Some new coupled fixed point theorems in ordered partial $b$-metric spaces

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Abstract

In this paper, we establish some new coupled fixed point theorems in ordered partial $b$-metric spaces. Also, an example is provided to support our new results. The results presented in this paper extend and improve several well-known comparable results. ©2016 All rights reserved.

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1. Introduction

Fixed point theory of nonlinear operators in metric spaces finds a lot of applications in convex optimization problems, see [13, 21, 27] and the references therein. In 1993, Czerwik [7] introduced the concept of the $b$-metric space. In 1994, Matthews [18] introduced the notion of partial metric spaces. After that, many researches have dealt with fixed point theories for various contraction mappings in $b$-metric spaces [5, 8, 11, 16, 22, 25, 26] and partial metric spaces [2, 3]. By combining these, Shukla [26] introduced a new generalization of metric space called partial $b$-metric space which was paid widespread attention immediately. Also, in [19] a modified version of partial $b$-metric space was introduced and many useful lemmas could be proved right away. Since then, several authors obtained more helpful results in this space [10, 19].

On the other hand, since the ordered set was introduced, many authors got many fixed point theorems in ordered metric space. In 2006, Bhaskar and Lakshmikantham [9] introduced the notion of a coupled fixed point and used the mixed monotone property to prove some coupled fixed point theorems. Three
years later, Lakshmikantham and Ćirić [14] introduced the new concepts of coupled coincidence and coupled common fixed and used a mixed $g$-monotone property to prove some coupled common fixed point theorems which extended Bhaskar and Lakshmikantham’s result from one mapping $F : X \times X \to X$ to two mapping $F : X \times X \to X$ and $g : X \to X$. Subsequently, many authors got a variety of coupled coincidence and coupled fixed point theorems in ordered metric spaces [11, 17, 20].

Recently, Aghajani and Arab [4] introduced a generalized contractive mapping with the altering distance functions and proved a new coupled common fixed point theorems in ordered $b$-metric space. Also, a number of articles on the topic of coupled fixed point theorems were obtained in ordered $b$-metric space and ordered partial metric space [1, 6, 23, 24]. But in ordered partial $b$-metric spaces, there are almost no research of them. In this paper, we use a more generalized contractive mapping to prove some coupled coincidence and coupled common fixed point theorems in ordered partial $b$-metric spaces. An example is provided to support our new results. The results presented in this paper extend and improve several well-known comparable results.

2. Preliminaries and definitions

First, we introduce some basic definitions and concepts as the following.

Definition 2.1 ([7]). A $b$-metric on nonempty set $X$ is a mapping $d : X \times X \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$,

1. $x = y \iff d(x, y) = 0$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$.

A $b$-metric space is a pair $(X, d)$ such that $X$ is a nonempty set and $d$ is a $b$-metric on $X$. The number $s$ is called the coefficient of $(X, d)$.

It is obvious that a $b$-metric space with coefficient $s = 1$ is a metric space. There are examples of $b$-metric spaces which are not metric spaces (see, e.g., Akkouchi [5]).

Definition 2.2 ([18]). A partial metric on a nonempty set $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$;
2. $p(x, x) \leq p(x, y)$;
3. $p(x, y) = p(y, x)$;
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair $(X, p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. If $p$ is a partial metric on $X$, then function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) := 2p(x, y) - p(x, x) - p(y, y),$$

is ordinary equivalent metric on $X$.

Definition 2.3 ([19]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A function $p_b : X \times X \to \mathbb{R}^+$ is a partial $b$-metric, if for all $x, y, z \in X$, the following conditions are satisfied:

1. $x = y \iff p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b2}) \ p_b(x, x) \leq p_b(x, y);

(p_{b3}) \ p_b(x, y) = p_b(y, x);

(p_{b4}) \ p_b(x, y) \leq s[p_b(x, z) + p_b(z, y) - p_b(z, z)] + (\frac{1}{s^2}) (p_b(x, x) + p_b(y, y)).

The pair \((X, p_b)\) is called a partial \(b\)-metric space.

**Example 2.4** ([20]). Let \(X = \mathbb{R}^+\), \(q > 1\) be a constant, and \(p_b : X \times X \to \mathbb{R}^+\) be defined by

\[ p_b(x, y) = \left[ \max\{x, y\}\right]^q + |x - y|^q, \quad \text{for all } x, y \in X. \]

Obviously, \((X, p_b)\) is a partial \(b\)-metric space with the coefficient \(s = 2^{q-1}\), but it is neither a partial metric space nor a \(b\)-metric space.

Other examples of partial \(b\)-metric can be constructed thank to the following propositions.

**Proposition 2.5** ([20]). Let \(X\) be a nonempty set and let \(p\) be a partial metric and \(d\) be a \(b\)-metric with the coefficient \(s \geq 1\) on \(X\). Then the function \(p_b : X \times X \to \mathbb{R}^+\) defined by

\[ p_b(x, y) = p(x, y) + d(x, y), \quad \text{for all } x, y \in X, \]

is a partial \(b\)-metric on \(X\) with the coefficient \(s\).

**Proposition 2.6** ([20]). Let \((X, p)\) be a partial metric space and \(q \geq 1\). Then \((X, p_b)\) is a partial \(b\)-metric space with the coefficient \(s = 2^{q-1}\), where \(p_b\) is defined by \(p_b(x, y) = [p(x, y)]^q\).

**Proposition 2.7** ([19]). Every partial \(b\)-metric \(p_b\) defines a \(b\)-metric \(d_{p_b}\), where

\[ d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, x) - p_b(y, y), \quad \text{for all } x, y \in X. \]

**Definition 2.8** ([19]). Let \((X, p_b)\) be a partial \(b\)-metric space with coefficient \(s \geq 1\). Let \(\{x_n\}\) be any sequence in \(X\) and \(x \in X\). Then

(i) the sequence \(\{x_n\}\) is said to be \(p_b\)-convergent to \(x\), if \(\lim_{n \to \infty} p_b(x_n, x) = p_b(x, x)\),

(ii) the sequence \(\{x_n\}\) is said to be \(p_b\)-Cauchy sequence in \((X, p_b)\), if \(\lim_{n,m \to \infty} p_b(x_n, x_m)\) exists and is finite.

(iii) \((X, p_b)\) is said to be a \(p_b\)-complete partial \(b\)-metric space, if for every Cauchy sequence \(\{x_n\}\) in \(X\), there exists \(x \in X\) such that

\[ \lim_{n,m \to \infty} p_b(x_n, x_m) = \lim_{n \to \infty} p_b(x_n, x) = p_b(x, x). \]

Thank to [19], we have the following important lemmas.

**Lemma 2.9** ([19]).

1. A sequence \(\{x_n\}\) is a \(p_b\)-Cauchy sequence in a partial \(b\)-metric space \((X, p_b)\), if and only if it is a \(b\)-Cauchy sequence in the \(b\)-metric space \((X, d_{p_b})\).

2. A partial \(b\)-metric space \((X, p_b)\) is \(p_b\)-complete, if and only if the \(b\)-metric space \((X, d_{p_b})\) is \(b\)-complete. Moreover, \(\lim_{n \to \infty} d_{p_b}(x, x_n) = 0\), if and only if

\[ \lim_{n,m \to \infty} p_b(x_n, x_m) = \lim_{n \to \infty} p_b(x_n, x) = p_b(x, x). \]

**Lemma 2.10** ([19]). Let \((X, p_b)\) be a partial \(b\)-metric space with the coefficient \(s \geq 1\) and suppose that \(\{x_n\}\) and \(\{y_n\}\) are convergent to \(x\) and \(y\), respectively. Then we have

\[ \frac{1}{s^2} p_b(x, y) - \frac{1}{s} p_b(x, x) - p_b(y, y) \leq \liminf_{n \to \infty} p_b(x_n, y_n) \leq \limsup_{n \to \infty} p_b(x_n, y_n) \leq s p_b(x, x) + s^2 p_b(y, y) + s^2 p_b(x, y). \]
In particular, if \( p_b(x, y) = 0 \), then we have \( \lim_{n \to \infty} d_{p_b}(x_n, y_n) = 0 \). Moreover, for each \( z \in X \), we have
\[
\frac{1}{s} p_b(x, z) - p_b(x, x) \leq \liminf_{n \to \infty} p_b(x_n, z) \leq \limsup_{n \to \infty} p_b(x_n, z) \leq s p_b(x, z) + s p_b(x, x).
\]
In particular, if \( p_b(x, x) = 0 \), then we have
\[
\frac{1}{s} p_b(x, z) \leq \liminf_{n \to \infty} p_b(x_n, z) \leq \limsup_{n \to \infty} p_b(x_n, z) \leq s p_b(x, z).
\]

**Definition 2.11** ([9]). An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \( F : X \times X \to X\), if \( F(x, y) = x \) and \( F(y, x) = y\).

**Definition 2.12** ([14]). An element \((x, y) \in X \times X\) is called a coupled coincidence point of the mapping \( F : X \times X \to X\) and \( g : X \to X\), if \( F(x, y) = gx \) and \( F(y, x) = gy\), and in this case, \((gx, gy)\) is called a coupled point of coincidence.

**Definition 2.13** ([1]). An element \((x, x) \in X \times X\) is called a common fixed point of the mapping \( F : X \times X \to X\) and \( g : X \to X\), if \( F(x, x) = gx = x\).

**Definition 2.14** ([14]). Let \( X \) be a nonempty set. Then we say that the mappings \( F : X \times X \to X\) and \( g : X \to X\) are commutative, if \( gF(x, y) = F(gx, gy)\).

**Definition 2.15** ([9]). Let \((X, \preceq)\) be a partially ordered set and \( F : X \times X \to X\). The mapping \( F\) is said to have the mixed monotone property, if \((F(x, y))\) is monotone non-decreasing in \( x\) and is monotone non-increasing in \( y\), that is, for any \( x, y \in X\), we have
\[
x_1, x_2 \in X, \ x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y),
\]
and
\[
y_1, y_2 \in X, \ y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2).
\]

**Definition 2.16** ([14]). Let \((X, \preceq)\) be a partially ordered set and \( F : X \times X \to X\) and \( g : X \to X\). The mapping \( F\) is said to have the mixed \( g\)-monotone property, if \((F(x, y))\) is monotone \( g\)-nondecreasing in its first argument and is monotone \( g\)-nonincreasing in its second argument, that is, for any \( x, y \in X\), we have
\[
x_1, x_2 \in X, \ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y),
\]
and
\[
y_1, y_2 \in X, \ g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2).
\]

**Definition 2.17** ([12]). A function \( \psi : [0, \infty) \to [0, \infty)\) is called an altering distance function, if the following properties are satisfied:

1. \( \psi \) is continuous and nondecreasing;
2. \( \psi(t) = 0 \), if and only if \( t = 0 \).

3. **Main results**

**Theorem 3.1.** Let \((X, \preceq, p_b)\) be a ordered partial \( b\)-metric space. Let \( F : X \times X \to X\) and \( g : X \to X\) be two mappings and \( F\) has the mixed \( g\)-monotone property with \( g\). Suppose that there exists an altering distance function \( \psi\) and \( \theta : [0, \infty)^{10} \to [0, \infty)\) is continuous with \( \theta(t_1, t_2, \ldots, t_{10}) = 0 \) implies \( t_1 = t_2 = t_5 = t_6 = 0 \)
such that
\[
\psi (sp_b(F(x, y), F(u, v))) \leq \psi (M(x, y, u, v) - \Theta(x, y, u, v))
\]
for all \((x, y), (u, v) \in X \times X\) with \(g(x) \preceq g(u)\) and \(g(y) \succeq g(v)\), where
\[
M(x, y, u, v) = \max \left\{ \frac{p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), p_b(gy, F(y, x))}{p_b(gx, F(u, v)) + p_b(gu, F(x, y))}, \frac{2s}{2s} \right\},
\]
and
\[
\Theta(x, y, u, v) = \theta \left( \frac{p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), p_b(gy, F(y, x))}{p_b(gy, F(u, v)) + p_b(gu, F(x, y))} \right).
\]

Further, suppose \(F(X \times X) \subseteq g(X)\) and \(g(X)\) is a \(p_b\)-complete subspace of \((X, p_b)\). Also suppose that \(X\) satisfies the following properties:

(i) if a non-decreasing sequence \(\{x_n\}\) in \(X\) converges to \(x \in X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\);

(ii) if a non-increasing sequence \(\{y_n\}\) in \(X\) converges to \(y \in X\), then \(y_n \succeq y\) for all \(n \in \mathbb{N}\).

If there exists \((x_0, y_0) \in X \times X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point.

Proof. By the given condition, there exists \((x_0, y_0) \in X \times X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\). Since \(F(X \times X) \subseteq g(X)\), we can define \((x_1, y_1) \in X \times X\) such that \(gx_1 = F(x_0, y_0)\) and \(gy_1 = F(y_0, x_0)\), then \(gx_0 \preceq F(x_0, y_0) = gx_1\) and \(gy_0 \succeq F(y_0, x_0) = gy_1\). Going on in this way, we can construct two sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that
\[
gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n), \quad \forall n \geq 0. \tag{3.2}
\]

Now we prove that
\[
gx_n \preceq gx_{n+1} \quad \text{and} \quad gy_n \succeq gy_{n+1}, \quad \forall n \geq 0.
\]

We will use the mathematical induction. The conclusion holds for \(n = 0\), suppose it holds for some \(n > 0\). Since \(F\) has the mixed \(g\)-monotone property, \(gx_n \preceq g(x_{n+1})\) and \(g(y_n) \succeq g(y_{n+1})\), from (3.2), we have
\[
\begin{cases}
gx_{n+1} = F(x_n, y_n) \preceq F(x_{n+1}, y_{n+1}) = gx_{n+2}, \quad \forall n \geq 0, \\
gy_{n+1} = F(y_n, x_n) \succeq F(y_{n+1}, x_{n+1}) = gy_{n+2}, \quad \forall n \geq 0.
\end{cases}
\]

Thus, by the mathematical induction, we conclude that
\[
\begin{cases}
gx_0 \preceq gx_1 \preceq gx_2 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots, \\
gy_0 \succeq gy_1 \succeq gy_2 \succeq \cdots \succeq gy_n \succeq gy_{n+1} \succeq \cdots. \tag{3.3}
\end{cases}
\]

From (3.2), (3.3), (3.1), and the property of \(\psi\) we have
\[
\begin{align*}
\psi (p_b(gx_n, gx_{n+1})) &\leq \psi (sp_b(gx_n, gx_{n+1})) = \psi (sp_b(F(x_n-1, y_{n-1}), F(x_n, y_n))) \\
&\leq \psi (M(x_n-1, y_{n-1}, x_n, y_n)) - \Theta(x_n-1, y_{n-1}, x_n, y_n), \tag{3.4}
\end{align*}
\]
where
\[
M(x_n-1, y_{n-1}, x_n, y_n)
= \max \left\{ \frac{p_b(gx_n-1, gx_n), p_b(gy_n-1, gy_n), p_b(gx_n-1, F(x_n-1, y_{n-1})), p_b(gy_n-1, F(y_n-1, x_n-1))}{p_b(gx_n, F(x_n, y_n)) + p_b(gy_n, F(y_n, x_n))}, \frac{2s}{2s} \right\},
\]
By combining (3.4) and (3.8), we get

\[
\frac{p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n)}{2s} \leq \frac{p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gx_n, gx_{n+1})}{2s}.
\]

Similarly, we can show that

\[
\frac{p_b(gx_{n-1}, gx_n) + p_b(gx_n, gx_{n+1})}{2s} \leq \frac{p_b(gx_{n-1}, gx_n) + p_b(gx_n, gx_{n+1}) + (1 - s)p_b(gx_n, gx_{n+1})}{2s} \leq \max\{p_b(gx_{n-1}, gx_n), p_b(gx_n, gx_{n+1})\}. \tag{3.6}
\]

By substituting (3.6) and (3.7) into (3.5), we obtain

\[
M(x_{n-1}, y_{n-1}, x_n, y_n) = \max \{p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})\} = \max \{\delta_{n-1}, \delta_n\}, \tag{3.8}
\]

where

\[
\delta_n = \max\{p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})\}.
\]

By combining (3.4) and (3.8), we get

\[
\psi(p_b(gx_n, gx_{n+1})) \leq \psi(\max\{\delta_{n-1}, \delta_n\}) - \Theta(x_{n-1}, y_{n-1}, x_n, y_n). \tag{3.9}
\]

By the same way as above, we can show that

\[
M(y_{n-1}, x_{n-1}, y_n, x_n) = \max \{\delta_{n-1}, \delta_n\},
\]

and

\[
\psi(p_b(gy_n, gy_{n+1})) \leq \psi(M(y_{n-1}, x_{n-1}, y_n, x_n)) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n) = \psi(\max\{\delta_{n-1}, \delta_n\}) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n), \tag{3.10}
\]

where

\[
\Theta(y_{n-1}, x_{n-1}, y_n, x_n) = \theta\left(\frac{p_b(gy_{n-1}, gy_n), p_b(gx_{n-1}, gx_n), p_b(gy_{n-1}, gy_n), p_b(gy_n, gy_{n+1}), p_b(gx_n, gx_{n+1})}{2s}\right).
\]
Next we prove that \( \delta_n \leq \delta_{n-1} \) for all \( n \in \mathbb{N} \). In fact, suppose that \( \delta_{n-1} < \delta_n \), then \( \delta_n > 0 \) (otherwise, \( \delta_{n-1} < \delta_n = 0 \), which is a contradiction). We consider the following two cases.

**Case 1.** \( \max\{\delta_{n-1}, \delta_n\} = \delta_n = p_b(gx_n, gx_{n+1}) > 0 \).

By (3.9) we have

\[
\psi(p_b(gx_n, gx_{n+1})) \leq \psi(p_b(gx_n, gx_{n+1})) - \Theta(x_{n-1}, y_{n-1}, x_n, y_n),
\]

which means \( \Theta(x_{n-1}, y_{n-1}, x_n, y_n) = 0 \). By the properties of \( \theta \), we can find \( p_b(gx_n, gx_{n+1}) = 0 \), which is a contradiction.

**Case 2.** \( \max\{\delta_{n-1}, \delta_n\} = \delta_n = p_b(gy_n, gy_{n+1}) > 0 \).

By (3.10) we have

\[
\psi(p_b(gy_n, gy_{n+1})) \leq \psi(p_b(gy_n, gy_{n+1})) - \Theta(y_{n-1}, x_{n-1}, y_n, x_n),
\]

which means \( \Theta(y_{n-1}, x_{n-1}, y_n, x_n) = 0 \). By the properties of \( \theta \) we can find \( p_b(gy_n, gy_{n+1}) = 0 \), which is a contradiction.

Therefore, we have \( \delta_n \leq \delta_{n-1} \) for all \( n \in \mathbb{N} \) holds, thus the sequence \( \{\delta_n\} \) is a non-increasing sequence of nonnegative real number, and so, there exists \( \delta \geq 0 \) such that

\[
\lim_{n \to \infty} \delta_n = \delta.
\]

Since \( \psi(\max\{x, y\}) = \max\{\psi(x), \psi(y)\} \), from (3.9) and (3.10) we have

\[
\begin{align*}
\psi(\delta_n) &= \max\{\psi(p_b(gx_n, gx_{n+1})), \psi(p_b(gy_n, gy_{n+1}))\} \\
&\leq \psi(\delta_{n-1}) - \min \left\{ \Theta(x_{n-1}, y_{n-1}, x_n, y_n), \Theta(y_{n-1}, x_{n-1}, y_n, x_n) \right\}.
\end{align*}
\]

(3.11)

By taking the upper limit as \( n \to \infty \) in (3.11), we have

\[
\begin{align*}
\psi(\delta) &\leq \psi(\delta) - \lim inf_{n \to \infty} \min \left\{ \Theta(x_{n-1}, y_{n-1}, x_n, y_n), \Theta(y_{n-1}, x_{n-1}, y_n, x_n) \right\} \\
&\leq \psi(\delta) - \min \left\{ \lim inf_{n \to \infty} \Theta(x_{n-1}, y_{n-1}, x_n, y_n), \lim inf_{n \to \infty} \Theta(y_{n-1}, x_{n-1}, y_n, x_n) \right\}.
\end{align*}
\]

Therefore,

\[
\lim inf_{n \to \infty} \Theta(x_{n-1}, y_{n-1}, x_n, y_n) = 0 \quad \text{or} \quad \lim inf_{n \to \infty} \Theta(y_{n-1}, x_{n-1}, y_n, x_n) = 0.
\]

Hence, by using the properties of \( \theta \), we get

\[
\lim inf_{n \to \infty} p_b(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim inf_{n \to \infty} p_b(gy_n, gy_{n+1}) = 0.
\]

So,

\[
\delta = \lim inf_{n \to \infty} \delta_n = \lim inf_{n \to \infty} \max\{p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})\} = 0.
\]

That is

\[
\lim_{n \to \infty} p_b(gx_n, gx_{n+1}) = 0 \quad \text{and} \quad \lim_{n \to \infty} p_b(gy_n, gy_{n+1}) = 0.
\]

(3.12)

From (p02) and (3.12), we have

\[
\lim_{n \to \infty} p_b(gx_n, gx_n) = 0 \quad \text{and} \quad \lim_{n \to \infty} p_b(gy_n, gy_n) = 0.
\]

(3.13)

Next we prove that \( \{gx_n\}, \{gy_n\} \) are \( p_b \)-Cauchy sequences in \( g(X) \). For this, we have to show that \( \{gx_n\}, \{gy_n\} \) are \( b \)-Cauchy sequences in \( g(X), d_{p_b} \). In other words, we need to show that for every \( \varepsilon > 0 \), there exists \( k \in \mathbb{N} \) such that for all \( m, n \geq k \),

\[
\max\{d_{p_b}(gx_m, gx_n), d_{p_b}(gy_m, gy_n)\} < \varepsilon.
\]
By the triangle inequality, we have
\[ \max\{d_{p_0}(g_{x_m_i}, g_{x_n}), d_{p_0}(g_{y_m}, g_{y_n})\} \geq \varepsilon. \quad (3.14) \]
That is,
\[ \max\{d_{p_0}(g_{x_m_i}, g_{x_{n-1}}), d_{p_0}(g_{y_m}, g_{y_{n-1}})\} < \varepsilon. \quad (3.15) \]
From the definition of \( d_{p_0} \), (3.12) and (3.13) we obtain
\[ \lim_{n \to \infty} d_{p_0}(g_{x_m}, g_{x_{n+1}}) = 2 \lim_{n \to \infty} p_0(g_{x_n}, g_{x_{n+1}}) - \lim_{n \to \infty} p_0(g_{x_n+1}, g_{x_{n+1}}) \]
\[ = 0. \]
Similarly, we have \( \lim_{n \to \infty} d_{p_0}(g_{y_n}, g_{y_{n+1}}) = 0 \). To sum up, we get
\[ \lim_{n \to \infty} d_{p_0}(g_{x_n}, g_{x_{n+1}}) = 0 \quad \text{and} \quad \lim_{n \to \infty} d_{p_0}(g_{y_n}, g_{y_{n+1}}) = 0. \quad (3.16) \]
By using the triangle inequality, we get
\[ d_{p_0}(g_{x_m}, g_{x_n}) \leq s d_{p_0}(g_{x_m}, g_{x_{n-1}}) + s d_{p_0}(g_{x_{n-1}}, g_{x_n}), \quad (3.17) \]
and
\[ d_{p_0}(g_{y_m}, g_{y_n}) \leq s d_{p_0}(g_{y_m}, g_{y_{n-1}}) + s d_{p_0}(g_{y_{n-1}}, g_{y_n}). \quad (3.18) \]
Hence from (3.14), (3.17) and (3.18), we have
\[ \varepsilon \leq \max\{d_{p_0}(g_{x_m_i}, g_{x_n}), d_{p_0}(g_{y_m}, g_{y_n})\} \]
\[ \leq s \max\{d_{p_0}(g_{x_m_i}, g_{x_{n-1}}), d_{p_0}(g_{y_m}, g_{y_{n-1}})\} + s \max\{d_{p_0}(g_{x_{n-1}}, g_{x_n}), d_{p_0}(g_{y_{n-1}}, g_{y_n})\}. \quad (3.19) \]
By taking the lower limit as \( i \to \infty \) in (3.19) and using (3.15), (3.16), we have
\[ \varepsilon \leq \liminf_{i \to \infty} \max\{d_{p_0}(g_{x_m_i}, g_{x_n}), d_{p_0}(g_{y_m}, g_{y_n})\} \]
\[ \leq s \liminf_{i \to \infty} \max\{d_{p_0}(g_{x_m_i}, g_{x_{n-1}}), d_{p_0}(g_{y_m}, g_{y_{n-1}})\} \quad (3.20) \]
\[ \leq s \limsup_{i \to \infty} \max\{d_{p_0}(g_{x_m_i}, g_{x_{n-1}}), d_{p_0}(g_{y_m}, g_{y_{n-1}})\} \leq s \varepsilon. \]
Also, by using (3.15) and (3.16), taking the upper limit as \( i \to \infty \) in (3.19), we obtain
\[ \varepsilon \leq \limsup_{i \to \infty} \max\{d_{p_0}(g_{x_m_i}, g_{x_n}), d_{p_0}(g_{y_m}, g_{y_n})\} \leq s \varepsilon. \quad (3.21) \]
By the triangle inequality, we have
\[ d_{p_0}(g_{x_m}, g_{x_n}) \leq s d_{p_0}(g_{x_m}, g_{x_{m+1}}) + s d_{p_0}(g_{x_{m+1}}, g_{x_n}), \quad (3.22) \]
and
\[ d_{p_0}(g_{y_m}, g_{y_n}) \leq s d_{p_0}(g_{y_m}, g_{y_{m+1}}) + s d_{p_0}(g_{y_{m+1}}, g_{y_n}). \quad (3.23) \]
Therefore, from (3.14), (3.22) and (3.23), we have
\[ \varepsilon \leq \max\{d_{p_0}(g_{x_m_i}, g_{x_n}), d_{p_0}(g_{y_m}, g_{y_n})\} \]
\[ \leq s \max\{d_{p_0}(g_{x_m_i}, g_{x_{m+1}}), d_{p_0}(g_{y_m}, g_{y_{m+1}})\} + s \max\{d_{p_0}(g_{x_{m+1}}, g_{x_n}), d_{p_0}(g_{y_{m+1}}, g_{y_n})\}. \]
By taking the upper limit as $i \to \infty$ in the above inequality, and using (3.16), we have
\[
\frac{\varepsilon}{s} \leq \limsup_{i \to \infty} \max \{d_{p_b}(gx_{m_i+1}, gx_{n_i}), d_{p_b}(gy_{m_i+1}, gy_{n_i})\}. \tag{3.24}
\]

Again, by the triangle inequality we have
\[
d_{p_b}(gx_{m_i+1}, gx_{n_i-1}) \leq sd_{p_b}(gx_{m_i+1}, gx_{m_i}) + sd_{p_b}(gx_{m_i}, gx_{n_i-1}), \tag{3.25}
\]
and
\[
d_{p_b}(gy_{m_i+1}, gy_{n_i-1}) \leq sd_{p_b}(gy_{m_i+1}, gy_{m_i}) + sd_{p_b}(gy_{m_i}, gy_{n_i-1}). \tag{3.26}
\]

From the inequality (3.25), (3.26) and (3.15), we have
\[
\max \{d_{p_b}(gx_{m_i+1}, gx_{n_i-1}), d_{p_b}(gy_{m_i+1}, gy_{n_i-1})\} \leq s \max \{d_{p_b}(gx_{m_i+1}, gx_{m_i}), d_{p_b}(gy_{m_i+1}, gy_{m_i})\}
+ s \max \{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\}
< s \max \{d_{p_b}(gx_{m_i+1}, gx_{m_i}), d_{p_b}(gy_{m_i+1}, gy_{m_i})\} + s\varepsilon.
\]

By taking the upper limit as $i \to \infty$ in the above inequality, and using (3.16), we get
\[
\limsup_{i \to \infty} \max \{d_{p_b}(gx_{m_i+1}, gx_{n_i-1}), d_{p_b}(gy_{m_i+1}, gy_{n_i-1})\} \leq s\varepsilon. \tag{3.27}
\]

On the other hand, because of the definition of $d_{p_b}$ and (3.16), we have
\[
\liminf_{i \to \infty} d_{p_b}(gx_{m_i}, gx_{n_i-1}) = 2 \liminf_{i \to \infty} p_b(gx_{m_i}, gx_{n_i-1}), \tag{3.28}
\]
and
\[
\liminf_{i \to \infty} d_{p_b}(gy_{m_i}, gy_{n_i-1}) = 2 \liminf_{i \to \infty} p_b(gy_{m_i}, gy_{n_i-1}). \tag{3.29}
\]

Hence, from (3.28), (3.29) and (3.20), we obtain
\[
\frac{\varepsilon}{s} \leq \liminf_{i \to \infty} \max \{d_{p_b}(gx_{m_i}, gx_{n_i-1}), d_{p_b}(gy_{m_i}, gy_{n_i-1})\}
= 2 \liminf_{i \to \infty} \max \{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \leq \varepsilon.
\]

Thus, we get
\[
\frac{\varepsilon}{2s} \leq \liminf_{i \to \infty} \max \{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \leq \frac{\varepsilon}{2}. \tag{3.30}
\]

Similarly, from (3.15), (3.21), (3.24), (3.27) and definition of $d_{p_b}$, we can show that
\[
\limsup_{i \to \infty} \max \{p_b(gx_{m_i}, gx_{n_i-1}), p_b(gy_{m_i}, gy_{n_i-1})\} \leq \frac{\varepsilon}{2}, \tag{3.31}
\]
\[
\limsup_{i \to \infty} \max \{p_b(gx_{m_i}, gx_{n_i}), p_b(gy_{m_i}, gy_{n_i})\} \leq \frac{s\varepsilon}{2}, \tag{3.32}
\]
\[
\frac{\varepsilon}{2s} \leq \limsup_{i \to \infty} \max \{p_b(gx_{m_i+1}, gx_{n_i}), p_b(gy_{m_i+1}, gy_{n_i})\}, \tag{3.33}
\]
\[
\limsup_{i \to \infty} \max \{p_b(gx_{m_i+1}, gx_{n_i-1}), p_b(gy_{m_i+1}, gy_{n_i-1})\} \leq \frac{s\varepsilon}{2}. \tag{3.34}
\]

By using (3.1) with $(x, y) = (x_{m_i}, y_{m_i})$ and $(u, v) = (x_{n_i-1}, y_{n_i-1})$, we get
\[
\psi(s_{p_b}(gx_{m_i+1}, gx_{n_i})) = \psi(s_{p_b}(F(x_{m_i}, y_{m_i}), F(x_{n_i-1}, y_{n_i-1})))
\leq \psi(M(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1})) - \Theta(x_{m_i}, y_{m_i}, x_{n_i-1}, y_{n_i-1}) \tag{3.35}
\]
where
\[ M(x_m, y_m, x_{n-1}, y_{n-1}) = \max \left\{ p_b(gx_{m-1}, gx_n) , p_b(gy_{m-1}, gy_n) , p_b(gx_{m-1}, F(x_m, y_m)) \right\} \]

and
\[ \Theta(x_m, y_m, x_{n-1}, y_{n-1}) = \theta \left( p_b(gx_{m-1}, gx_n) , p_b(gy_{m-1}, gy_n) , p_b(gx_{m-1}, F(x_m, y_m)) \right) . \]

Similarly, we have
\[ \psi(s p_b(gy_{m+1}, gy_n)) = \psi(s p_b(F(y_m, x_m), F(y_{n+1}, x_{n+1}))) \leq \psi(M(y_m, x_m, y_{n+1}, x_{n+1})) - \Theta(y_m, x_m, y_{n+1}, x_{n+1}) , \tag{3.36} \]

where
\[ M(y_m, x_m, y_{n+1}, x_{n+1}) = \max \left\{ p_b(gy_{m+1}, gy_n) , p_b(gx_{m+1}, gy_n) , p_b(gx_{m+1}, F(y_m, x_m)) \right\} \]

and
\[ \Theta(y_m, x_m, y_{n+1}, x_{n+1}) = \theta \left( p_b(gy_{m+1}, gy_n) , p_b(gx_{m+1}, gy_n) , p_b(gx_{m+1}, F(y_m, x_m)) \right) . \]

By combining (3.35) and (3.36), we have
\[ \psi(s \max\{p_b(gx_{m+1}, gx_n) , p_b(gy_{m+1}, gy_n)\}) = \max\{\psi(s p_b(gx_{m+1}, gx_n)) , \psi(s p_b(gy_{m+1}, gy_n))\} \leq \psi(M(x_m, y_m, x_{n+1}, y_{n+1})) - \min \{\Theta(x_m, y_m, x_{n+1}, y_{n+1}) , \Theta(y_m, x_m, y_{n+1}, x_{n+1})\} . \]

By taking the upper limit as \( i \to \infty \) in the above inequality and using (3.12), (3.31), (3.32), (3.33) and (3.34), we have
\[ \psi\left( \frac{\varepsilon}{2} \right) = \psi\left( s \cdot \frac{\varepsilon}{2s} \right) \leq \psi(s \limsup_{i \to \infty} \max\{p_b(gx_{m+1}, gx_n) , p_b(gy_{m+1}, gy_n)\}) \]
\[ \leq \psi\left( \max\left\{ \frac{\varepsilon}{2}, 0, 0, 0, \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right\} \right) - \liminf_{i \to \infty} \min\{\Theta(x_m, y_m, x_{n+1}, y_{n+1}) , \Theta(y_m, x_m, y_{n+1}, x_{n+1})\} \]
\[ = \psi\left( \frac{\varepsilon}{2} \right) - \min \left\{ \liminf_{i \to \infty} \Theta(x_m, y_m, x_{n+1}, y_{n+1}) , \liminf_{i \to \infty} \Theta(y_m, x_m, y_{n+1}, x_{n+1}) \right\} , \]

which implies that
\[ \liminf_{i \to \infty} \Theta(x_m, y_m, x_{n+1}, y_{n+1}) = 0 \quad \text{or} \quad \liminf_{i \to \infty} \Theta(y_m, x_m, y_{n+1}, x_{n+1}) = 0 . \]
Hence, by using the properties of $\theta$, we get
\[ \liminf_{i \to \infty} p_b(gx_{m_i}, gx_{n_i-1}) = 0 \quad \text{and} \quad \liminf_{i \to \infty} p_b(gy_{m_i}, gy_{n_i-1}) = 0, \]
which is a contradiction to (3.30). Thus, $\{gx_n\}, \{gy_n\}$ are $b$-Cauchy sequences in $(g(X), d_{p_b})$. By Lemma 2.9, $\{gx_n\}, \{gy_n\}$ are $p_b$-Cauchy sequences in $(g(X), p_b)$. Since $g(X)$ is $p_b$-complete subspace of $(X, p_b)$, there exist $gx, gy \in g(X)$, such that $\{gx_n\}$ and $\{gy_n\}$ $p_b$-converges to $gx$ and $gy$, respectively. By using Lemma 2.9 again, we have
\[ \lim_{n \to \infty} p_b(gx_n, gx) = \lim_{n,m \to \infty} p_b(gx_n, gx_m) = p_b(gx, gx), \quad (3.37) \]
and
\[ \lim_{n \to \infty} p_b(gy_n, gy) = \lim_{n,m \to \infty} p_b(gy_n, gy_m) = p_b(gy, gy). \]
Since $\{gx_n\}$ is $b$-Cauchy sequence in $(X, d_{p_b})$, so $\lim_{n,m \to \infty} d_{p_b}(gx_n, gx_m) = 0$. By using
\[ d_{p_b}(gx_n, gx_m) = 2p_b(gx_n, gx_m) - p_b(gx_n, gx) - p_b(gx_m, gx_n), \]
and (3.13) we obtain that $\lim_{n,m \to \infty} p_b(gx_n, gx_m) = 0$. Thus, it follows from (3.37) that
\[ \lim_{n \to \infty} p_b(gx_n, gx) = \lim_{n,m \to \infty} p_b(gx_n, gx_m) = p_b(gx, gx) = 0. \quad (3.38) \]
On using similar steps as above we can show that
\[ \lim_{n \to \infty} p_b(gy_n, gy) = \lim_{n,m \to \infty} p_b(gy_n, gy_m) = p_b(gy, gy) = 0. \quad (3.39) \]
By (3.3) and the properties (i) and (ii), we have $gx_n \preceq gx, \ gy_n \preceq gy$ for all $n \in N$. From (3.1), we have
\[ \psi(s p_b(gx_{n+1}, F(x, y))) = \psi(s p_b(F(x, y_n), F(x, y))) \leq \psi(M(x_n, y_n, x, y)) - \Theta(x_n, y_n, x, y), \quad (3.40) \]
where
\begin{equation}
M(x_n, y_n, x, y) = \max \left\{ \frac{p_b(gx_n, gx), p_b(gy_n, gy), p_b(gx_n, gx_{n+1}), p_b(gx, F(x, y))}{2}, \frac{p_b(gx, F(x, y)), p_b(gy, F(y, x))}{2} \right\}, \quad (3.41)
\end{equation}
and
\[ \Theta(x_n, y_n, x, y) = \theta \left( \frac{p_b(gx_n, gx), p_b(gy_n, gy), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1}), p_b(gx, F(x, y)), p_b(gy, F(y, x)), p_b(gx_n, F(x, y)), p_b(gy_n, F(y, x)), p_b(gx_n, gx_{n+1}), p_b(gy_n, gy_{n+1})}{2} \right). \]
By taking the upper limit as $n \to \infty$ in (3.41), and using (3.12), (3.38), (3.39) and Lemma 2.10 we obtain
\[ \limsup_{n \to \infty} M(x_n, y_n, x, y) \leq \max \left\{ 0, 0, 0, 0, \frac{p_b(gx, F(x, y))}{2s}, \frac{p_b(gy, F(y, x))}{2s}, \frac{sp_b(gx, F(x, y)) + 0}{2s}, \frac{sp_b(gy, F(y, x)) + 0}{2s} \right\} \]
\[ \leq \max \{ p_b(gx, F(x, y)), p_b(gy, F(y, x)) \}. \quad (3.42) \]
By using Lemma 2.10 (3.42) and the properties of $\psi$, and taking the upper limit as $n \to \infty$ in (3.40), we obtain
\[ \psi(p_b(gx, F(x, y))) = \psi \left( s \cdot \frac{p_b(gx, F(x, y))}{s} \right) \leq \psi \left( s \limsup_{n \to \infty} p_b(gx_{n+1}, F(x, y)) \right) \]
\[= \limsup_{n \to \infty} \psi \left( sp_b(gx_{n+1}, F(x,y)) \right) \]
\[\leq \limsup_{n \to \infty} \psi \left( M(x_n, y_n, x, y) \right) - \liminf_{n \to \infty} \Theta(x_n, y_n, x, y) \]
\[\leq \psi \left( \max \{ p_b(gx, F(x,y)), p_b(gy, F(y,x)) \} \right) - \liminf_{n \to \infty} \Theta(x_n, y_n, x, y). \]

Similarly, we can show that
\[\psi \left( p_b(gy, F(y,x)) \right) \leq \psi \left( \max \{ p_b(gx, F(x,y)), p_b(gy, F(y,x)) \} \right) - \liminf_{n \to \infty} \Theta(y_n, x_n, y, x), \]
where
\[\Theta(y_n, x_n, y, x) = \psi \left( \begin{pmatrix} p_b(gy_n, gy), p_b(gx_n, gx), p_b(gy_n, gy_{n+1}), p_b(gx_n, gx_{n+1}), \\
p_b(gy, F(y,x)), p_b(gx, F(x,y)); p_b(gy_n, F(y,x)), \\
p_b(gx_n, F(x,y)), p_b(gy, gy_{n+1}), p_b(gx, gx_{n+1}) \end{pmatrix} \right). \]

By combining (3.43) and (3.44) we obtain
\[\psi \left( \max \{ p_b(gx, F(x,y)), p_b(gy, F(y,x)) \} \right) = \max \{ \psi \left( p_b(gx, F(x,y)) \right), \psi \left( p_b(gy, F(y,x)) \right) \} \]
\[\leq \psi \left( \max \{ p_b(gx, F(x,y)), p_b(gy, F(y,x)) \} \right) - \liminf_{n \to \infty} \Theta(x_n, y_n, x, y), \liminf_{n \to \infty} \Theta(y_n, x_n, y, x). \]

Accordingly, we get
\[\liminf_{n \to \infty} \Theta(x_n, y_n, x, y) = 0 \quad \text{or} \quad \liminf_{n \to \infty} \Theta(y_n, x_n, y, x) = 0. \]

By using the properties of \( \theta \), we get \( gx = F(x,y) \) and \( gy = F(y,x) \). That is, \((x,y)\) is a coupled coincidence point of the mappings \( F \) and \( g \).

\[\Box\]

\textbf{Remark 3.2.} The contractive conditions of Theorem 3.1 is new. As far as now, no author has investigated the problems. Theorem 3.1 improves and extends several well-known comparable results from \( b \)-metric spaces and partial metric spaces to ordered partial \( b \)-metric spaces.

\textbf{Corollary 3.3.} Let \((X, \preceq, p_b)\) be a ordered partial \( b \)-metric space. Let \( F : X \times X \to X \) and \( g : X \to X \) be two mappings and \( F \) has the mixed \( g \)-monotone property with \( g \). Suppose that there exists an altering distance function \( \psi \) and \( \phi : [0, \infty) \to [0, \infty) \) is continuous with \( \phi(t) = 0 \) implies \( t = 0 \) such that
\[\psi \left( sp_b(F(x,y), F(u,v)) \right) \leq \psi \left( M(x,y,u,v) \right) - \phi \left( M(x,y,u,v) \right) \]
for all \((x,y),(u,v)\) \in \( X \times X \) with \( g(x) \preceq g(u) \) and \( g(y) \succeq g(v) \), where
\[M(x,y,u,v) = \max \left\{ \begin{array}{l} p_b(gx,gu), p_b(gy,gv), p_b(gx,F(x,y)), \\
p_b(gy,F(y,x)), \frac{p_b(gx,F(u,v))}{p_b(gy,F(v,u)) + \frac{1}{2s} p_b(gx,F(v,u)),} \\
\end{array} \right\} \]
Further, suppose \( F(X \times X) \subseteq g(X) \) and \( g(X) \) is a \( p_b \)-complete subspace of \((X, p_b)\). Also, suppose that \( X \) satisfies the following properties:
\begin{enumerate}
\item[(i)] if a non-decreasing sequence \( x_n \) in \( X \) converges to \( x \in X \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \);
\item[(ii)] if a non-increasing sequence \( y_n \) in \( X \) converges to \( y \in X \), then \( y_n \succeq y \) for all \( n \in \mathbb{N} \).
\end{enumerate}
If there exists \((x_0, y_0)\) \in \( X \times X \) such that \( gx_0 \preceq F(x_0,y_0) \) and \( gy_0 \succeq F(y_0,x_0) \), then \( F \) and \( g \) have a coupled coincidence point.
Proof. Take
\[ \theta(t_1, t_2, \cdots, t_{10}) = \phi \left( \max \left\{ t_1, t_2, t_3, t_4, \frac{t_5}{2s}, \frac{t_6}{2s}, \frac{t_7 + t_8}{2s}, \frac{t_9 + t_{10}}{2s} \right\} \right), \]
in Theorem 3.1 then Corollary 3.3 holds. \qed

Remark 3.4. Corollary 3.3 improves and extends Theorem 2.2 in [4] from ordered b-metric space to ordered partial b-metric space.

Corollary 3.5. Let \((X, \preceq, p_b)\) be a \(p_b\)-complete ordered partial b-metric space. Let \(F : X \times X \to X\) be a mapping and \(F\) has the mixed monotone property on \(X\). Suppose that there exists an altering distance function \(\psi\) and \(\phi : [0, \infty) \to [0, \infty)\) is continuous with \(\phi(t) = 0 \Rightarrow t = 0\) such that
\[ \psi \left( s \left( p_b(F(x, y), F(u, v)) \right) \right) \leq \psi \left( M(x, y, u, v) \right) - \phi \left( M(x, y, u, v) \right) \]
for all \((x, y), (u, v) \in X \times X\) with \(x \preceq u\) and \(y \succeq v\), where
\[ M(x, y, u, v) = \max \left\{ \frac{p_b(x, u) + p_b(y, v) + p_b(x, F(x, y))}{2s}, \frac{p_b(y, F(y, x)) + p_b(u, F(u, v))}{2s} \right\}. \]

Further, suppose that \(X\) satisfies the following properties:

(i) if a non-decreasing sequence \(x_n\) in \(X\) converges to \(x \in X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\);

(ii) if a non-increasing sequence \(y_n\) in \(X\) converges to \(y \in X\), then \(y_n \succeq y\) for all \(n \in \mathbb{N}\).

If there exists \((x_0, y_0) \in X \times X\) such that \(x_0 \preceq F(x_0, y_0)\) and \(y_0 \succeq F(y_0, x_0)\), then \(F\) has a coupled fixed point.

Proof. It suffices to take \(g = I_x\) in Corollary 3.3 \qed

Remark 3.6. Corollary 3.5 improves and extends Corollary 2.3 in [4] from ordered b-metric space to ordered partial b-metric space.

Corollary 3.7. Let \((X, \preceq, p_b)\) be a ordered partial b-metric space. Let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings and \(F\) has the mixed g-monotone property with \(g\). Suppose that there exists \(k \in [0, 1)\) such that
\[ p_b(F(x, y), F(u, v)) \leq \frac{k}{s} \max \left\{ p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), \frac{p_b(gy, F(y, x))}{2s}, \frac{p_b(gx, F(x, y)) + p_b(gu, F(u, v))}{2s}, \frac{p_b(gu, F(u, v)) + p_b(gv, F(y, x))}{2s} \right\} \]
for all \((x, y), (u, v) \in X \times X\) with \(g(x) \preceq g(u)\) and \(g(y) \succeq g(v)\). Further, suppose \(F(X \times X) \subset g(X)\) and \(g(X)\) is a \(p_b\)-complete subspace of \((X, p_b)\). Also, suppose that \(X\) satisfies the following properties:

(i) if a non-decreasing sequence \(x_n\) in \(X\) converges to \(x \in X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\);

(ii) if a non-increasing sequence \(y_n\) in \(X\) converges to \(y \in X\), then \(y_n \succeq y\) for all \(n \in \mathbb{N}\).

If there exists \((x_0, y_0) \in X \times X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point.

Proof. It suffices to take
\[ \theta(t_1, t_2, \cdots, t_{10}) = (1 - k) \max \left\{ \frac{t_1}{2s}, \frac{t_2}{2s}, \frac{t_3 + t_4}{2s}, \frac{t_5}{2s}, \frac{t_6}{2s}, \frac{t_7 + t_8}{2s}, \frac{t_9 + t_{10}}{2s} \right\}, \]
and \(\psi(t) = t\) in Theorem 3.1 \qed
Corollary 3.8. Let \((X, \preceq, p_b)\) be a ordered partial \(b\)-metric space. Let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings and \(F\) has the mixed \(g\)-monotone property with \(g\). Suppose that there exist non-negative real numbers \(\alpha_1, \alpha_2, \cdots, \alpha_{10}\) with
\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2s(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) < 1,
\]
such that
\[
sp_b(F(x, y), F(u, v)) \leq \alpha_1 p_b(gx, gu) + \alpha_2 p_b(gy, gv) + \alpha_3 p_b(gx, F(x, y)) + \alpha_4 p_b(gy, F(y, x)) + \alpha_5 p_b(gu, F(u, v)) + \alpha_6 p_b(gv, F(v, u)) + \alpha_7 p_b(gx, F(u, v)) + \alpha_8 p_b(gy, F(v, u)) + \alpha_{10} p_b(gy, F(y, x))
\]
(3.45)
for all \((x, y), (u, v) \in X \times X\) with \(g(x) \leq g(u)\) and \(g(y) \geq g(v)\). Further, suppose \(F(X \times X) \subset g(X)\) and \(g(X)\) is a \(p_b\)-complete subspace of \((X, p_b)\). Also, suppose that \(X\) satisfies the following properties:
(i) if a non-decreasing sequence \(x_n\) in \(X\) converges to \(x \in X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\);
(ii) if a non-increasing sequence \(y_n\) in \(X\) converges to \(y \in X\), then \(y_n \succeq y\) for all \(n \in \mathbb{N}\).
If there exists \((x_0, y_0) \in X \times X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point.

Proof. By noting that \(\alpha_i, i = 1, 2, \cdots, 10\) are non-negative real numbers, from (3.45) we have
\[
sp_b(F(x, y), F(u, v)) \leq \alpha_1 p_b(gx, gu) + \alpha_2 p_b(gy, gv) + \alpha_3 p_b(gx, F(x, y)) + \alpha_4 p_b(gy, F(y, x)) + \alpha_5 p_b(gu, F(u, v)) + \alpha_6 p_b(gv, F(v, u)) + \alpha_7 p_b(gx, F(u, v)) + \alpha_8 p_b(gy, F(v, u)) + \alpha_{10} p_b(gy, F(y, x))
\]
\[
\leq k \max \left\{ \frac{p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), p_b(gy, F(y, x))}{2s}, \frac{p_b(gu, F(u, v))}{2s}, \frac{p_b(gv, F(v, u))}{2s}, \frac{p_b(gx, F(u, v))}{2s}, \frac{p_b(gy, F(v, u))}{2s}, \frac{p_b(gy, F(y, x))}{2s} \right\},
\]
where
\[
k = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2s(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 + \alpha_9 + \alpha_{10}) < 1.
\]
From Corollary 3.7 we can find that \(F\) and \(g\) have a coupled coincidence point. \(\square\)

Corollary 3.9. Let \((X, \preceq, p_b)\) be a ordered partial \(b\)-metric space. Let \(F : X \times X \to X\) and \(g : X \to X\) be two mappings and \(F\) has the mixed \(g\)-monotone property with \(g\). Suppose that there exists \(k \in [0, 1)\) such that
\[
p_b(F(x, y), F(u, v)) \leq \frac{k}{s} \max \{p_b(gx, gu), p_b(gy, gv)\}
\]
for all \((x, y), (u, v) \in X \times X\) with \(g(x) \leq g(u)\) and \(g(y) \geq g(v)\). Further, suppose \(F(X \times X) \subset g(X)\) and \(g(X)\) is a \(p_b\)-complete subspace of \((X, p_b)\). Also, suppose that \(X\) satisfies the following properties:
(i) if a non-decreasing sequence \(x_n\) in \(X\) converges to \(x \in X\), then \(x_n \preceq x\) for all \(n \in \mathbb{N}\);
(ii) if a non-increasing sequence \(y_n\) in \(X\) converges to \(y \in X\), then \(y_n \succeq y\) for all \(n \in \mathbb{N}\).
If there exists \((x_0, y_0) \in X \times X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \succeq F(y_0, x_0)\), then \(F\) and \(g\) have a coupled coincidence point.

Proof. Since
\[
p_b(F(x, y), F(u, v)) \leq \frac{k}{s} \max \{p_b(gx, gu), p_b(gy, gv)\}
\]
\[
\leq \frac{k}{s} \max \left\{ \frac{p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x, y)), p_b(gy, F(y, x))}{2s}, \frac{p_b(gu, F(u, v))}{2s}, \frac{p_b(gv, F(v, u))}{2s}, \frac{p_b(gx, F(u, v))}{2s}, \frac{p_b(gy, F(v, u))}{2s}, \frac{p_b(gy, F(y, x))}{2s} \right\}.
\]
From Corollary 3.7 we can find that $F$ and $g$ have a coupled coincidence point.

**Corollary 3.10.** Let $(X, \preceq, p_b)$ be a ordered partial $b$-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and $F$ has the mixed $g$-monotone property with $g$. Suppose that there exists $k \in [0,1)$ such that

$$p_b(F(x,y), F(u,v)) \leq \frac{k}{2s}(p_b(gx, gu) + p_b(gy, gv))$$

for all $(x,y), (u,v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further, suppose $F(X \times X) \subset g(X)$ and $g(X)$ is a $p_b$-complete subspace of $(X, p_b)$. Also, suppose that $X$ satisfies the following properties:

(i) if a non-decreasing sequence $x_n$ in $X$ converges to $x \in X$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;

(ii) if a non-increasing sequence $y_n$ in $X$ converges to $y \in X$, then $y_n \succeq y$ for all $n \in \mathbb{N}$.

If there exists $(x_0, y_0) \in X \times X$ such that $gx_0 \preceq F(x_0, y_0)$ and $gy_0 \succeq F(y_0, x_0)$, then $F$ and $g$ have a coupled coincidence point.

**Proof.** Since

$$p_b(F(x,y), F(u,v)) \leq \frac{k}{2s}(p_b(gx, gu) + p_b(gy, gv)) \leq \frac{k}{s} \max \{p_b(gx, gu), p_b(gy, gv)\}.$$ 

From Corollary 3.9 we can find that $F$ and $g$ have a coupled coincidence point.

**Remark 3.11.** If we define $g = I_x, s = 1$, in Corollaries 3.9 and 3.10, then we can get some new results, which improve and extend Theorem 2.2 in [9], and Corollary 2 in [6] from ordered partial metric space to ordered partial $b$-metric space. These results also extend and generalize the corresponding results of [11, 17, 20].

Now we give an example to show the usability of Theorem 3.1

**Example 3.12.** Let $X = [0,1]$ with usual ordering. Define $p_b(x,y) = (\max \{x,y\})^2$. Then $(X, \preceq, p_b)$ is a complete ordered partial $b$-metric space with coefficient $s = 2$.

Next we define

$$F(x,y) = \frac{1}{3}x(1-y), \quad \text{and} \quad g(x) = \frac{2}{3}x \quad \text{for all} \quad x,y \in X,$$

$$\psi(t) = t, \quad \text{and} \quad \theta(t_1, t_2, \cdots, t_{10}) = \frac{1}{3} \max \left\{ t_1, t_2, \cdots, t_6, \frac{t_7 + t_8}{2s}, \frac{t_9 + t_{10}}{2s} \right\}.$$ 

Clearly, $F$ has the mixed $g$-monotone property with $g$ and $F(X \times X) \subset g(X)$. Otherwise,

$$p_b(F(x,y), F(u,v)) = \left( \max \left\{ \frac{1}{3}(1-y), \frac{1}{3}(1-v) \right\} \right)^2 = \max \left\{ \frac{1}{9}x^2(1-y)^2, \frac{1}{9}y^2(1-v)^2 \right\},$$

$$M(x,y,u,v) = \max \left\{ \frac{p_b(gx, gu), p_b(gy, gv), p_b(gx, F(x,y)), p_b(gy, F(y,x)), p_b(gy, F(x,y)), p_b(gx, F(y,x)), p_b(gx, F(y,x)), p_b(gy, F(x,y)), p_b(gx, F(y,x)), p_b(gy, F(x,y))}{2s, 2s, 2s, 2s, 2s, 2s, 2s, 2s, 2s} \right\}$$

$$= \max \left\{ \frac{1}{9}x^2(1-y)^2, \frac{1}{9}y^2(1-v)^2, \frac{1}{9}x^2(1-y)^2, \frac{1}{9}y^2(1-v)^2, \frac{1}{9}x^2(1-y)^2, \frac{1}{9}y^2(1-v)^2, \frac{1}{9}x^2(1-y)^2, \frac{1}{9}y^2(1-v)^2 \right\}$$

$$= \frac{4}{9} \max \{ x^2, u^2, v^2, v^2 \},$$

$$\Theta(x,y,u,v) = \frac{1}{3} M(x,y,u,v) = \frac{4}{27} \max \{ x^2, u^2, v^2, v^2 \}.$$
Therefore,
\[
\psi(M(x, y, u, v)) - \Theta(x, y, u, v) = \frac{2}{3} \cdot \frac{4}{9} \max \{x^2, u^2, y^2, v^2\} = \frac{8}{27} \max \{x^2, u^2, y^2, v^2\}.
\]

Then
\[
\psi(s_p(F(x, y), F(u, v))) = \max \left\{ \frac{2}{9} x^2(1 - y)^2, \frac{2}{9} u^2(1 - v)^2 \right\} \\
\leq \max \left\{ \frac{2}{9} x^2, \frac{2}{9} y^2 \right\} \\
\leq \frac{2}{9} \max \{x^2, u^2, y^2, v^2\} \\
\leq \frac{8}{27} \max \{x^2, u^2, y^2, v^2\} \\
= \psi(M(x, y, u, v)) - \Theta(x, y, u, v).
\]

At last, define \(x_0 = 0, y_0 = 0\), then \(g x_0 \preceq F(x_0, y_0)\) and \(g y_0 \succeq F(y_0, x_0)\). So, the conditions of Theorem 3.1 are all satisfied. Since \(F(0, 0) = g(0)\) and \(F(0, 0) = g(0), (0, 0)\) is the coupled coincidence point of \(F\) and \(g\).

4. Uniqueness of common fixed points

In this section we prove the existence and uniqueness of common fixed point. If \((X, \preceq)\) is a partially ordered set, first we define product space \(X \times X\) with a partial order relation in the following way. For all \((x, y), (u, v) \in X \times X\),
\[(x, y) \preceq (u, v) \iff x \preceq u, y \succeq v.\]

We say that \((x, y)\) and \((u, v)\) are comparable, if \((x, y) \preceq (u, v)\) or \((x, y) \succeq (u, v)\).

**Theorem 4.1.** In addition to hypotheses of Theorem 3.1, suppose that for every \((x, y)\) and \((x^*, y^*)\) in \(X \times X\), there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y), F(y, x))\) and to \((F(x^*, y^*), F(y^*, x^*))\). Also we assume that \(F\) commutes with \(g\). Then \(F\) and \(g\) have a unique common fixed point, that is, there exists \(x \in X\) such that \(x = gx = F(x, x)\).

**Proof.** From Theorem 3.1, there exists at least a coupled coincidence point. Suppose \((x, y)\) and \((x^*, y^*)\) are coupled coincidence points of \(F\) and \(g\), that is, \(gx = F(x, y), gy = F(y, x), gx^* = F(x^*, y^*)\) and \(gy^* = F(y^*, x^*)\). Next we prove \(gx = gx^*, gy = gy^*\). By the assumptions, there exists \((u, v) \in X \times X\) such that \((F(u, v), F(v, u))\) is comparable to \((F(x, y), F(y, x))\) and to \((F(x^*, y^*), F(y^*, x^*))\). Without loss of generality, we can assume that
\[
(F(x, y), F(y, x)) \preceq (F(u, v), F(v, u)), (F(x^*, y^*), F(y^*, x^*)) \preceq (F(u, v), F(v, u)). \tag{4.1}
\]

Put \(u_0 = u, v_0 = v\) and choose \(u_1, v_1 \in X\) such that \(g(u_1) = F(u_0, v_0)\) and \(g(v_1) = F(v_0, u_0)\). By continuing this process, we can define sequences \(\{gu_n\}, \{gv_n\}\) such that
\[gu_{n+1} = F(u_n, v_n) \text{ and } gv_{n+1} = F(v_n, u_n), \forall n \geq 0.\]

Since
\[
(F(x, y), F(y, x)) = (gx, gy), \\
(F(u, v), F(v, u)) = (gu_1, gv_1).
\]
By using (4.1) we have \( gx \leq gu_1 \) and \( gy \geq gv_1 \). By using the mixed \( g \)-monotone property, we have

\[
\begin{align*}
 gx &= F(x, y) \leq F(u_1, y) \leq F(u_1, v_1) = gu_2, \\
 gy &= F(y, x) \geq F(v_1, x) \geq F(v_1, u_1) = gv_2.
\end{align*}
\]

By going on this, we can show that \( gx \leq gu_n \) and \( gy \geq gv_n \), for all \( n \geq 1 \). Thus from (3.1) we have

\[
\psi(gx, gu_{n+1}) = \psi(gx, gu_n) \\
\leq \psi(M(x, y, u_n, v_n)) - \Theta(x, y, u_n, v_n),
\]

where

\[
M(x, y, u_n, v_n) = \max\left\{ \frac{p_b(gx, gu_n), p_b(gy, gv_n), p_b(gx, F(x, y)), p_b(gy, F(y, x))}{2s}, \frac{p_b(gu_n, F(u_n, v_n)) + p_b(gv_n, F(v_n, u_n))}{2s}, \frac{p_b(gx, F(u_n, v_n)) + p_b(gu_n, F(u_n, v_n))}{2s}, \frac{p_b(gy, F(v_n, u_n)) + p_b(gv_n, F(v_n, u_n))}{2s} \right\}.
\]

It follows from (p4) that

\[
\frac{p_b(gu_n, gu_{n+1})}{2s} \leq \frac{sp_b(gx, gu_n) + sp_b(gx, gu_{n+1})}{2s} \leq \max\{p_b(gx, gu_n), p_b(gx, gu_{n+1})\}.
\]

Similarly, we can show that

\[
\frac{p_b(gv_n, gv_{n+1})}{2s} \leq \max\{p_b(gy, gv_n), p_b(gy, gv_{n+1})\}.
\]

Therefore

\[
M(x, y, u_n, v_n) \leq \max\{p_b(gx, gu_n), p_b(gy, gv_n), p_b(gx, gu_{n+1}), p_b(gy, gv_{n+1})\} = \max\{\gamma_{n-1}, \gamma_n\},
\]

where \( \gamma_n = \max\{p_b(gx, gu_{n+1}), p_b(gy, gv_{n+1})\} \). Hence

\[
\psi(gx, gu_{n+1}) \leq \psi(\max\{\gamma_{n-1}, \gamma_n\}) - \Theta(x, y, u_n, v_n),
\]

where

\[
\Theta(x, y, u_n, v_n) = \theta\left( p_b(gx, gu_n), p_b(gy, gv_n), p_b(gx, F(x, y)), p_b(gy, F(y, x)), p_b(gu_n, F(u_n, v_n)), p_b(gv_n, F(v_n, u_n)), p_b(gx, F(u_n, v_n)), p_b(gy, F(v_n, u_n)), p_b(gu_n, F(x, y)), p_b(gv_n, F(y, x)), p_b(gy, F(v_n, u_n)), p_b(gv_n, F(y, x)) \right).
\]

Similarly,

\[
\psi(gy, gv_{n+1}) \leq \psi(\max\{\gamma_{n-1}, \gamma_n\}) - \Theta(y, x, v_n, u_n),
\]

where

\[
\Theta(y, x, v_n, u_n) = \theta\left( p_b(gy, gv_n), p_b(gx, gu_n), p_b(gy, F(y, x)), p_b(gx, F(x, y)), p_b(gv_n, F(u_n, v_n)), p_b(gu_n, F(v_n, u_n)), p_b(gy, F(v_n, u_n)), p_b(gu_n, F(y, x)), p_b(gv_n, F(y, x)), p_b(gu_n, F(u_n, v_n)), p_b(gv_n, F(y, x)) \right).
\]

In the same way of Theorem 3.1 (Case 1 and Case 2), we can prove that \( \gamma_n \leq \gamma_{n-1} \) for all \( n \in \mathbb{N} \) holds. Therefore, the sequence \( \{\gamma_n\} \) is a non-increasing sequence of nonnegative real number, and so, there exists \( \gamma \geq 0 \) such that \( \lim_{n \to \infty} \gamma_n = \gamma \). Next we prove \( \gamma = 0 \).
By combining (4.2) and (4.3), we get
\[ \psi(\gamma_n) \leq \psi(s\gamma_n) \leq \psi(\gamma_{n-1}) - \min \left\{ \Theta(x, y, u_n, v_n), \Theta(y, x, v_n, u_n) \right\}. \] \quad (4.4)

By taking the upper limit as \( n \to \infty \) in (4.4), we have
\[ \psi(\gamma) \leq \psi(\gamma) - \min \left\{ \liminf_{n \to \infty} \Theta(x, y, u_n, v_n), \liminf_{n \to \infty} \Theta(y, x, v_n, u_n) \right\}. \]

So,
\[ \liminf_{n \to \infty} \Theta(x, y, u_n, v_n) = 0, \quad \text{or} \quad \liminf_{n \to \infty} \Theta(y, x, v_n, u_n) = 0. \]

Hence, by using the properties of \( \theta \), we get
\[ \liminf_{n \to \infty} p_b(gx, gu_n) = 0, \quad \text{and} \quad \liminf_{n \to \infty} p_b(gy, gv_n) = 0. \]

That is,
\[ \gamma = \liminf_{n \to \infty} \gamma_{n-1} = \max \{ \liminf_{n \to \infty} p_b(gx, gu_n), \liminf_{n \to \infty} p_b(gy, gv_n) \} = 0, \]
which concludes
\[ \lim_{n \to \infty} p_b(gx, gu_{n+1}) = 0, \quad \text{and} \quad \lim_{n \to \infty} p_b(gy, gv_{n+1}) = 0. \] \quad (4.5)

In the same way, we can get
\[ \lim_{n \to \infty} p_b(gx^*, gu_{n+1}) = 0, \quad \text{and} \quad \lim_{n \to \infty} p_b(gy^*, gv_{n+1}) = 0. \] \quad (4.6)

From (4.5) and (4.6), we have
\[ p_b(gx, gx^*) \leq \lim_{n \to \infty} sp_b(gx, gu_{n+1}) + \lim_{n \to \infty} sp_b(gu_{n+1}, gx^*) = 0. \]

That is \( gx = gx^* \). Similarly, \( gy = gy^* \). This implies the uniqueness of coupled coincidence point. On the other hand, \( (y, x) \) is also the coupled coincidence point of \( F \) and \( g \). So, \( gx = gy \).

Define \( t = gx \). By the commutativity of \( F \) and \( g \), we have
\[ gt = g(gx) = gF(x, y) = F(gx, gy) = F(t, t). \]

Thus, \( (gt, gt) \) is a coupled coincidence point. It follows that \( gt = gx = t \), that is, \( t = gt = F(t, t) \).

Therefore, \( (t, t) \) is a common fixed point of \( F \) and \( g \). Finally, we prove the uniqueness, assume that \( (s, s) \) is another common fixed point, that is \( s = gs = F(s, s) \). Since \( (gs, gs) \) is a coupled coincidence point of \( F \) and \( g \), we have \( gs = gt \), that is \( s = t \), which is the desired result.

\[ \square \]

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