := \frac{xy - yx}{2}
\]
on \(C\), is called a fuzzy Lie \(C^*\)-algebra.

**Definition 3.1.** Let \(A\) be a fuzzy Lie \(C^*\)-algebra. A \(\mathbb{C}\)-linear mapping \(\delta : A \to A\) is called a fuzzy Lie \(C^*\)-algebra derivation if
\[
\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]
\]
for all \(x, y \in A\).

Throughout this section, assume that \(A\) is a fuzzy Lie \(C^*\)-algebra with norm \(N_A\). We prove the Hyers-Ulam stability of fuzzy Lie \(C^*\)-algebra derivations on fuzzy Lie \(C^*\)-algebras for the functional equation
\[
D_\mu f(x_1, \ldots, x_m) = 0.
\]
Theorem 3.2. Let $f: A \to A$ be a mapping for which there are two functions $\varphi: A^m \times (0, \infty) \to [0, 1]$ and $\psi: A^2 \times (0, \infty) \to [0, 1]$ such that
\begin{align*}
\lim_{j \to \infty} \varphi(m^j x_1, \ldots, m^j x_m, m^j t) &= 1, \\
N_A(D_\mu f(x_1, \ldots, x_m), t) &\geq \varphi(x_1, \ldots, x_m, t), \\
N_A(f([x, y]) - [f(x), y] - [x, f(y)], t) &\geq \psi(x, y, t), \\
\lim_{j \to \infty} \psi(m^j x, m^j y, m^j t) &= 1
\end{align*}
for all $\mu \in \mathbb{T}^1, x_1, \ldots, x_m, x, y \in A$ and $t > 0$. If there exists an $L < 1$ such that
\[ \varphi(mx, 0, \ldots, 0, mx) \geq \varphi(x, 0, \ldots, 0, t) \]
for all $x \in A$ and $t > 0$, then there exists a unique fuzzy Lie $C^*$-algebra derivation $\delta: A \to A$ such that
\[ N_A(f(x) - \delta(x), t) \geq \varphi(x, 0, \ldots, 0, (m - mL)t) \]
for all $x \in A$ and $t > 0$.

Proof. By the same reasoning as in the proof of Theorem 2.1, we can find the mapping $\delta: A \to A$ given by
\[ \delta(x) = \lim_{n \to \infty} \frac{f(m^n x)}{m^n} \]
for all $x \in A$. It follows from (3.3) that
\begin{align*}
N_A(\delta([x, y]) - [\delta(x), y] - [x, \delta(y)], t) &= \lim_{n \to \infty} N_A(f(m^{2n} x, y) - [f(m^n x), y] - [m^n x, f(m^n y)], m^{2n} t) \\
&\geq \lim_{n \to \infty} \psi(m^n x, m^n y, m^{2n} t) = 1
\end{align*}
for all $x, y \in A$ and $t > 0$. So
\[ \delta([x, y]) = [\delta(x), y] + [x, \delta(y)] \]
for all $x, y \in A$. Thus $\delta: A \to B$ is a fuzzy Lie $C^*$-algebra derivation satisfying (3.5). This completes the proof. 

Corollary 3.3. Let $A$ be a normed fuzzy Lie $C^*$-algebra with norm $\| \cdot \|$. Let $0 < r < 1$ and $\theta$ be nonnegative real numbers, and let $f: A \to A$ be a mapping such that
\[ N_A(D_\mu f(x_1, \ldots, x_m), t) \geq \frac{t}{t + \theta(\|x_1\|_A^r + \|x_2\|_A^r + \cdots + \|x_m\|_A^r)} \]
and
\[ N_A(f([x, y]) - [f(x), y] - [x, f(y)], t) \geq \frac{t}{t + \theta \cdot \|x\|_A^r \cdot \|y\|_A^r} \]
for all $\mu \in \mathbb{T}^1, x_1, \ldots, x_m, x, y \in A$ and $t > 0$. Then there exists a unique fuzzy Lie $C^*$-algebra derivation $\delta: A \to A$ such that
\[ N_A(f(x) - \delta(x), t) \leq \frac{t}{t + \frac{\theta}{m - mL} \|x\|_A^r} \]
for all $x \in A$ and $t > 0$. 

Proof. The proof follows from Theorem 3.2 by taking
\[
\varphi(x_1, \cdots, x_m, t) = \frac{t}{t + \theta \left( \|x_1\|^r_A + \|x_2\|^r_A + \cdots + \|x_m\|^r_A \right)}
\]
and
\[
\psi(x, y, t) := \frac{t}{t + \theta \cdot \|x\|^r_A \cdot \|y\|^r_A}
\]
for all \(x_1, \cdots, x_m, x, y \in A\) and \(t > 0\). Putting \(L = m^{r-1}\), we get the desired result. \(\square\)

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