On the Ulam stability of a quadratic set-valued functional equation

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Abstract

In this paper, we prove the Ulam stability of the following set-valued functional equation by employing the direct method and the fixed point method, respectively,

\[ f \left( x - \frac{y + z}{2} \right) \oplus f \left( x + \frac{y - z}{2} \right) \oplus f(x + z) = 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z). \]

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1. Introduction and Preliminaries

The investigation of the Ulam stability problems of functional equations originated from a question of Ulam \cite{Ulam} concerning the stability of group homomorphisms, i.e.,

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there is a homomorphism \( H : G_1 \to G_2 \) such that \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)?
The following year, Hyers \cite{7} gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hereafter, the theorem of Hyers was generalized by Aoki \cite{11} for additive mappings and by Rassias \cite{14} for linear mappings by allowing an unbounded Cauchy difference. It should be pointed out that Rassias’s work has a great influence on the development of the Ulam stability theory of functional equations. Afterwards, Găvruta \cite{6} generalized the Rassas’s theorem by using a general control function. Since then, the Ulam stability of various types of functional equations has been widely and extensively studied. For more details, the reader is referred to \cite{5,9,15,17}.

As a generalization of the stability of single-valued functional equations, Lu and Park \cite{11} initiated the study of the Ulam stability of set-valued functional equations, in which the functional inequality is replaced by an appropriate inclusion relation. In the following, various authors considered the Ulam stability problems of several types of set-valued functional equations by using a similar method \cite{12,13}. Unlike the previous approach, Kenary et al. \cite{10} applied the Hausdorff metric defined on all closed convex subsets of a Banach space to characterize the functional inequality and investigated the Ulam stability of several types of set-valued functional equations by using a fixed point technique, which is used to deal with the stability of single-valued functional equations. Recently, Jang et al. \cite{8} and Chu et al. \cite{3} further studied the Ulam stability of some generalized set-valued functional equations in a similar way.

In \cite{18}, Shen and Lan constructed the following functional equation:

\[
f(x - y + z) + f(x + y - z) + f(x + z) = 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z),
\]

they proved that the general solution of the preceding functional equation on an Abelian group is equivalent to the solution of the classic quadratic functional equation

\[
f(x + y) + f(x - y) = 2f(x) + 2f(y),
\]

it is natural to say that the above functional equation constructed by Shen and Lan is a quadratic functional equation.

Throughout this paper, unless otherwise stated, let \(X\) be a real vector space and \(Y\) be a Banach space with the norm \(\|\cdot\|_Y\). We denote by \(C_b(Y), C_c(Y)\) and \(C_{cb}(Y)\) the set of all closed bounded subsets of \(Y\), the set of all closed convex subsets of \(Y\) and the set of all closed convex bounded subsets of \(Y\), respectively.

Let \(A\) and \(B\) be two nonempty subsets of \(Y\), \(\lambda \in \mathbb{R}\). The addition and the scalar multiplication can be defined as follows

\[
A + B = \{a + b|a \in A, b \in B\}, \quad \lambda A = \{\lambda a|a \in A\}.
\]

Furthermore, for the subsets \(A, B \in C_c(Y)\), we write \(A \oplus B = \overline{A + B}\), where \(\overline{A + B}\) denotes the closure of \(A + B\).

Generally, for arbitrary \(\lambda, \mu \in \mathbb{R}^+\), we can obtain that

\[
\lambda A + \lambda B = \lambda(A + B), \quad (\lambda + \mu)A \subseteq \lambda A + \mu A.
\]

In particular, if \(A\) is convex, then we have \((\lambda + \mu)A = \lambda A + \mu A\).

For \(A, B \in C_b(Y)\), the Hausdorff distance between \(A\) and \(B\) is defined by

\[
h(A, B) := \inf\{\varepsilon > 0|A \subseteq B + c\overline{S}_Y, B \subseteq A + c\overline{S}_Y\},
\]

where \(\overline{S}_Y\) denotes the closed unit ball in \(Y\), i.e., \(\overline{S}_Y = \{y \in Y||y||_Y \leq 1\}\). Since \(Y\) is a Banach space, it is proved that \((C_{cb}(Y), \oplus, h)\) is a complete metric semigroup \cite{2}. Rådström \cite{10} proved that \((C_{cb}(Y), \oplus, h)\) can be isometrically embedded in a Banach space.

The main purpose of this paper is to establish the Ulam stability of the following quadratic set-valued functional equation by employing the direct method and the fixed point method, respectively.

\[
f(x - y + z) \oplus f(x + y - z) \oplus f(x + z) = 3f(x) \oplus \frac{1}{2}f(y) \oplus \frac{3}{2}f(z) \tag{1.1}
\]

The following are some properties of the Hausdorff distance.
Lemma 1.1 (Castaing and Valadier [2]). For any $A_1, A_2, B_1, B_2, C \in C_{cb}(Y)$ and $\lambda \in \mathbb{R}^+$, the following expressions hold:

(i) $h(A_1 \oplus A_2, B_1 \oplus B_2) \leq h(A_1, B_1) + h(A_2, B_2)$;
(ii) $h(\lambda A_1, \lambda B_1) = \lambda h(A_1, B_1)$;
(iii) $h(A_1 \oplus C, B_1 \oplus C) = h(A_1, B_1)$.

In the following, we recall a fundamental result in the fixed point theory to be used.

Lemma 1.2 (Diaz and Margolis [4]). Let $(X, d)$ be a complete generalized metric space, i.e., one for which $d$ may assume infinite values. Suppose that $J : X \to X$ is a strictly contractive mapping with Lipschitz constant $L < 1$. Then for every element $x \in X$, either

$$d(J^n x, J^{n+1} x) = \infty$$

for all $n \geq 0$ or there exists an $n_0 \in \mathbb{N}$ such that

(i) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;

(ii) The sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;

(iii) $y^*$ is the unique fixed point of $J$ in the set $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;

(iv) $d(y, y^*) \leq \frac{1}{1-\lambda}d(y, Jy)$ for all $y \in Y$.

2. Ulam stability of the quadratic set-valued functional equation (1.1): The direct method

In this section, we shall consider the Ulam stability of the set-valued equation (1.1) by employing the direct method.

Theorem 2.1. Let $\varphi : X^3 \to [0, \infty)$ be a function such that

$$\Phi(x, y, z) = \sum_{k=0}^{\infty} \frac{1}{4^k} \varphi(2^k x, 2^k y, 2^k z) < \infty$$ (2.1)

for all $x, y, z \in X$. Suppose that $f : X \to C_{cb}(Y)$ is the mapping with $f(0) = \{0\}$ and satisfies

$$h\left(f \left(x - \frac{y + z}{2}\right) \oplus f \left(x + \frac{y - z}{2}\right) \oplus f(x + z), 3f(x) \oplus \frac{1}{2} f(y) \oplus \frac{3}{2} f(z)\right) \leq \varphi(x, y, z)$$ (2.2)

for all $x, y, z \in X$. Then

$$Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}$$

exists for every $x \in X$ and defines a unique quadratic mapping $Q : X \to C_{cb}(Y)$ such that

$$h(f(x), Q(x)) \leq \frac{1}{4} \Phi(x, x, x)$$ (2.3)

for all $x \in X$.

Proof. Putting $y = z = x$ in (2.2). Since $f(0) = \{0\}$, by Lemma 1.1, we can get that

$$h\left(\frac{1}{4} f(2x), f(x)\right) \leq \frac{1}{4} \varphi(x, x, x)$$ (2.4)

for all $x \in X$. Replacing $x$ by $2^{n-1} x$ and dividing by $4^{n-1}$ in (2.4), we have

$$h\left(\frac{1}{4^n} f(2^n x), \frac{1}{4^{n-1}} f(x)\right) \leq \frac{1}{4^n} \varphi(2^{n-1} x, 2^{n-1} x, 2^{n-1} x)$$ (2.5)
for all $x \in X$ and $n \in \mathbb{N}$. From (2.4) and (2.5), it follows that
\[
\frac{1}{2^n} f(2^n x) \leq \frac{1}{4^n} \varphi(2^{k-1} x, 2^{k-1} x, 2^{k-1} x)
\]
(2.6)
for all $x \in X$ and $n \in \mathbb{N}$. Now we claim that the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is a Cauchy sequence in $(C_{cb}(Y), h)$. Indeed, for all $m, n \in \mathbb{N}$, by (2.6), we can obtain that
\[
\frac{1}{4^n} h\left(\frac{1}{4^n} f(2^{n+m} x), \frac{1}{4^n} f(2^{n+m} x)\right)
= \frac{1}{4^n} \frac{1}{4^n} h(\frac{1}{4^n} f(2^{n+m} x), f(2^{n+m} x))
\leq \frac{1}{4^n} \sum_{k=1}^{n} \frac{1}{4^k} \varphi(2^{m+k-1} x, 2^{m+k-1} x, 2^{m+k-1} x)
\leq \frac{1}{4^n} \sum_{k=0}^{n-1} \frac{1}{4^k+1} \varphi(2^{m+k} x, 2^{m+k} x, 2^{m+k} x)
\]
(2.7)
for all $x \in X$. From the condition (2.1), it follows that the last expression tends to zero as $m \to \infty$. Then, the sequence $\{\frac{1}{2^n} f(2^n x)\}$ is Cauchy. Therefore, the completeness of $C_{cb}(Y)$ implies that the following expression is well-defined, that is, we can define
\[
Q(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]
for all $x \in X$.

Next, we show that $Q$ satisfies the set-valued equality (1.1). Replacing $x, y, z$ by $2^n x, 2^n y, 2^n z$ in (2.2), respectively, and dividing both sides by $4^n$, we get
\[
\frac{1}{4^n} h\left(\frac{1}{2^n} f\left(2^n x - \frac{y + z}{2}\right) \oplus f\left(2^n x + \frac{y - z}{2}\right)\right) \oplus f(2^n(x + z)),
3f(2^n x) \oplus \frac{1}{2} f(2^n y) \oplus \frac{3}{2} f(2^n z) \leq \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z).
\]
By letting $n \to \infty$, since the right-hand side in the preceding expression tends to zero, we obtain that $Q$ is a quadratic set-valued mapping. Moreover, letting $n \to \infty$ in (2.6), we get the desired inequality (2.3).

To prove the uniqueness of $Q$. Assume that $Q'$ is another quadratic set-valued mapping satisfying the inequality (2.3). Thus we can infer that
\[
\frac{1}{4^n} h(Q(x), Q'(x)) = \frac{1}{4^n} h(Q(2^n x), Q'(2^n x))
\leq \frac{1}{4^n} (h(Q(2^n x), f(2^n x)) + h(f(2^n x), Q'(2^n x))
\leq \frac{2}{4^n+1} \Phi(2^n x, 2^n x, 2^n x).
\]
It is easy to see from the condition (2.1) that the last expression tends to zero as $n \to \infty$. Then, we obtain that $Q(x) = Q'(x)$ for all $x \in X$. This completes the proof of the theorem. \qed

**Corollary 2.2.** Let $0 < p < 2$ and $\theta \geq 0$ be real numbers, and let $X$ be a real normed space. Suppose that $f : X \to C_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies
\[
h\left(f\left(x - \frac{y + z}{2}\right) \oplus f\left(x + \frac{y - z}{2}\right) \oplus f(x + z), 3f(x) \oplus \frac{1}{2} f(y) \oplus \frac{3}{2} f(z)\right)
\leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)
\]
for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and
\[
h(f(x), Q(x)) \leq \frac{3\theta \|x\|^p}{4 - 2^p}
\]
for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and the result follows directly from Theorem 2.1.

\[ \tag{2.8} \]

Corollary 2.3. Let $0 < p < \frac{2}{3}$ and $\theta \geq 0$ be real numbers, and let $X$ be a real normed space. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies
\[
h(f\left( x - \frac{y + z}{2} \right) + f\left( x + \frac{y - z}{2} \right) + f(x + z), 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z)) \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)
\]
for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q : X \to \mathcal{C}_{cb}(Y)$ that satisfies the equality (1.1) and
\[
h(f(x), Q(x)) \leq \frac{\theta \|x\|^p}{4 - 2^p}
\]
for all $x \in X$.

Proof. Letting $\varphi(x, y, z) = \theta(\|x\|^p + \|y\|^p + \|z\|^p)$ and the result follows directly from Theorem 2.1.

\[ \tag{2.9} \]

Theorem 2.4. Let $\varphi : X^3 \to [0, \infty)$ be a function such that
\[
\Psi(x, y, z) = \sum_{k=0}^{\infty} 4^k \psi(2^{-k}x, 2^{-k}y, 2^{-k}z) < \infty
\]
for all $x, y, z \in X$. Suppose that $f : X \to \mathcal{C}_{cb}(Y)$ is the mapping satisfying
\[
h\left( f\left( x - \frac{y + z}{2} \right) + f\left( x + \frac{y - z}{2} \right) + f(x + z), 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z) \right) \leq \psi(x, y, z)
\]
for all $x, y, z \in X$. Then
\[
Q(x) = \lim_{n \to \infty} 4^n f(2^{-n}x)
\]
exists for every $x \in X$ and defines a unique quadratic mapping $Q : X \to \mathcal{C}_{cb}(Y)$ such that
\[
h(f(x), Q(x)) \leq \Psi\left( \frac{x}{2}, \frac{x}{2}, \frac{x}{2} \right)
\]
for all $x \in X$.

Proof. Letting $x = y = z = 0$ in (2.9), we get $f(0) = \{0\}$, since the condition $\Psi(0, 0, 0) = \sum_{k=0}^{\infty} 4^k \psi(0, 0, 0)$ implies that $\psi(0, 0, 0) = 0$.

Setting $y = z = x$ in (2.9), we have
\[
h(4f(x), f(2x)) \leq \psi(x, x, x)
\]
for all $x \in X$. Replacing $x$ by $\frac{x}{2}$ in (2.11), we get
\[
h\left( 4f\left( \frac{x}{2} \right), f(x) \right) \leq \psi\left( \frac{x}{2}, \frac{x}{2}, \frac{x}{2} \right)
\]
for all $x \in X$. Replacing $x$ by $\frac{x}{2n-1}$ and multiplying both sides by $4^{n-1}$ in (2.12), we can obtain that
\[
h\left( 4^n f\left( \frac{x}{2^n} \right), 4^{n-1} f\left( \frac{x}{2^{n-1}} \right) \right) \leq 4^{n-1} \psi\left( \frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n} \right)
\]
(2.13)
for all \( x \in X \) and \( n \in \mathbb{N} \). Combing the inequalities (2.12) and (2.13) gives

\[
h\left(4^n f\left(\frac{x}{2^n}\right), f(x)\right) \leq \sum_{k=0}^{n-1} 4^k \psi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}} \right)
\]

(2.14)

for all \( x \in X \) and \( n \in \mathbb{N} \). The rest of the proof is analogous to the proof of Theorem 2.1.

\[\square\]

**Corollary 2.5.** Let \( p > 2 \) and \( \theta \geq 0 \) be real numbers, and let \( X \) be a real normed space. Suppose that \( f : X \to \mathcal{C}_{cb}(Y) \) is a set-valued mapping satisfying

\[
h\left(f\left(x - \frac{y + z}{2}\right) + f\left(x + \frac{y - z}{2}\right) + f(x + z), 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z)\right)
\]

\[
\leq \theta\left(||x||^p + ||y||^p + ||z||^p\right)
\]

for all \( x, y, z \in X \). Then there exists a unique quadratic set-valued mapping \( Q : X \to \mathcal{C}_{cb}(Y) \) that satisfies the equality (1.1) and

\[
h(f(x), Q(x)) \leq \frac{3\theta||x||^p}{2^p - 4}
\]

for all \( x \in X \).

**Proof.** Letting \( \psi(x, y, z) = \theta(||x||^p + ||y||^p + ||z||^p) \) and the result follows directly from Theorem 2.4.

\[\square\]

**Corollary 2.6.** Let \( p > \frac{2}{3} \) and \( \theta \geq 0 \) be real numbers, and let \( X \) be a real normed space. Suppose that \( f : X \to \mathcal{C}_{cb}(Y) \) is a set-valued mapping satisfying

\[
h\left(f\left(x - \frac{y + z}{2}\right) + f\left(x + \frac{y - z}{2}\right) + f(x + z), 3f(x) + \frac{1}{2}f(y) + \frac{3}{2}f(z)\right)
\]

\[
\leq \theta||x||^p ||y||^p ||z||^p
\]

for all \( x, y, z \in X \). Then there exists a unique quadratic set-valued mapping \( Q : X \to \mathcal{C}_{cb}(Y) \) that satisfies the equality (1.1) and

\[
h(f(x), Q(x)) \leq \frac{\theta||x||^{3p}}{2^{3p} - 4}
\]

for all \( x \in X \).

**Proof.** Letting \( \psi(x, y, z) = \theta||x||^p ||y||^p ||z||^p \) and the result follows directly from Theorem 2.4.

\[\square\]

3. Ulam stability of the quadratic set-valued functional equation (1.1): The fixed point method

In this section, we will investigate the Ulam stability of the set-valued functional equation (1.1) by using the fixed point technique.

**Theorem 3.1.** Let \( \varphi : X^3 \to [0, \infty) \) be a function such that there exists a positive constant \( L < 1 \) satisfying

\[
\varphi(2x, 2y, 2z) \leq 4L \varphi(x, y, z)
\]

(3.1)

for all \( x, y, z \in X \). Assume that \( f : X \to \mathcal{C}_{cb}(Y) \) is a set-valued mapping with \( f(0) = \{0\} \) and satisfies the inequality (2.2) for all \( x, y, z \in X \). Then there exists a unique quadratic set-valued mapping \( Q \) defined by \( Q(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n} \) such that

\[
h(f(x), Q(x)) \leq \frac{1}{4(1 - L)} \varphi(x, x, x)
\]

(3.2)

for all \( x \in X \).
Proof. Consider the set $S = \{g|g : X \rightarrow C_{cb}(Y), g(0) = \{0\}\}$ and introduce the generalized metric $d$ on $S$, which is defined by

$$d(g_1, g_2) = \inf\{\mu \in (0, \infty)|h(g_1(x), g_2(x)) \leq \mu \varphi(x, x, x), \forall x \in X\},$$

where, as usual, $\inf\emptyset = \infty$. It can easily be verified that $(S, d)$ is a complete generalized metric space (see [10]).

Now, we define an operator $T : S \rightarrow S$ by

$$Tg(x) = \frac{1}{4}g(2x)$$

for all $x \in X$.

Let $g_1, g_2 \in S$ be given such that $d(g_1, g_2) = \epsilon$. Then

$$h(g_1(x), g_2(x)) \leq \epsilon \varphi(x, x, x)$$

for all $x \in X$. Thus, we can obtain that

$$h(Tg_1(x), Tg_2(x)) = h\left(\frac{1}{4}g_1(2x), \frac{1}{4}g_2(2x)\right)$$

$$= \frac{1}{4}h(g_1(2x), g_2(2x))$$

$$\leq \frac{1}{4}\epsilon \varphi(2x, 2x, 2x)$$

$$\leq L\epsilon \varphi(x, x, x)$$

for all $x \in X$. Hence, $d(g_1, g_2) = \epsilon$ implies that $d(Tg_1, Tg_2) \leq L\epsilon$. Therefore, we know that $d(Tg_1, Tg_2) \leq Ld(g_1, g_2)$, which means that $T$ is a strictly contractive mapping with the Lipschitz constant $L < 1$.

Moreover, we can infer from (2.4) that $d(Tf, f) \leq \frac{1}{4}$. By Lemma 1.2, there exists a set-valued mapping $Q : X \rightarrow C_{cb}(Y)$ satisfying the following:

(i) $Q$ is a fixed point of $T$, i.e., $4Q(x) = Q(2x)$ for all $x \in X$. Further, $Q$ is the unique fixed point of $T$ in the set $\{g \in S|d(f, g) < \infty\}$, which means that there exists an $\eta \in (0, \infty)$ such that

$$h(f(x), Q(x)) \leq \eta \varphi(x, x, x)$$

for all $x \in X$.

(ii) $d(T^n f, Q) \rightarrow 0$ as $n \rightarrow \infty$. Then we get

$$\lim_{n \rightarrow \infty} \frac{f(2^n x)}{4^n} = Q(x)$$

for all $x \in X$.

(iii) $d(f, Q) \leq \frac{1}{4^n}d(f, T f)$. Then we have $d(f, Q) \leq \frac{1}{4(1-L)}$, which implies the inequality (3.2) holds.

Finally, we replace $x, y, z$ by $2^n x, 2^n y, 2^n z$ in (2.2), respectively, and divide both sides by $4^n$, we obtain that

$$\frac{1}{4^n}h\left(f\left(2^n \left(x - \frac{y + z}{2}\right)\right) + f\left(2^n \left(x + \frac{y - z}{2}\right)\right) \oplus f\left(2^n (x + z)\right)\right)$$

$$\leq \frac{1}{4^n} \varphi(2^n x, 2^n y, 2^n z)$$

$$\leq \frac{1}{4^n} \cdot 4^n L^n \varphi(x, y, z)$$

$$= L^n \varphi(x, y, z).$$

Since $L < 1$, the last expression tends to zero as $n \rightarrow \infty$. By (ii), we conclude that $Q$ is a quadratic set-valued mapping satisfying (1.1).

\cbox
Remark 3.2. Based on Theorem 3.1, the corollaries 2.2 and 2.3 can also be directly obtained by choosing $L = 2^{p-2}$ and $L = 2^{p-\frac{3}{2}}$, respectively.

Theorem 3.3. Let $\varphi : X^3 \to [0, \infty)$ be a function such that there exists a positive constant $L < 1$ satisfying

$$\varphi(x, y, z) \leq \frac{1}{4}L\varphi(2x, 2y, 2z)$$  \hspace{1cm} (3.3)

for all $x, y, z \in X$. Assume that $f : X \to C_{cb}(Y)$ is a set-valued mapping with $f(0) = \{0\}$ and satisfies the inequality (2.2) for all $x, y, z \in X$. Then there exists a unique quadratic set-valued mapping $Q$ defined by $Q(x) = \lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right)$ such that

$$h(f(x), Q(x)) \leq \frac{L}{4(1-L)}\varphi(x, x, x)$$  \hspace{1cm} (3.4)

for all $x \in X$.

Proof. Let us consider the set $S$ and introduce the generalized metric $d$ on $S$ given as in Theorem 3.1.

Define a mapping $T : S \to S$ by

$$Tg(x) = 4g \left( \frac{x}{2} \right)$$

for all $x \in X$. By a similar argument as in Theorem 3.1, we can obtain that $T$ is a strictly contractive mapping with the Lipschitz constant $L$. From (2.12) and the condition (3.3), we can infer that $d(Tf, f) \leq \frac{L}{4}$. According to Lemma 1.2, there exists a set-valued mapping $Q : X \to C_{cb}(Y)$ such that the following results hold.

(i) $Q$ is a fixed point of $T$, i.e., $Q(x) = 4Q \left( \frac{x}{2} \right)$ for all $x \in X$. Moreover, $Q$ is the unique fixed point of $T$ in the set $\{ g \in S | d(g, f) < \infty \}$, which means that there exists an $\eta \in (0, \infty)$ such that

$$h(f(x), Q(x)) \leq \eta \varphi(x, x, x)$$

for all $x \in X$.

(ii) $d(T^n f, Q) \to 0$ as $n \to \infty$. Then we can obtain

$$\lim_{n \to \infty} 4^n f \left( \frac{x}{2^n} \right) = Q(x)$$

for all $x \in X$.

(iii) $d(f, Q) \leq \frac{1}{4^n}d(f, Tf)$. Then we get $d(f, Q) \leq \frac{L}{4(1-L)}$ and hence the inequality (3.4) holds.

Replacing $x, y, z$ by $2^{-n}x, 2^{-n}y, 2^{-n}z$ in (2.2), respectively, and multiplying both sides by $4^n$, we have

$$4^n h \left( f \left( 2^{-n} \left( x - \frac{y + z}{2} \right) \right) \right) + f \left( 2^{-n} \left( x + \frac{y - z}{2} \right) \right) \leq 4^n \varphi(2^n x, 2^n y, 2^n z)$$

$$\leq 4^n f(2^{-n}x) + \frac{1}{2}f(2^{-n}y) + \frac{3}{2}f(2^{-n}z)$$

$$\leq 4^n \cdot \frac{1}{4^n} L^n \varphi(x, y, z)$$

Since $L < 1$, the last expression tends to zero as $n \to \infty$. By (ii), we conclude that $Q$ is a quadratic set-valued mapping satisfying (1.1).

Remark 3.4. In view of Theorem 3.3, the corollaries 2.5 and 2.6 can also be directly obtained by taking $L = 2^{2-p}$ and $L = 2^{\frac{3}{2}-p}$, respectively.
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References


