Existence of solutions for quasi-linear impulsive functional integrodifferential equations in Banach spaces

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Abstract

We study the existence of mild solutions for quasilinear impulsive integrodifferential equation in Banach spaces. The results are established by using Hausdorff’s measure of noncompactness and fixed point theorem. Application is provided to illustrate the theory. ©2014 All rights reserved.

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1. Introduction

Measures of noncompactness are a very useful tool in many branches of mathematics. They are used in the fixed point theory, linear operators theory, theory of differential and integral equations and others\textsuperscript{4}. There are two measures which are the most important ones. The Kuratowski measure of noncompactness $\sigma(X)$ of a bounded set $X$ in a metric space is defined as infimum of numbers $r > 0$ such that $X$ can be covered with a finite number of sets of diameter smaller than $r$. The Hausdorff measure of noncompactness $\Psi(X)$ defined as infimum of numbers $r > 0$ such that $X$ can be covered with a finite number of balls of radii smaller than $r$. The Hausdorff measure is convenient in applications. There exist many formulae on $\Psi(X)$ in various spaces\textsuperscript{4,5}. However, there are some differences between the Kuratowski measure and the Hausdorff measure.

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One of these differences is that $\sigma(X)$ depends on the set $X$ only, while $\Psi(X)$ also depends on the space in which $X$ is included. Let $E$ be a Banach space and $F$ be a subspace of $E$. Let $\Psi_E(X), \Psi_F(X), \sigma_E(X), \sigma_F(X)$ denote Hausdorff and Kuratowski measures in spaces $E,F$, respectively. Then, for any bounded $X \subset F$ we have

$$\Psi_E(X) \leq \Psi_F(X) \leq \sigma_F(X) = \sigma_E(X) \leq 2\Psi_E(X).$$

The notion of a measure of noncompactness turns out to be a very important and useful tool in many branches of mathematical analysis. The notion of a measure of weak compactness was introduced by De Blasi [10] and was subsequently used in numerous branches of functional analysis and the theory of differential and integral equations. Several authors have studied the measures of noncompactness in Banach spaces [4, 14, 19, 3, 12, 11].

On the other hand, the study of the impulsive differential equations has attract a great deal of attention. Many evolution processes are characterized by the fact that at certain moments of time they experience a change of state abruptly. These processes are subject to short-term perturbations whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. The theory of impulsive differential equations is an important branch of differential equations have studied by many authors [16, 19, 20, 21].

The existence of solution to evolution equations with nonlocal conditions in Banach space was studied first by Byszewski [7, 8]. Byszewski and Lakshmikanthan [9] proved an existence and uniqueness of solutions of a nonlocal Cauchy problem in Banach spaces. Ntouyas and Tsamatos [23] studied the existence for semilinear perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many processes whose duration is negligible in comparison with the duration of the process. Consequently, it is natural to assume that these perturbations act instantaneously, that is, in the form of impulses. It is known, for example, that many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control model in economics, pharmacokinetics and frequency modulated systems, do exhibit impulsive effects. The theory of impulsive differential equations is an important branch of differential equations have studied by many authors [16, 19, 20, 21].

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In this paper, we consider the quasilinear integrodifferential equations with impulsive and nonlocal condition of the form

$$u'(t) + A(t, u(t))u(t) = f(t, u(t)) + \int_0^t g(t, s, u(s))ds, \quad t \in [0, b], \quad t \neq t_i$$

$$u(0) + h(u) = u_0 \quad (1.2)$$

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, 3, \ldots, n, \quad 0 < t_1 < t_2 < \ldots < t_n < b, \quad (1.3)$$

where $A : [0, b] \times X \to X$ are continuous functions in Banach space $X$, $u_0 \in X$, $f : [0, b] \times X \to X$, $g : \Omega \times X \to X$, $h : PC([0, b]; X) \to X$ and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. Here $\Omega = \{(t,s) : 0 \leq s \leq t \leq b\}$.

2. Preliminaries

Let $X$ be a Banach space with norm $\|\cdot\|$. Let $PC([0, b]; X)$ consist of functions $u$ from $[0, b]$ into $X$, such that $u(t)$ is continuous at $t \neq t_i$ and left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, 3, \ldots, n$. Evidently $PC([0, b]; X)$ is a Banach space with the norm

$$\|u\|_{PC} = \sup_{t \in [0, b]} \|u(t)\|,$$

and denoted $L([0, b]; X)$ by the space of $X$-valued Bochner integrable functions on $[0, b]$ with the form

$$\|u\|_L = \int_0^b \|u(t)\| dt.$$
The Hausdorff’s measure of noncompactness $\Psi_Y$ is defined by

$$\Psi(Y) = \inf \{ r > 0, Y \text{ can be covered by finite number of balls with radii } r \}$$

for bounded set $Y$ in a Banach space $X$.

**Lemma 2.1** ([4]). Let $Y$ be a real Banach space and $B, E \subseteq Y$ be bounded, with the following properties:

(i) $B$ is precompact if and only if $\Psi_X(B) = 0$.

(ii) $\Psi_Y(B) = \Psi_Y(\bar{B}) = \Psi_Y(\text{conv} B)$, where $\bar{B}$ and $\text{conv} B$ mean the closure and convex hull of $B$ respectively.

(iii) $\Psi_Y(B) \leq \Psi_Y(E)$, where $B \subseteq E$.

(iv) $\Psi_Y(B + E) \leq \Psi_Y(B) + \Psi_Y(E)$, where $B + E = \{ x + y : x \in B, y \in E \}$

(v) $\Psi_Y(B \cup E) \leq \max \{ \Psi_Y(B), \Psi_Y(E) \}$.

(vi) $\Psi_Y(\lambda B) \leq |\lambda| \Psi_Y(B)$ for any $\lambda \in \mathbb{R}$.

(vii) If the map $F : D(F) \subseteq Y \to Z$ is Lipschitz continuous with constant $r$, then $\Psi_Z(FB) \leq r \Psi_Y(B)$ for any bounded subset $B \subseteq D(F)$, where $Z$ be a Banach space.

(viii) $\Psi_Y(B) = \inf \{ d_Y(B, E) : E \subseteq Y \text{ is precompact} \} = \inf \{ d_Y(B, E) : E \subseteq Y \text{ is finite valued} \}$, where $d_Y(B, E)$ means the nonsymmetric (or symmetric) Hausdorff distance between $B$ and $E$ in $Y$.

(ix) If $\{ \mathcal{W}_n \}_{n=1}^{\infty}$ is decreasing sequence of bounded closed nonempty subsets of $Y$ and $\lim_{n \to \infty} \Psi_Y(\overline{\mathcal{W}_n}) = 0$, then $\bigcap_{n=1}^{\infty} \mathcal{W}_n$ is nonempty and compact in $Y$.

The map $F : \mathcal{W} \subseteq Y \to Y$ is said to be a $\Psi_Y$-contraction if there exists a positive constant $r < 1$ such that $\Psi_Y(FB) \leq r \Psi_Y(B)$ for any bounded closed subset $B \subseteq \mathcal{W}$, where $Y$ be a Banach space.

**Lemma 2.2** (Darbo-Sadovskii ([4]). If $\mathcal{W} \subseteq Y$ is bounded closed and convex, the continuous map $F : \mathcal{W} \to \mathcal{W}$ is a $\Psi_Y$-contraction, the map $F$ has at least one fixed point in $\mathcal{W}$.

We denote by $\Psi$ the Hausdorff’s measure of noncompactness of $X$ and also denote $\Psi_c$ by the Hausdorff’s measure of noncompactness of $\mathcal{PC}([0, b]; X)$.

Before we prove the existence results, we need the following Lemmas.

**Lemma 2.3** ([4]). If $\mathcal{W} \subseteq \mathcal{PC}([0, b]; X)$ is bounded, then $\Psi(\mathcal{W}(t)) \leq \Psi_c(\mathcal{W})$ for all $t \in [0, b]$, where $\mathcal{W}(t) = \{ u(t) : u \in \mathcal{W} \} \subseteq X$. Furthermore if $\mathcal{W}$ is equicontinuous on $[a, b]$, then $\Psi(\mathcal{W}(t))$ is continuous on $[a, b]$ and $\Psi(\mathcal{W}) = \sup \{ \Psi(\mathcal{W}(t)) \}$, $t \in [a, b]$.

**Lemma 2.4** ([17], [18]). If $\{ u_n \}_{n=1}^{\infty} \subset L^1([a, b]; X)$ is uniformly integrable, then the function $\Psi(\{ u_n(t) \}_{n=1}^{\infty})$ is measurable and

$$\Psi \left( \left\{ \int_0^t u_n(s)ds \right\}_{n=1}^{\infty} \right) \leq 2 \int_0^t \Psi(\{ u_n(s) \}_{n=1}^{\infty})ds. \quad (2.1)$$

**Lemma 2.5** ([4]). If $\mathcal{W} \subseteq \mathcal{PC}([0, b]; X)$ is bounded and equicontinuous, then $\Psi(\mathcal{W}(t))$ is continuous and

$$\Psi(\int_0^t \mathcal{W}(s)ds) \leq \int_0^t \Psi(\mathcal{W}(s))ds. \quad (2.2)$$

for all $t \in [0, b]$, where $\int_0^t \mathcal{W}(s)ds = \{ \int_0^t u(s)ds : u \in \mathcal{W} \}$.

The $C_0$ semigroup $U(t, s)$ is said to be equicontinuous if $t, s \to \{ U(t, s)u(s) : u \in B \}$ is equicontinuous for $t > 0$ for all bounded set $B$ in $X$. The following lemma is obvious.

**Lemma 2.6.** If the evolution family $\{ U(t, s) \}_{0 \leq s \leq t \leq b}$ is equicontinuous and $\eta \in L([0, b]; \mathbb{R}^+)$, then the set $\{ \int_0^t U(t, s)u(s)ds, \| u(s) \| \leq \eta(s) \}$ for a.e. $s \in [0, b]$ is equicontinuous for $t \in [0, b]$.
From [3], we know that for any fixed \( u \in \mathcal{PC}([0, b] ; X) \) there exist a unique continuous function \( U_u : [0, b] \times [0, b] \to B(X) \) defined on \([0, b] \times [0, b] \) such that
\[
U_u(t, s) = I + \int_s^t A_u(w)U_u(w, s)dw,
\]
where \( B(X) \) denote the Banach space of bounded linear operators from \( X \) to \( X \) with the norm \( \| F \| = \sup \{\| Fu \| : \| u \| = 1 \} \), and \( I \) stands for the identity operator on \( X \), \( A_u(t) = A(t, u(t)) \), we have
\[
U_u(t, t) = I, \quad U_u(t, s)U_u(s, r) = U_u(t, r), \quad (t, s, r) \in [0, b] \times [0, b] \times [0, b],
\]
\[
\frac{\partial U_u(t, s)}{\partial t} = A_u(t)U_u(t, s) \quad \text{for almost all} \quad t, s \in [0, b]
\]

3. The Existence of Mild Solution

**Definition 3.1.** A function \( u \in \mathcal{PC}([0, b] ; X) \) is said to be a mild solution of (1.1) - (1.3) if it satisfies the integral equation
\[
u(t) = U_u(t, 0)u_0 - U_u(t, 0)h(u) + \int_0^t U_u(t, s)f(s, u(s)) + \int_s^t g(s, r, u(r))d\tau ds
+ \sum_{0 < t_i < t} U_u(t, t_i)I_i(u(t_i)), \quad 0 \leq t \leq b.
\]

In this paper, we denote \( M_0 = \sup \{\| U_u(t, s) \| : (t, s) \in [0, b] \times [0, b] \} \) for all \( u \in X \). Without loss of generality, we let \( u_0 = 0 \). Assume the following conditions:

(H1) The evolution family \( \{U_u(t)\}_{0 \leq s \leq t \leq b} \) generated by \( A(t, u(t)) \) is equicontinuous and \( \| U_u(t, s) \| \leq M_0 \) for almost \( t, s \in [0, b] \).

(H2) (a) The function \( h : \mathcal{PC}([0, b] \times X \to X \) is continuous and compact.

(b) There exist \( N_0 > 0 \) such that \( \| h(u) \| \leq N_0 \) for all \( u \in \mathcal{PC}([0, b]) \).

(H3) (i) The nonlinear function \( f : [0, b] \times X \to X \) satisfies the Caratheodory-type conditions; i.e., \( f(\cdot, u) \) is measurable for all \( u \in X \) and \( f(t, \cdot) \) is continuous for a.e \( t \in [a, b] \).

(ii) There exists a function \( \alpha \in \mathcal{L}([0, b]; \mathcal{R}^+) \) such that for every \( u \in X \), we have
\[
\| f(t, u) \| \leq \alpha(t)(1 + \| u \|)
\]
a.e \( t \in [0, b] \).

(iii) There exists a function \( k_1 \in \mathcal{L}([0, b]; \mathcal{R}^+) \) such that, for every bounded \( D \subset X \), we have
\[
\Psi(f(t, D)) \leq k_1(t)\Psi(D)
\]
a.e \( t \in [0, b] \).

(H4) (i) The nonlinear function \( g : [0, b] \times [0, b] \times X \to X \) satisfies the Caratheodory-type conditions; i.e., \( g(\cdot, \cdot, u) \) is measurable for all \( u \in X \) and \( g(t, s, \cdot) \) is continuous for a.e \( t \in [a, b] \).

(ii) There exist two functions \( \beta_1 \in \mathcal{L}([0, b]; \mathcal{R}^+) \) and \( \beta_2 \in \mathcal{L}([0, b]; \mathcal{R}^+) \) such that for every \( u \in X \), we have
\[
\| g(t, s, u(s)) \| \leq \beta_1(t)\beta_2(s)(1 + \| u(s) \|)
\]
a.e \( t \in [0, b] \).

(iii) There exists a function \( k_2 \in \mathcal{L}([0, b]; \mathcal{R}^+) \) such that, for every bounded \( D \subset X \), we have
\[
\Psi(g(t, s, D)) \leq k_2(t)k_3(s)\Psi(D)
\]
a.e \( t \in [0, b] \). Assume that the finite bound of \( \int_0^t k_2(s)ds \) is \( G_0 \).
(H$_5$) For every $t \in [0,b]$ and there exist positive constants $N_1$ and $N_2$, the scalar equation

$$m(t) = M_0N_1 + M_0N_2 \left[ \int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i \right].$$

(H$_5$) $I_i : X \rightarrow X$ is continuous. There exists a constant $d_i > 0$ $i = 1, 2, 3, \ldots , n$ such that

$$\|I_i(u(t_i))\| \leq \sum_{i=1}^n d_i, \text{ where, } i = 1, 2, 3, \ldots , n.$$ 

For any bounded subset $D \subset X$, and there is a constant $l_i > 0$ such that

$$\Psi(I_i(D)) \leq \sum_{i=1}^n l_i \Psi(D), \text{ } i = 1, 2, \ldots , n.$$ 

**Theorem 3.2.** Assumptions (H$_1$) - (H$_5$) holds, then the impulsive nonlocal problem (1.1) - (1.3) has at least one mild solution.

**Proof.** Let $m(t)$ be a solution of the scalar equation

$$m(t) = M_0N_0 + M_0 \left[ \int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i \right]. \quad (3.2)$$

Let us assume that the finite bound of $\int_0^t \beta_1(s)ds$ is $C_0$ for $t \in [0,b]$. Consider the map $F : \mathcal{PC}([0,b]; X) \rightarrow \mathcal{PC}([0,b]; X)$ defined by

$$(Fu)(t) = U_u(t,0)h(u) + \int_0^t U_u(t,s) \left[ f(s,u(s)) + \int_s^t g(s,\tau,u(\tau))d\tau \right] ds$$

$$+ \sum_{0 < t_i < t} U_u(t,t_i)I_i(u(t_i)) \quad (3.3)$$

for all $u \in \mathcal{PC}([0,b]; X)$. Let us take $\mathcal{W}_0 = \{ u \in \mathcal{PC}([0,b]; X), \|u(t)\| \leq m(t) \text{ for all } t \in [0,b] \}$. Then $\mathcal{W}_0 \subseteq \mathcal{PC}([0,b]; X)$ is bounded and convex.

We define $\mathcal{W}_1 = \text{cconv}K(\mathcal{W}_0)$, where $\text{cconv}$ means the closure of the convex hull in $\mathcal{PC}([0,b]; X)$. As $U_u(t,s)$ is equicontinuous, $h$ is compact and $\mathcal{W}_0 \subseteq \mathcal{PC}([0,b]; X)$ is bounded, due to Lemma 2.6 and using the assumptions, $\mathcal{W}_1 \subseteq \mathcal{PC}([0,b]; X)$ is bounded closed convex nonempty and equicontinuous on $[0,b]$. For any $u \in F(\mathcal{W}_0)$, we know

$$\|u(t)\| \leq \left\| U_u(t,0)h(u) \right\| + \int_0^t \left\| U_u(t,s) \left[ f(s,u(s)) + \int_s^t g(s,\tau,u(\tau))d\tau \right] ds \right\| ds$$

$$+ \sum_{0 < t_i < t} \|U_u(t,t_i)I_i(u(t_i))\|$$

$$\leq M_0N_0 + M_0 \left[ \int_0^t \|f(s,u(s))\|ds + \int_0^t \int_s^t \|g(s,\tau,u(\tau))\|d\tau ds \right] + M_0 \sum_{i=1}^n \|I_i(u(t_i))\|$$

$$\leq M_0N_0 + M_0 \int_0^t \alpha(s)(1 + \|u(s)\|)ds + M_0 \int_0^t \int_s^t \beta_1(s)\beta_2(\tau)(1 + \|u(\tau)\|)d\tau ds + M_0 \sum_{i=1}^n d_i$$

$$\leq M_0N_0 + M_0 \int_0^t \alpha(s)(1 + m(s))ds + M_0C_0 \int_0^t \beta_2(s)(1 + m(s))ds + M_0 \sum_{i=1}^n d_i$$

$$\leq M_0N_0 + M_0 \left[ \int_0^t [\alpha(s) + C_0\beta_2(s)](1 + m(s))ds + \sum_{i=1}^n d_i \right], \text{ for } t \in [0,b].$$

$$= m(t).$$
It follows that $\mathcal{W}_1 \subset \mathcal{W}_0$. We define $\mathcal{W}_{n+1} = \psi(\mathcal{F}(\mathcal{W}_n))$, for $n = 1, 2, 3, \ldots$. From above we know that $\{\mathcal{W}_n\}_{n=1}^\infty$ is a decreasing sequence of bounded, closed, convex, equicontinuous on $[0, b]$ and nonempty subsets in $\mathcal{P}C([0, b]; X)$.

Now for $n \geq 1$ and $t \in [0, b]$, $\mathcal{W}_n(t)$ and $\mathcal{F}(\mathcal{W}_n(t))$ are bounded subsets of $X$, hence, for any $\epsilon > 0$, there is a sequence $\{u_k\}_{k=1}^\infty \subset \mathcal{W}_n$ such that (see, e.g. [6], pp125).

$$\psi(\mathcal{W}_{n+1}(t)) = \psi(\mathcal{F}(\mathcal{W}_n(t)))$$

$$\leq 2M_0 \int_0^t \psi\left(f(s, \{u_k(s)\}_{k=1}^\infty)\right) ds + 4M_0 \int_0^t \int_0^s \psi\left(g(s, \tau, \{u_k(\tau)\}_{k=1}^\infty)\right) d\tau ds$$

$$+ 2M_0 \sum_{i=1}^n \psi\left(I_i(\{u_k(t_i)\}_{k=1}^\infty)\right) + \epsilon$$

$$\leq 2M_0 \int_0^t k_1(s) \psi\left(\{u_k(s)\}_{k=1}^\infty\right) ds + 4M_0 \int_0^t \int_0^s k_2(s)k_3(\tau) \psi\left(\{u_k(\tau)\}_{k=1}^\infty\right) d\tau ds$$

$$+ 2M_0 \sum_{i=1}^n I_i(\{u_k(t_i)\}_{k=1}^\infty) + \epsilon$$

$$\leq 2M_0 \left[ \int_0^t k_1(s) \psi(\mathcal{W}_n(s)) ds + 2G_0 \int_0^t k_3(s) \psi(\mathcal{W}_n(s)) ds + \sum_{i=1}^n I_i(\mathcal{W}_n(t_i)) \right] + \epsilon$$

Since $\epsilon > 0$ is arbitrary, it follows that from the above inequality that

$$\psi(\mathcal{W}_{n+1}(t)) \leq 2M_0 \left[ \int_0^t [k_1(s) + 2G_0k_3(s)] \psi(\mathcal{W}_n(s)) ds + \sum_{i=1}^n I_i(\mathcal{W}_n(t_i)) \right]$$

for all $t \in [0, b]$. Because $\mathcal{W}_n$ is decreasing for $n$, we have

$$\gamma(t) = \lim_{n \to \infty} \psi(\mathcal{W}_n(t))$$

for all $t \in [0, b]$. From (3.4), we have

$$\gamma(t) \leq 2M_0 \left[ \int_0^t [k_1(s) + 2G_0k_3(s)] \gamma(s) ds + \sum_{i=1}^n I_i(\gamma(t_i)) \right]$$

for $t \in [0, b]$, which implies that $\gamma(t) = 0$ for all $t_0 \in [0, b]$. By Lemma 2.3, we know that $\lim_{n \to \infty} \psi(\mathcal{W}_n(t)) = 0$. Using Lemma 2.1 we know that $\mathcal{W} = \bigcap_{n=1}^\infty \mathcal{W}_n$ is convex compact and nonempty in $\mathcal{P}C([0, b]; X)$ and $\mathcal{F}(\mathcal{W}) \subset \mathcal{W}$. By the Schauder fixed point theorem, there exist at least one mild solution $u$ of the initial value problem [1.1] - [1.3], where $u \in \mathcal{W}$ is a fixed point of the continuous map $\mathcal{F}$.

**Remark 3.3.** If the functions $f, g$ and $I_i$ are compact or Lipschitz continuous (see e.g [7, 9], then $(H_3) - (H_5)$ is automatically satisfied.

In some of the early related results in references and above results, it is supposed that the map $h$ is uniformly bounded. In fact, if $h$ is compact, then it must be bounded on bounded set. Here we give an existence result under growth condition of $f, g$ and $I_i$, when $h$ is not uniformly bounded. Precisely, we replace the assumptions $(H_3) - (H_4)$ by

$(H_6)$ There exists a function $p \in \mathcal{L}([0, b]; R^+)$ and a increasing function $\phi : R^+ \to R^+$ such that

$$\|f(t, u)\| \leq p(t)\phi(\|u\|),$$

for a.e $t \in [0, b]$ and forall $u \in \mathcal{P}C([0, b]; X)$.
(H₇) There exist two functions \( q \in \mathcal{L}([0, b]; R^+) \) and \( \tilde{q} \in \mathcal{L}([0, b]; R^+) \) and an increasing function \( \chi : R^+ \to R^+ \) such that
\[
\|g(t, s, u)\| \leq q(t)\tilde{q}(\|u\|),
\]
for a.e \( t \in [0, b] \) and for all \( u \in \mathcal{C}([0, b]; X) \). Assume that the finite bound of \( \int_0^t q(s)ds \) is \( G_1 \).

**Theorem 3.4.** Suppose that the assumptions \((H_1) - (H_7)\) and \((H_6) - (H_7)\) are satisfied, then the equation \( (1.1) - (1.3) \) has at least one mild solution if
\[
\lim_{r \to \infty} \sup_r \frac{M_0}{r} \left( \phi(r) + \varphi(r) \int_0^t (p(s)ds + G_1\chi(r)\int_0^t \tilde{q}(s)ds + \sum_{i=1}^n d_i) \right) < 1,
\]
where \( \varphi(r) = \sup\{\|h(u)\|, \|u\| \leq r\} \).

**Proof.** The inequality (3.5) implies that there exist a constant \( r > 0 \) such that
\[
M_0 \left[ \phi(r) + \varphi(r) \int_0^t p(s)ds + G_1\chi(r)\int_0^t \tilde{q}(s)ds + \sum_{i=1}^n d_i \right] < r.
\]
As in the proof of Theorem 3.1, let \( \mathcal{W}_0 = \{u \in \mathcal{C}([0, b]; X), \|u(t)\| \leq r\} \) and \( \mathcal{W}_1 = \varTheta M \mathcal{F} \mathcal{W}_0 \). Then for any \( u \in \mathcal{W}_1 \), we have
\[
\|u(t)\| \leq \|U_a(t, 0)h(u)\| + \int_0^t \|U_a(t, s)\left[ f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau \right] ds
\]
\[
+ \sum_{0 < s_i < t} \|U_a(t, s_i)I_i(u(s_i))\|
\]
\[
\leq M_0 \phi(r) + M_0 \int_0^t p(s)\phi(\|u(s)\|)ds + \int_0^t \int_0^s q(s)\tilde{q}(\tau)\chi(\|u(\tau)\|)d\tau ds + \sum_{i=1}^n d_i
\]
\[
\leq M_0 \phi(r) + M_0 \int_0^t p(s)\phi(\|u(s)\|)ds + G_1\int_0^t \tilde{q}(s)\chi(\|u(s)\|)ds + \sum_{i=1}^n d_i
\]
\[
\|u(t)\| \leq M_0 \left[ \phi(r) + \varphi(r) \int_0^t p(s)ds + G_1\chi(r)\int_0^t \tilde{q}(s)ds + \sum_{i=1}^n d_i \right] < r
\]
for \( t \in [0, b] \). It means that \( \mathcal{W}_1 \subset \mathcal{W}_0 \). So we can complete the proof similarly to Theorem 3.1.

4. When \( h \) is Lipschitz

In this section, we discuss the equation \( (1.1) - (1.3) \) when \( h \) is Lipschitz and \( f, g \) and \( I_i \) are not Lipschitz. Assume that

(H₈) The function \( h \) is a Lipschitz continuous in \( X \), there exist a constant \( L_0 > 0 \) such that
\[
\|h(u) - h(v)\| \leq L_0\|u - v\|, \ u, v \in \mathcal{C}([0, b]; X).
\]

**Theorem 4.1.** Suppose that the assumptions \((H_1) - (H_8)\) are satisfied, then the equation \( (1.1) - (1.3) \) has at least one mild solution provided that
\[
M_0 \left[ L_0 + 2\int_0^t (k_1(s) + 2G_0)ds + \sum_{i=1}^n l_i \right] < 1.
\]
Theorem 4.2. Suppose that the assumptions which is a solution of (1.1) - (1.3). This completes the proof.

Proof. Consider the map \( F : \mathcal{PC}([0, B]; X) \to \mathcal{PC}([0, B]; X) \) is defined by \( F = F_1 + F_2 \), where

\[
(F_1u)(t) = U_u(t, 0)h(u),
\]

\[
(F_2u)(t) = \int_0^t U_u(t, s)\left[f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau \right]ds + \sum_{0 < t_i < t} U_u(t, t_i)I_i(u(t_i))
\]

for \( u \in \mathcal{PC}([0, B]; X) \). As defined in the proof of Theorem 3.1. We define \( W_0 = \{ u \in \mathcal{PC}([0, B]; X) : ||u(t)|| \leq m(t) \} \) for all \( t \in [0, b] \) and let \( W = \text{conv}F\mathcal{W}_0 \). Then from the proof of Theorem 3.1 we know that \( W \) is a bounded closed convex and equicontinuous subset of \( \mathcal{PC}([0, b]; X) \) and \( F\mathcal{W} \subset \mathcal{W} \). We shall prove that \( F \) is \( \Psi_\epsilon \)-contraction on \( \mathcal{W} \). Then Darbo-Sadovskii’s fixed point theorem can be used to get a fixed point of \( F \) in \( \mathcal{W} \), which is a mild solution of (1.1) - (1.2).

First, for every bounded subset \( B \subset \mathcal{W} \), from the (H8) and Lemma 2.1 we have

\[
\Psi_\epsilon(F_1B) = \Psi_\epsilon(U_B(t, 0)h(B)) \leq M_0\Psi_\epsilon(h(B)) \leq M_0L_0\Psi_\epsilon(B).
\]  

Next, for every bounded subset \( B \subset \mathcal{W} \), for \( t \in [0, b] \) and every \( \epsilon > 0 \), there is a sequence \( \{u_k\}_{k=1}^\infty \subset B \), such that

\[
\Psi(F_2(B(t))) \leq 2\Psi(\{F_2u_k(t)\}_{n=1}^\infty) + \epsilon.
\]

Note that \( B \) and \( F_2B \) are equicontinuous, we can get from Lemma 2.1, Lemma 2.4, Lemma 2.5 and using the assumptions we get

\[
\Psi(F_2B(t)) \leq 2M_0 \int_0^t \Psi\left(f(s, \{u_k(s)\}_{k=1}^\infty)\right)ds + 4M_0 \int_0^t \int_0^s \Psi\left(g(s, \tau, \{u_k(\tau)\}_{k=1}^\infty)\right)d\tau ds + 2M_0 \sum_{0 < t_i < t} l_i \Psi\left(\{u_k(t_i)\}_{k=1}^\infty\right) + \epsilon
\]

\[
\leq 2M_0 \int_0^t k_1(s)\Psi\left(\{u_k(s)\}_{k=1}^\infty\right)ds + 4M_0 \int_0^t \int_0^s k_2(s)k_3(\tau)\Psi\left(\{u_k(\tau)\}_{k=1}^\infty\right)d\tau ds + 2M_0 \sum_{i=1}^n l_i \Psi(B) + \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that from the above inequality that

\[
\Psi_\epsilon(F_2B(t)) \leq 2M_0 \left[\int_0^t [k_1(s) + 2G_0k_3(s)]ds + \sum_{i=1}^n l_i \right] \Psi_\epsilon(B)
\]  

(4.3)

for any bounded \( B \subset \mathcal{W} \).

Now, for any subset \( B \subset \mathcal{W} \), due to Lemma 2.1, (4.2) and (4.3) we have

\[
\Psi_\epsilon(FB) \leq \Psi_\epsilon(F_1B) + \Psi_\epsilon(F_2B)
\]

\[
\leq M_0\left[L_0 + 2\int_0^t [k_1(s) + 2G_0k_3(s)]ds + \sum_{i=1}^n l_i \right] \Psi_\epsilon(B).
\]  

(4.4)

By (4.4) we know that \( F \) is a \( \Psi_\epsilon \)-contraction on \( \mathcal{W} \). By Lemma 2.2, there is a fixed point \( u \) of \( F \) in \( \mathcal{W} \), which is a solution of (1.1) - (1.3). This completes the proof.

Theorem 4.2. Suppose that the assumptions (H1) – (H8) are satisfied, then the equation (1.1) - (1.3) has at least one mild solution if (4.4) and the following condition is satisfied.

\[
M_0L_0 + \lim_{r \to \infty} \sup_r \frac{M_0}{r} \left(\phi(r) \int_0^t p(s)ds + \chi(r)G_1 \int_0^t q(s)ds + \sum_{i=1}^n d_i \right) < 1.
\]  

(4.5)
Proof. From the equation (4.5) and fact that $L_0 < 1$, there exist a constant $r > 0$ such that

$$M_0(\alpha L_0 + \|h(0)\| + \varphi(r) \int_0^t p(s)ds + \chi(r)G_1 \int_0^t \hat{q}(s)ds + \sum_{i=1}^n d_i < r.$$  

We define $W_0 = \{u \in PC([0, b]; X), \|u(t)\| \leq r, \text{ for all } t \in [0, b]\}$. Then for every $u \in W_0$ we have

$$\|Fu(t)\| \leq \|U_0(t, 0)h(u)\| + \int_0^t \|U_0(t, s)f(s, u(s)) + \int_0^s g(s, \tau, u(\tau))d\tau\|ds$$

$$+ \sum_{0 < t_i < t} \|U_0(t, t_i)I_i(u(t_i))\|$$

$$\leq M_0(\|h(u) - h(0) + h(0)\| + M_0 \int_0^t p(s) \varphi(\|u(s)\|)ds$$

$$+ \int_0^t \int_0^s q(s, \hat{q}(\tau))\chi(\|u(\tau)\|)d\tau ds + \sum_{i=1}^n d_i)$$

$$\|Fu(t)\| \leq M_0(\alpha L_0 + \|h(0)\| + \varphi(r) \int_0^t p(s)ds + \chi(r)G_1 \int_0^t \hat{q}(s)ds + \sum_{i=1}^n d_i < r.$$  

for $t \in [0, b]$. This means that $FW_0 \subset W_0$. Define $W = \text{com} FW_0$. The above proof also implies that $FW \subset W$. So we can prove the theorem similar with Theorem 4.1 and hence we omit it.

5. Application

As an application of Theorem 3.1 we shall consider the system (1.1) - (1.3) with a control parameter such as

$$u'(t) + A(t, u)u(t) = f(t, u(t)) + Cv(t) + \int_0^t g(t, s, u(s))ds, \quad t \in J = [0, b], \quad t \neq t_i$$

$$u(0) + h(u) = u_0.$$  

$$\Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \cdots, n, \quad 0 < t_1 < \cdots < t_n < b,$$  

where $A, f, g, h$ and $I_i$ are as before and $C$ is a bounded linear operator from a Banach space $V$ into $X$ and $v \in L^2(J, V)$. The mild solution of (5.1) - (5.3) is given by

$$u(t) = U_0(t, 0)[u_0 - h(u)] + \int_0^t U_0(t, s)f(s, u(s)) + Cv(s) + \int_0^s g(s, \tau, u(\tau))d\tau ds$$

$$+ \sum_{0 < t_i < t} U_0(t, t_i)I_i(u(t_i)), \quad 0 \leq t \leq b.$$  

Definition 5.1 (2). System (5.1) - (5.3) is said to be controllable on the interval $J$ if for every $u_0, u_1 \in X$, there exists a control $v \in L^2(J, V)$ such that the mild solution $u(\cdot)$ of (5.1) - (5.3) satisfies

$$u(0) + h(u) = u_0 \text{ and } u(b) = u_1.$$  

To study the controllability, we need the following additional condition

$(H_0)$ The linear operator $W : L^2(J, V) \to X$, defined by

$$Wv = \int_0^b U_0(b, s)Cv(s)ds,$$  

induces an inverse operator $\hat{W}^{-1}$ defined an $L^2(J,V)/\ker W$ and there exists a positive constant $M_1 > 0$ such that

$$\|C \hat{W}^{-1}\| \leq M_1.$$  

**Theorem 5.2.** If the assumptions $(H_1) - (H_9)$ are satisfied, then the system $(5.1)-(5.3)$ is controllable on $J$.

**Proof.** Using the assumption $(H_9)$, for an arbitrary function $u(\cdot)$, define the control

$$v(t) = \hat{W}^{-1}\left[ u_1 - U_u(b,0)[u_0 - h(u)] - \int_0^b U_u(t,s)\left( f(s,u(s)) + \int_0^s g(s,\tau,u(\tau))d\tau\right)ds - \sum_{0 < t_i < t} U_u(t,t_i)I_i(u(t_i))\right](t).$$

We shall show that when using this control, the operator $\mathcal{H}: Z \to Z$ defined by

$$(\mathcal{H}v)(t) = U_u(t,0)[u_0 - h(u)] + \int_0^t U_u(t,s)\left( f(s,u(s)) + Cv(s) + \int_s^\tau g(s,\tau,u(\tau))d\tau\right)ds + \sum_{0 < t_i < t} U_u(t,t_i)I_i(u(t_i)).$$

has a fixed point. This fixed point is, then a solution of $(5.1)-(5.3)$. Clearly, $(\mathcal{H}v)(b) = u_1$, which means that the control $v$ steers the system $(5.1)-(5.3)$ from the initial state $u_0$ to $u_1$ in time $b$, provided we can obtain a fixed point of the nonlinear operator $\mathcal{H}$. The remaining part of the proof is similar to Theorem 3.1 and hence, it is omitted.

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**References**


