Weak convergence theorems for two asymptotically quasi-nonexpansive non-self mappings in uniformly convex Banach spaces

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Abstract

The purpose of this paper is to establish some weak convergence theorems of modified two-step iteration process with errors for two asymptotically quasi-nonexpansive non-self mappings in the setting of real uniformly convex Banach spaces if $E$ satisfies Opial’s condition or the dual $E^*$ of $E$ has the Kedec-Klee property. Our results extend and improve some known corresponding results from the existing literature. ©2014 All rights reserved.

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1. Introduction

Let $K$ be a nonempty subset of a real Banach space $E$. Let $T: K \to K$ be a mapping, then we denote the set of all fixed points of $T$ by $F(T)$. The set of common fixed points of two mappings $S$ and $T$ will be denoted by $F = F(S) \cap F(T)$. A mapping $T: K \to K$ is said to be:

(i) asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \quad (1.1)$$

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for all \( x, y \in K \) and \( n \geq 1 \),

(ii) asymptotically quasi-nonexpansive if \( F(T) \neq \emptyset \) and there exists a sequence \( \{k_n\} \in [1, \infty) \) such that 
\[
\lim_{n \to \infty} k_n = 1 \quad \text{and} \quad \|T^n x - p\| \leq k_n \|x - p\| \quad (1.2)
\]
for all \( x \in K \), \( p \in F(T) \) and \( n \geq 1 \),

(iii) uniformly \( L \)-Lipschitzian if there exists a constant \( L > 0 \) such that 
\[
\|T^n x - T^n y\| \leq L \|x - y\| \quad (1.3)
\]
for all \( x, y \in K \) and \( n \geq 1 \).

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk \([8]\) as an important generalization of the class of nonexpansive maps (i.e., mappings \( T: K \to K \) such that \( \|Tx - Ty\| \leq \|x - y\| \), \( \forall x, y \in K \)) who proved that if \( K \) is a nonempty closed convex subset of a real uniformly convex Banach space and \( T \) is an asymptotically nonexpansive self-mapping of \( K \), then \( T \) has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authors (see e.g., \([2]\)-\([4]\),\([7]\),\([10]\),\([12]\),\([15]\)-\([18]\), \([20]\)-\([25]\)) using the Mann iteration method (see e.g.,\([13]\)) or the modified Mann iteration method or the Ishikawa iteration method (see e.g.,\([9]\)) or the modified Ishikawa iteration method. (See, also \([19]\) and \([26]\)).

In 1978, Bose \([1]\) proved that if \( K \) is a bounded closed convex nonempty subset of a uniformly convex Banach space \( E \) satisfying Opial’s \([14]\) condition and \( T: K \to K \) is an asymptotically nonexpansive mapping, then the sequence \( \{T^n x\} \) converges weakly to a fixed point of \( T \) provided \( T \) is asymptotically regular at \( x \in K \), i.e., \( \lim_{n \to \infty} \|T^n x - T^{n+1} x\| = 0 \). Passty \([16]\) and also Xu \([27]\) proved that the requirement that \( E \) satisfies Opial’s condition can be replaced by the condition that \( E \) has a Frechet differentiable norm. Furthermore, Tan and Xu \([22]\) \([23]\) later proved that the asymptotic regularity of \( T \) can be weakened to the weakly asymptotic regularity of \( T \) at \( x \), i.e., \( \omega - \lim_{n \to \infty} \|T^n x - T^{n+1} x\| = 0 \).

In \([20]\), Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subset of a Hilbert space \( H \). In 1994, Rhoades \([18]\) extended the Schu’s result to uniformly convex Banach space using a modified Ishikawa iteration scheme.

In all the above results, the operator \( T \) remains a self-mapping of a nonempty closed convex subset \( K \) of a uniformly convex Banach space \( E \). If, however, the domain of \( T \), \( D(T) \), is a proper subset of \( E \) (and this is the case in several applications), and \( T \) maps \( D(T) \) into \( E \), then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

The aim of this paper is to establish some weak convergence theorems for two asymptotically quasi-nonexpansive non-self mappings in the framework of real uniformly convex Banach spaces. Our results extend, improve and unify some known corresponding results from the existing literature.
2. Preliminaries

Let $E$ be a real normed linear space. The modulus of convexity of $E$ is the function $\delta_E: (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| = \|y\| = 1, \epsilon = \|x - y\| \right\}$$

$E$ is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$.

A subset $K$ of $E$ is said to be a retract of $E$ if there exists a continuous map $P: E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A map $P: E \to E$ is said to be a retraction if $P^2 = P$. It follows that if a map $P$ is a retraction, then $Py = y$ for all $y$ in the range of $P$.

**Definition 2.1.** Let $E$ be a real normed linear space, $K$ a nonempty subset of $E$. Let $P: E \to K$ be the nonexpansive retraction of $E$ onto $K$. A map $T: K \to E$ is said to be:

(i) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq k_n\|x - y\|, \quad \forall n \geq 1,$$

(ii) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$ and $x^* \in F(T)$, the following inequality holds:

$$\|T(PT)^{n-1}x - T(PT)^{n-1}x^*\| \leq k_n\|x - x^*\|, \quad \forall n \geq 1,$$

(iii) uniformly $L$-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|T(PT)^{n-1}x - T(PT)^{n-1}y\| \leq L\|x - y\|, \quad \forall n \geq 1.$$  

Let $K$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. The iteration scheme: $x_1 \in K$ and

$$x_{n+1} = P(a_n x_n + b_n T_1(PT_1)^{n-1} y_n + c_n l_n), \quad \forall n \geq 1,$$

$$y_n = P(\bar{a}_n x_n + \bar{b}_n T_2(PT_2)^{n-1} x_n + \bar{c}_n m_n), \quad \forall n \geq 1,$$

where $l_n, m_n \in K$ and $\{l_n\}_{n=1}^{\infty}, \{m_n\}_{n=1}^{\infty}$ are bounded, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n, 0 \leq a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \leq 1$, for all $n \geq 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and $P$ is as in definition 2.1, is called modified Ishikawa iteration scheme with errors in the sense of Xu [28] for two mappings.

**Remark 2.2.** If $T$ is a self map, then $P$ becomes the identity map so that (2.1), (2.2) and (2.3) coincide with (1.1), (1.2) and (1.3) respectively. Moreover, iteration scheme (2.4) reduces to the modified Ishikawa iteration scheme with errors.

Now, we study the iteration scheme which is independent of (2.4) as follows:

$$x_{n+1} = P(a_n T_1(PT_1)^{n-1} x_n + b_n T_2(PT_2)^{n-1} y_n + c_n l_n), \quad \forall n \geq 1,$$

$$y_n = P(\bar{a}_n x_n + \bar{b}_n T_1(PT_1)^{n-1} x_n + \bar{c}_n m_n), \quad \forall n \geq 1,$$

where $l_n, m_n \in K$ and $\{l_n\}_{n=1}^{\infty}, \{m_n\}_{n=1}^{\infty}$ are bounded, $a_n + b_n + c_n = 1 = \bar{a}_n + \bar{b}_n + \bar{c}_n, 0 \leq a_n, b_n, c_n, \bar{a}_n, \bar{b}_n, \bar{c}_n \leq 1$, for all $n \geq 1$, $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} \bar{c}_n < \infty$, and $P$ is as in definition 2.1, is called modified Ishikawa type iteration scheme with errors in the sense of Xu [28] for two mappings.

In the sequel, we shall need the following lemmas.
Lemma 2.3. (See [24]). Let \( \{r_n\}, \{s_n\} \) and \( \{t_n\} \) be sequences of nonnegative real numbers satisfying
\[
r_{n+1} \leq (1 + s_n)r_n + t_n, \quad \forall \ n \geq 1.
\]
If \( \sum_{n=1}^{\infty} s_n < \infty \) and \( \sum_{n=1}^{\infty} t_n < \infty \), then \( \lim_{n \to \infty} r_n \) exists. In particular, if \( \{r_n\} \) has a subsequence converging to zero, then \( \lim_{n \to \infty} r_n = 0 \).

Lemma 2.4. (See [27]). Let \( E \) be a uniformly convex Banach space and \( 0 < a \leq t_n \leq b < 1 \) for all \( n \geq 1 \). Suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences in \( E \) satisfying \( \limsup_{n \to \infty} \|x_n\| \leq r \), \( \limsup_{n \to \infty} \|y_n\| \leq r \), \( \lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r \) for some \( r \geq 0 \). Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

Lemma 2.5. (See [14]) Let \( E \) be a real reflexive Banach space with its dual \( E^* \) has the Kadec-Klee property. Let \( \{x_n\} \) be a bounded sequence in \( E \) and \( p, q \in w_w(x_n) \) (where \( w_w(x_n) \) denotes the set of all weak subsequential limits of \( \{x_n\} \)). Suppose \( \lim_{n \to \infty} \|tx_n + (1-t)p - q\| \) exists for all \( t \in [0, 1] \). Then \( p = q \).

Lemma 2.6. (See [2]) Let \( K \) be a nonempty bounded closed convex subset of a uniformly convex Banach space \( E \). Then there exists a strictly increasing continuous convex function \( \phi : [0, \infty) \to [0, \infty) \) with \( \phi(0) = 0 \) such that for any Lipschitzian mapping \( T : K \to E \) with the Lipschitz constant \( L \geq 1 \), and element \( \{x_j\}_{j=1}^{n} \in K \) and any nonnegative number \( \{t_j\}_{j=1}^{n} \) with \( \sum_{j=1}^{n} t_j = 1 \), the following inequality holds:
\[
\|T \left( \sum_{j=1}^{n} t_j x_j \right) - \sum_{j=1}^{n} t_j Tx_j \| \leq L \phi^{-1} \left\{ \max_{1 \leq j, k \leq n} \left( \|x_j - x_k\| - \frac{1}{L} \|Tx_j - Tx_k\| \right) \right\}.
\]

We recall that a Banach space \( E \) is said to satisfy Opial’s condition if, for any sequence \( \{x_n\} \) in \( E \), \( x_n \to x \) weakly implies that
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|
\]
for all \( y \in E \) with \( y \neq x \).

A Banach space \( E \) has the Kadec-Klee property if for every sequence \( \{x_n\} \) in \( E \), \( x_n \to x \) weakly and \( \|x_n\| \to \|x\| \) it follows that \( \|x_n - x\| \to 0 \).

3. Main Results

In this section, we establish some weak convergence theorems of the iteration scheme (2.5) by using Opial condition and Kadec-Klee property in the framework of real uniformly convex Banach space. First we need the following lemma to prove our main results of this paper.

Lemma 3.1. Let \( E \) be a real uniformly convex Banach space and \( K \) be a nonempty closed convex subset which is also a nonexpansive retract of \( E \). Let \( T_1, T_2 : K \to E \) be two uniformly L-Lipschitzian asymptotically quasi-nonexpansive non-self mappings with sequences \( \{k_n\}, \{h_n\} \subset [1, \infty) \) such that \( F = \cap_{i=1}^{2} F(T_i) \neq \emptyset \). Suppose \( N_1 = \lim_{n \to \infty} k_n \geq 1 \) and \( N_2 = \lim_{n \to \infty} h_n \geq 1 \) such that \( \sum_{n=1}^{\infty} (k_n h_n - 1) < \infty \). From arbitrary \( x_1 \in K \), the sequence \( \{x_n\} \) defined iteratively by (2.5) with the restrictions \( \sum_{n=1}^{\infty} e_n < \infty \) and \( \sum_{n=1}^{\infty} b_n e_n < \infty \). Let \( \{a_n\} \) and \( \{\bar{a}_n\} \) be sequences in \( [\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). Then we have the following:

(a) \( \lim_{n \to \infty} \|x_n - x^*\| \) exists for all \( x^* \in F \).

(b) \( \lim_{n \to \infty} \|x_n - T_1 x_n\| = 0 \) and \( \lim_{n \to \infty} \|x_n - T_2 x_n\| = 0 \).
Proof. For all $x^* \in F$, we set

$$M_1 = \max\{\sup_{n \geq 1} \|l_n - x^*\|, \sup_{n \geq 1} \|m_n - x^*\|\}.$$ 

Then from (2.5), we have

$$\|x_{n+1} - x^*\| = \|P(a_nT_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_nl_n) - Px^*\| \leq \|a_nT_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_nl_n - x^*\| \leq a_n\|T_1(PT_1)^{n-1}x_n - x^*\| + b_n\|T_2(PT_2)^{n-1}y_n - x^*\| + c_n\|l_n - x^*\| \leq a_nk_n\|x_n - x^*\| + b_nh_n\|y_n - x^*\| + c_nM_1$$

(3.1)

and

$$\|y_n - x^*\| = \|P(\bar{a}_nT_1(PT_1)^{n-1}x_n + \bar{c}_nm_n) - Px^*\| \leq \|\bar{a}_nx_n + \bar{b}_nT_1(PT_1)^{n-1}x_n + \bar{c}_nm_n - x^*\| \leq \bar{a}_n\|x_n - x^*\| + \bar{b}_n\|T_1(PT_1)^{n-1}x_n - x^*\| + \bar{c}_n\|m_n - x^*\| \leq \bar{a}_n\|x_n - x^*\| + \bar{b}_nk_n\|x_n - x^*\| + \bar{c}_nM_1 \leq [\bar{a}_n + \bar{b}_n]k_n\|x_n - x^*\| + \bar{c}_nM_1 = [1 - \bar{c}_n]k_n\|x_n - x^*\| + \bar{c}_nM_1 \leq k_n\|x_n - x^*\| + \bar{c}_nM_1$$

(3.2)

which implies that

$$\|y_n - x^*\| \leq k_n\|x_n - x^*\| + \bar{c}_nM_1 \leq k_nh_n\|x_n - x^*\| + \bar{c}_nM_1.$$  

(3.3)

Using (3.1) and (3.3), we obtain that

$$\|x_{n+1} - x^*\| \leq a_nk_n\|x_n - x^*\| + b_nh_n[k_nh_n\|x_n - x^*\| + \bar{c}_nM_1] + c_nM_1 \leq a_nk_nh_n\|x_n - x^*\| + b_nh_n[k_nh_n\|x_n - x^*\| + \bar{c}_nM_1] + c_nM_1 \leq (a_n + b_n)k_n^2h_n^2\|x_n - x^*\| + b_nh_n\|x_n - x^*\| + \bar{c}_nM_1 \leq k_n^2h_n^2\|x_n - x^*\| + b_n\bar{c}_n + c_nM_1 \leq k_n^2h_n^2\|x_n - x^*\| + b_n\bar{c}_n + c_nM_1 \leq k_n^2h_n^2\|x_n - x^*\| + (b_n\bar{c}_n + c_n)M_2 \leq [1 + (k_n^2h_n^2 - 1)]\|x_n - x^*\| + A_n$$

(3.4)

where

$$M_2 = \sup_{n \geq 1} \{k_nh_n\}M_1, \quad A_n = (b_n\bar{c}_n + c_n)M_2.$$ 

By putting $\lambda_n = (k_n^2h_n^2 - 1)$, the inequality (3.4) can be written as follows

$$\|x_{n+1} - x^*\| \leq (1 + \lambda_n)\|x_n - x^*\| + A_n.$$  

(3.5)
By hypothesis of the theorem, we find
\[ \sum_{n=1}^{\infty} \lambda_n = \sum_{n=1}^{\infty} (k_n^2 h_n^2 - 1) = \sum_{n=1}^{\infty} (k_n h_n + 1)(k_n h_n - 1) \leq (N_1 N_2 + 1) \sum_{n=1}^{\infty} (k_n h_n - 1) < \infty. \]

Since by assumptions of the theorem \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n \bar{c}_n < \infty \), it follows that \( \sum_{n=1}^{\infty} A_n < \infty \) and \( \sum_{n=1}^{\infty} \lambda_n < \infty \), thus by Lemma 2.3, we have \( \lim_{n \to \infty} \|x_n - x^*\| \) exists. Let \( \lim_{n \to \infty} \|x_n - x^*\| = r \) for some \( r \geq 0 \). From (3.3), we have
\[ \|y_n - x^*\| \leq k_n h_n \|x_n - x^*\| + \bar{c}_n M_1, \quad \forall n \geq 1. \]

Taking \( \limsup_{n \to \infty} \) in both sides, we obtain
\[ \limsup_{n \to \infty} \|y_n - x^*\| \leq \limsup_{n \to \infty} \|x_n - x^*\| = \lim_{n \to \infty} \|x_n - x^*\| = r. \quad (3.6) \]

Since \( T_1 \) is asymptotically quasi-nonexpansive non-self mapping, we have
\[ \|T_1(P T_1)^{n-1} x_n - x^*\| \leq k_n \|x_n - x^*\|, \quad \forall n \geq 1. \]

Taking \( \limsup_{n \to \infty} \) in both sides, we obtain
\[ \limsup_{n \to \infty} \|T_1(P T_1)^{n-1} x_n - x^*\| \leq r. \quad (3.7) \]

In a similar way, we have
\[ \|T_2(P T_2)^{n-1} y_n - x^*\| \leq h_n \|y_n - x^*\|, \quad \forall n \geq 1. \]

By using (3.6), we obtain
\[ \limsup_{n \to \infty} \|T_2(P T_2)^{n-1} y_n - x^*\| \leq r. \quad (3.8) \]

Also, it follows from
\[
\begin{align*}
r &= \lim_{n \to \infty} \|x_{n+1} - x^*\| \\
&= \lim_{n \to \infty} \|a_n T_1(P T_1)^{n-1} x_n + b_n T_2(P T_2)^{n-1} y_n + c_n h_n - x^*\| \\
&= \lim_{n \to \infty} \|a_n[(T_1(P T_1)^{n-1} x_n - x^*) + \frac{c_n}{2a_n}(l_n - x^*)] \\
&\quad + b_n[(T_2(P T_2)^{n-1} y_n - x^*) + \frac{c_n}{2b_n}(l_n - x^*)]] \\
&= \lim_{n \to \infty} \|a_n[(T_1(P T_1)^{n-1} x_n - x^*) + \frac{c_n}{2a_n}(l_n - x^*)] \\
&\quad + (1 - a_n)[(T_2(P T_2)^{n-1} y_n - x^*) + \frac{c_n}{2b_n}(l_n - x^*)]]
\end{align*}
\]

and Lemma 2.4 that
\[ \lim_{n \to \infty} \|T_1(P T_1)^{n-1} x_n - T_2(P T_2)^{n-1} y_n + \left(\frac{c_n}{2a_n} - \frac{c_n}{2b_n}\right)(l_n - x^*)\| = 0. \quad (3.9) \]

Since \( \lim_{n \to \infty} \|\left(\frac{c_n}{2a_n} - \frac{c_n}{2b_n}\right)(l_n - x^*)\| = 0 \), we obtain that
\[ \lim_{n \to \infty} \|T_1(P T_1)^{n-1} x_n - T_2(P T_2)^{n-1} y_n\| = 0. \quad (3.10) \]
Now
\[
\|x_{n+1} - x^*\| = \|a_n T_1(P_T)^{n-1} x_n + b_n T_2(P_T)^{n-1} y_n + c_n l_n - x^*\|
\]
\[
= \|(T_1(P_T)^{n-1} x_n - x^*) + b_n (T_2(P_T)^{n-1} y_n - T_1(P_T)^{n-1} x_n)\|
\]
\[
\leq \|T_1(P_T)^{n-1} x_n - x^*\| + b_n\|T_2(P_T)^{n-1} y_n - T_1(P_T)^{n-1} x_n\|
\]
\[
+ c_n\|l_n - T_1(P_T)^{n-1} x_n\|
\]

yields that
\[
r \leq \liminf_{n \to \infty} \|T_1(P_T)^{n-1} x_n - x^*\|
\]
so that (3.7) gives
\[
\lim_{n \to \infty} \|T_1(P_T)^{n-1} x_n - x^*\| = r. \tag{3.11}
\]

On the other hand,
\[
\|T_1(P_T)^{n-1} x_n - x^*\| \leq \|T_1(P_T)^{n-1} x_n - T_2(P_T)^{n-1} y_n\| + \|T_2(P_T)^{n-1} y_n - x^*\|
\]
\[
\leq \|T_1(P_T)^{n-1} x_n - T_2(P_T)^{n-1} y_n\| + h_n\|y_n - x^*\|
\]
so we have
\[
r \leq \liminf_{n \to \infty} \|y_n - x^*\|. \tag{3.12}
\]

By using (3.6) and (3.12), we obtain
\[
\lim_{n \to \infty} \|y_n - x^*\| = r. \tag{3.13}
\]

Thus
\[
r = \lim_{n \to \infty} \|y_n - x^*\|
\]
\[
= \lim_{n \to \infty} \|\bar{a}_n x_n + \bar{b}_n T_1(P_T)^{n-1} x_n + \bar{c}_n m_n - x^*\|
\]
\[
= \lim_{n \to \infty} \|\bar{b}_n[(T_1(P_T)^{n-1} x_n - x^*) + \frac{\bar{c}_n}{2\bar{b}_n}(m_n - x^*)]
\]
\[
+ \bar{a}_n[x_n - x^*] + \frac{\bar{c}_n}{2\bar{a}_n}(m_n - x^*)]\|
\]
\[
= \lim_{n \to \infty} \|\bar{b}_n[(T_1(P_T)^{n-1} x_n - x^*) + \frac{\bar{c}_n}{2\bar{b}_n}(m_n - x^*)]
\]
\[
+ (1 - \bar{b}_n)[x_n - x^*] + \frac{\bar{c}_n}{2\bar{a}_n}(m_n - x^*)]\|
\]

Using (3.11), (3.13) and Lemma 2.4, the above inequality gives
\[
\lim_{n \to \infty} \|T_1(P_T)^{n-1} x_n - x_n\| = 0. \tag{3.14}
\]

Now
\[
\|y_n - x_n\| \leq \bar{b}_n\|T_1(P_T)^{n-1} x_n - x_n\| + \bar{c}_n\|m_n - x_n\|. \tag{3.15}
\]

Using (3.14) and by hypothesis of the theorem in (3.15), we obtain
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.16}
\]
Also note that
\[
\|x_{n+1} - x_n\| = \|a_nT_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_n l_n - x_n\|
\]
\[
= \|(1 - b_n - c_n)T_1(PT_1)^{n-1}x_n + b_nT_2(PT_2)^{n-1}y_n + c_n l_n - x_n\|
\]
\[
\leq \|T_1(PT_1)^{n-1}x_n - x_n\| + b_n\|T_2(PT_2)^{n-1}y_n - T_2(PT_2)^{n-1}x_n\|
+ c_n\|l_n - T_1(PT_1)^{n-1}x_n\|
\]
\[
\quad \text{→ 0 as } n \to \infty, \quad (3.17)
\]
so that
\[
\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \to 0 \text{ as } n \to \infty. \quad (3.18)
\]
Furthermore, from
\[
\|x_{n+1} - T_2(PT_2)^{n-1}y_n\| \leq \|x_{n+1} - x_n\| + \|x_n - T_1(PT_1)^{n-1}x_n\|
+ \|T_1(PT_1)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\|
\]
we find that
\[
\lim_{n \to \infty} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| = 0. \quad (3.19)
\]
Then
\[
\|x_{n+1} - T_1x_{n+1}\| \leq \|x_{n+1} - T_1(PT_1)^n x_{n+1}\| + \|T_1(PT_1)^n x_{n+1} - T_1(PT_1)^n x_n\|
+ \|T_1(PT_1)^n x_n - T_1x_{n+1}\|
\]
\[
\leq \|x_{n+1} - T_1(PT_1)^n x_{n+1}\| + L\|x_{n+1} - x_n\|
+ L\|T_1(PT_1)^{n-1}x_n - x_{n+1}\|
\]
\[
\leq \|x_{n+1} - T_1(PT_1)^n x_{n+1}\| + L\|x_{n+1} - x_n\|
+ Lb_n\|T_1(PT_1)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\|
+ Lc_n\|T_1(PT_1)^{n-1}x_n - l_n\|
\]
yields
\[
\lim_{n \to \infty} \|x_n - T_1x_n\| = 0. \quad (3.20)
\]
Now
\[
\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^{n-1}y_n\|
+ \|T_2(PT_2)^{n-1}y_n - T_2(PT_2)^{n-1}x_n\|
\]
\[
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_2(PT_2)^{n-1}y_n\|
+ L\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \quad (3.21)
\]
Thus
\[
\|x_{n+1} - T_2x_{n+1}\| \leq \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| + \|T_2(PT_2)^n x_{n+1} - T_2x_{n+1}\|
\leq \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| + L\|T_2(PT_2)^{n-1}x_{n+1} - x_{n+1}\|
\leq \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| + L\left(\|T_2(PT_2)^{n-1}x_{n+1} - T_2(PT_2)^{n-1}y_n\|
+ \|T_2(PT_2)^{n-1}y_n - x_{n+1}\|\right)
\leq \|x_{n+1} - T_2(PT_2)^n x_{n+1}\| + L^2\|x_{n+1} - y_n\|
+ L\|T_2(PT_2)^{n-1}y_n - x_{n+1}\|
implies
\[ \lim_{n \to \infty} \| x_n - T_2 x_n \| = 0. \]  
(3.22)\]
This completes the proof. \qed

**Theorem 3.2.** Let \( E \) be a real uniformly convex Banach space satisfying Opial’s condition and \( K, T_i \) (\( i = 1, 2 \)) and \( \{ x_n \} \) be as in Lemma 3.1. If \( F = F(T_1) \cap F(T_2) \neq \emptyset \), then the sequence \( \{ x_n \} \) converges weakly to a common fixed point of the mappings \( T_1 \) and \( T_2 \).

**Proof.** Let \( p \in F = F(T_1) \cap F(T_2) \neq \emptyset \). Then, by Lemma 3.1 \( \| x_n - p \| \) exists. Assume that \( x_n \to u \) weakly and \( x_n \to v \) weakly as \( n \to \infty \). Then \( u, v \in F \). We prove that \( u = v \). If \( u \neq v \), by Opial’s condition, we have
\[
\lim_{n \to \infty} \| x_n - u \| = \lim_{i \to \infty} \| x_{n_i} - u \| < \lim_{i \to \infty} \| x_{n_i} - v \| = \lim_{n \to \infty} \| x_n - v \| < \lim_{j \to \infty} \| x_{n_j} - u \| = \lim_{n \to \infty} \| x_n - u \|
\]
which is a contradiction. Therefore, we have the conclusion i.e. \( u = v \). Thus the sequence \( \{ x_n \} \) converges weakly to a common fixed point of the mappings \( T_1 \) and \( T_2 \). This completes the proof. \qed

**Lemma 3.3.** Let \( E \) be a real uniformly convex Banach space and \( K \) be a nonempty closed convex subset which is also a nonexpansive retract of \( E \). Let \( T_1, T_2 : K \to E \) be two uniformly \( L \)-Lipschitzian asymptotically quasi-nonexpansive nonself mappings with sequences \( \{ k_n \}, \{ h_n \} \subset [1, \infty) \) such that \( F = \cap_{i=1}^{2} F(T_i) \neq \emptyset \). Suppose \( N_1 = \lim_n k_n \geq 1 \) and \( N_2 = \lim_n h_n \geq 1 \) such that \( \sum_{n=1}^{\infty} (k_n h_n - 1) < \infty \). From arbitrary \( x_1 \in K \), the sequence \( \{ x_n \} \) defined iteratively by \((3.2)\) with the restrictions \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n \tilde{\lambda}_n < \infty \). Let \( \{ a_n \} \) and \( \{ \tilde{a}_n \} \) be sequences in \( [\delta, 1 - \delta] \) for some \( \delta \in (0, 1) \). Then \( \lim_{n \to \infty} \| t x_n + (1 - t) p - q \| \) exists for all \( p, q \in F \) and \( t \in [0, 1] \).

**Proof.** By Lemma 3.1 we know that \( \{ x_n \} \) is bounded. Letting
\[ a_n(t) = \| t x_n + (1 - t) p - q \| \]
for all \( t \in [0, 1] \). Then \( \lim_{n \to \infty} a_n(0) = \| p - q \| \) and \( \lim_{n \to \infty} a_n(1) = \| x_n - q \| \) exists by Lemma 3.1. It, therefore, remains to prove the Lemma 3.3 for \( t \in (0, 1) \). For all \( x \in K \), we define the mapping \( W_n : K \to K \) by
\[ W_n x = P(a_n T_1 (PT_1)^{n-1} x + b_n T_2 (PT_2)^{n-1} P(\tilde{a}_n x + \tilde{b}_n T_1 (PT_1)^{n-1} x + \tilde{c}_n m_n) + c_n l_n). \]
Then
\[ \| W_n x - W_n y \| \leq [1 + (k_n^2 h_n^2 - 1)] \| x - y \| = [1 + \lambda_n] \| x - y \| = H_n \| x - y \| \]  
(3.23)
for all \( x, y \in K \), where \( H_n = [1 + \lambda_n] \) and \( \lambda_n = (k_n^2 h_n^2 - 1) \) with \( \sum_{n=1}^{\infty} \lambda_n < \infty \) and \( H_n \to 1 \) as \( n \to \infty \). Setting
\[ S_{n, m} = W_{n+m-1} W_{n+m-2} \ldots W_n, \ n \geq 1 \]  
(3.24)
and
\[ b_{n,m} = \|S_{n,m}(tx_n + (1-t)p) - (tS_{n,m}x_n + (1-t)S_{n,m}q)\|. \] (3.25)

From (3.23) and (3.24), we have
\[ \|S_{n,m}x - S_{n,m}y\| \leq H_nH_{n+1} \ldots H_{n+m-1}\|x - y\| \]
\[ \leq \left( \prod_{j=n}^{n+m-1} H_j \right)\|x - y\| \]
\[ = \sigma_n\|x - y\| \] (3.26)

for all \( x, y \in K \), where \( \sigma_n = \prod_{j=n}^{n+m-1} H_j \) and \( S_{n,m}x_n = x_{n+m}, S_{n,m}p = p \) for all \( p \in F \). Thus
\[ a_{n+m}(t) = \|tx_{n+m} + (1-t)p - q\| \leq b_{n,m} + \|S_{n,m}(tx_n + (1-t)p) - q\| \leq b_{n,m} + \sigma_n a_n(t). \] (3.27)

It follows from (3.25), (3.26) and Lemma 2.6 that
\[ b_{n,m} \leq \sigma_n \phi^{-1}((\|x_n - p\| - \sigma_n^{-1}\|x_{n+m} - p\|)). \]

By Lemma 3.1 and \( \lim_{n \to \infty} \sigma_n = 1 \), we have \( \lim_{n \to \infty} b_{n,m} = 0 \) and so
\[ \limsup_{n \to \infty} a_n(t) \leq \lim_{n \to \infty} b_{n,m} + \liminf_{n \to \infty} \sigma_n a_n(t) = \liminf_{n \to \infty} a_n(t). \]

This shows that \( \lim_{n \to \infty} a_n(t) \) exists, that is,
\[ \lim_{n \to \infty} \|tx_n + (1-t)p - q\| \]
exists for all \( t \in [0,1] \). This completes the proof. \( \square \)

**Theorem 3.4.** Let \( E \) be a real uniformly convex Banach space such that its dual \( E^* \) has the Kadec-Klee property and \( K \) be a nonempty closed convex subset which is also a nonexpansive retract of \( E \). Let \( T_1, T_2 : K \to E \) be two uniformly \( \mathcal{L} \)-Lipschitzian asymptotically quasi-nonexpansive nonself mappings with sequences \( \{k_n\}, \{h_n\} \subset [1, \infty) \) such that \( F = \cap_{i=1}^{2} F(T_i) \neq \emptyset \). Suppose \( N_1 = \lim_n k_n \geq 1 \) and \( N_2 = \lim_n h_n \geq 1 \) such that \( \sum_{n=1}^{\infty} (k_nh_n - 1) < \infty \). From arbitrary \( x_1 \in K \), the sequence \( \{x_n\} \) defined iteratively by (2.3) with the restrictions \( \sum_{n=1}^{\infty} c_n < \infty \) and \( \sum_{n=1}^{\infty} b_n < \infty \). Let \( \{a_n\} \) and \( \{\bar{a}_n\} \) be sequences in \( [\delta, 1 - \delta] \) for some \( \delta \in (0,1) \). If the mappings \( I - T_1 \) and \( I - T_2 \), where \( I \) denotes the identity mapping, are demiclosed at zero. Then \( \{x_n\} \) converges weakly to a common fixed point of the mappings \( T_1 \) and \( T_2 \).

**Proof.** By Lemma 3.1, we know that \( \{x_n\} \) is bounded and since \( E \) is reflexive, there exists a subsequence \( \{x_{n_j}\} \) of \( \{x_n\} \) which converges weakly to some \( p \in K \). By Lemma 3.1, we have
\[ \lim_{n \to \infty} \|x_{n_j} - T_1x_{n_j}\| = 0, \quad \lim_{n \to \infty} \|x_{n_j} - T_2x_{n_j}\| = 0. \]

Since the mappings \( I - T_1 \) and \( I - T_2 \) are demiclosed at zero, therefore \( T_1p = p \) and \( T_2p = p \), which means \( p \in F \). Now, we show that \( \{x_n\} \) converges weakly to \( p \). Suppose \( \{x_n\} \) is another subsequence of \( \{x_n\} \) converges weakly to some \( q \in K \). By the same method as above, we have \( q \in F \) and \( p, q \in w_e(x_n) \). By Lemma 3.3 the limit
\[ \lim_{n \to \infty} \|tx_n + (1-t)p - q\| \]
exists for all \( t \in [0,1] \) and so \( p = q \) by Lemma 2.5. Thus, the sequence \( \{x_n\} \) converges weakly to \( p \in F \). This completes the proof. \( \square \)
Remark 3.5. If we put $c_n = \tilde{c}_n = 0$, $T_1 = I$ and $T_2 = T$ then Theorem 3.2 extends Theorem 2.1 of Schu [21] to the case of more general class of non-self maps considered in this paper.

Remark 3.6. Theorem 3.4 extends Theorem 3.10 of Chidume et al. [5] to the case of modified Ishikawa type iteration process with errors in the sense of Xu [28] for two mappings considered in this paper.

References