On Opial-Rozanova type inequalities

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Abstract

In the present paper we establish some inverses of Rozanova’s type integral inequalities. The results in special cases yield reverse Rozanova’s, Godunova’s and Pólya’s inequalities. ©2016 All rights reserved.

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1. Introduction

The well-known inequality due to Opial can be stated as follows (see [12]).

\textbf{Theorem 1.1.} Suppose \( f \in C^1[0,h] \) satisfies \( f(0) = f(h) = 0 \) and \( f(x) > 0 \) for all \( x \in (0,h) \). Then

\[
\int_0^h |f(x)f'(x)| \, dx \leq \frac{h}{4} \int_0^h (f'(x))^2 \, dx.
\] (1.1)

The first Opial’s type inequality was established by Willett [16] as follows:

\textbf{Theorem 1.2.} If \( x(t) \) be absolutely continuous in \([0,a]\), and \( x(0) = 0 \), then

\[
\int_0^a |x(t)x'(t)| \, dt \leq \frac{a}{2} \int_0^a |x'(t)|^2 \, dt.
\] (1.2)

A non-trivial generalization of Theorem 1.2 was established by Hua [10] as follows:

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Theorem 1.3. Let \( x(t) \) be absolutely continuous in \([0, a]\) and \( x(0) = 0 \). If \( l \) be a positive integer, then
\[
\int_0^a |x(t)x'(t)|dt \leq \frac{a^l}{l+1} \int_0^a |x'(t)|^{l+1}dt.
\] (1.3)

A sharper inequality was established by Godunova [9] as follows:

Theorem 1.4. Let \( f(t) \) be convex and increasing function on \([0, \infty)\) with \( f(0) = 0 \). If \( x(t) \) is absolutely continuous on \([0, \tau]\), and \( x(\alpha) = 0 \), then
\[
\int_\alpha^\tau f'(|x(t)|)|x'(t)|dt \leq f \left( \int_\alpha^\tau |x'(t)|dt \right).
\] (1.4)

Rozanova [14] proved an extension of Inequality (1.5) which is embodied in the following:

Theorem 1.5. Let \( f(t) \) and \( g(t) \) be convex and increasing functions on \([0, \infty)\) with \( f(0) = 0 \) and let \( p(t) \geq 0 \), \( p'(t) > 0 \), \( t \in [\alpha, a] \) with \( p(\alpha) = 0 \). If \( x(t) \) is absolutely continuous on \([\alpha, \alpha] \) and \( x(\alpha) = 0 \), then
\[
f \left( \int_\alpha^a p'(t) \cdot g \left( \frac{|x'(t)|}{p'(t)} \right) dt \right) \geq \int_\alpha^a p'(t) \cdot g \left( \frac{|x'(t)|}{p'(t)} \right) \left[ f' \left( p(t) \cdot g \left( \frac{|x(t)|}{p(t)} \right) \right) \right] dt.
\] (1.5)

The Inequality (1.5) will be called as Rozanova’s inequality in the paper.

Opial’s inequality and its generalizations, extensions and discretizations play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations [11, 12, 15, 16, 17, 18, 19]. For Opial type integral inequalities involving high-order partial derivatives see [3] and [18]. For an extensive survey on these inequalities, see [2].

The aim of the present paper is to establish some inverses of the Rozanova’s Inequality (1.3) as follows.

Theorem 1.6. Let \( f(t) \) and \( g(t) \) be convex and decreasing functions on \([0, \infty)\) with \( f(0) = 0 \) and let \( p(t) \geq 0 \), \( p'(t) > 0 \), \( t \in [\alpha, \tau] \) with \( p(\alpha) = 0 \). If \( x(t) \) is absolutely continuous on \([\alpha, \tau] \) and \( x(\alpha) = 0 \), then there exists \( \lambda \) (\( 0 \leq \lambda \leq 1 \)), following inequality holds
\[
f \left( \int_\alpha^\tau p'(t)g \left( \frac{|x'(t)|}{p'(t)} \right) dt \right) \leq \int_\alpha^\tau p'(t)g \left( \frac{|x'(t)|}{p'(t)} \right) f' \left( C_{g,\lambda}(\alpha, t) \cdot p(t)g \left( \frac{|x(t)|}{p(t)} \right) \right) dt.
\] (1.6)

where
\[
C_{g,\lambda}(\alpha, t) = \frac{\lambda g(\alpha) + (1 - \lambda)g(t)}{g(\alpha) + (1 - \lambda)t}.
\]

Remark 1.7. The reverse inequality in Theorem 1.6 is achieved. Moreover, in Theorem 1.5 we deal with convex and increasing functions \( f \) and \( g \), while the reverse inequality in Theorem 1.6 is achieved for convex and decreasing functions \( f \) and \( g \).

Theorem 1.8. Assume that
\begin{enumerate}
  \item \( f(t), g(t) \) and \( x(t) \) are as in Theorem 1.6
  \item \( p(t) \) is increasing on \([0, \tau]\) with \( p(0) = 0 \),
  \item \( h(t) \) is concave and increasing on \([0, \infty)\),
  \item \( \phi(t) \) is increasing on \([0, a]\) with \( \phi(0) = 0 \),
  \item For \( y(t) = \int_0^t p'(s)g \left( \frac{|x'(s)|}{p'(s)} \right) ds \),
\end{enumerate}
\[
f' \left( y(t) \right) y'(t) \cdot \phi \left( \frac{1}{y'(t)} \right) \geq f(y(\tau)) \cdot \phi' \left( \frac{t}{y(\tau)} \right).
\] (1.7)
Then there exists λ and μ (0 ≤ λ, μ ≤ 1), following inequality holds
\[ \omega \left( \int_0^\tau p'(t)g \left( \frac{|x(t)|}{p(t)} \right) dt \right) \leq E_{h,\mu}(0, \tau) \int_0^\tau f' \left( \frac{E^{-1}_{g,\lambda}(0, t)p(t)g \left( \frac{|x(t)|}{p(t)} \right)}{p(t)} \right) \cdot v \left( p(t)g \left( \frac{|x'(t)|}{p'(t)} \right) \right) dt, \] (1.8)

where
\[ E_{g,\lambda}(0, t) = \frac{g((1 - \lambda)t)}{\lambda g(0) + (1 - \lambda)g(t)}, \]
\[ E_{h,\mu}(0, \tau) = \frac{h((1 - \mu)\tau)}{\mu h(0) + (1 - \mu)h(\tau)}, \]
\[ v(z) = zh \left( \phi \left( \frac{1}{z} \right) \right), \] (1.9)
and
\[ w(z) = f(z)h \left( \phi \left( \frac{z}{2} \right) \right). \] (1.10)

Remark 1.9. Inequality (1.8) just is an inverse of the following inequality established by Rozanova [13].
\[ \omega \left( \int_0^\tau p'(t)g \left( \frac{|x(t)|}{p(t)} \right) dt \right) \geq \int_0^\tau f' \left( p(t)g \left( \frac{|x(t)|}{p(t)} \right) \right) \cdot v \left( p(t)g \left( \frac{|x'(t)|}{p'(t)} \right) \right) dt. \]

On the other hand, for \( x(t) = x_1(t), x_1'(t) > 0, x'_1(0) = 0, x(\tau) = b, g(t) = t, f(t) = \phi(t) = t^2 \) and \( h(t) = \sqrt{1 + t} \), the inequality (1.8) reduces to an inverse of the following inequality established by Pólya [13].
\[ 2 \int_0^\tau x_1(t) \left( 1 + (x'_1(t))^2 \right)^{1/2} dt \leq b(\tau^2 + b^2)^{1/2}. \]

2. Proof of main results

Lemma 2.1. Let \( p \) be a positive continuous function and \( \phi \) be continuous function on \([a, b]\). Let \( f \) be a positive, convex and continuous function on an interval containing both \([a, b]\) and \(\phi[a, b]\) as subsets. Then there exist \( \lambda \) (0 ≤ \( \lambda \) ≤ 1) such that
\[ f \left( \int_a^b p(x)\phi(x)dx \right) \geq E_{f,\lambda}(a, b) \int_a^b p(x)f(\phi(x))dx, \] (2.1)

where
\[ E_{f,\lambda}(a, b) = \frac{f(\lambda a + (1 - \lambda)b)}{\lambda f(a) + (1 - \lambda)f(b)}. \] (2.2)

Proof. For any finite sequence of real numbers \( \{u_i\} \) in a fixed closed interval \([a, b]\) and any sequence of positive numbers \( \{q_i\} \), since \( a \leq u_i \leq b \), there is a sequence \( t_i \in [0, 1] \) such that \( u_i = t_i a + (1 - t_i)b \). Therefore
\[ \frac{\sum_{i=1}^n q_i f(u_i)}{\sum_{i=1}^n q_i} = \frac{\sum_{i=1}^n q_i f(t_i a + (1 - t_i)b)}{\sum_{i=1}^n q_i}, \]
\[ \leq \frac{\sum_{i=1}^n q_i (t_i f(a) + (1 - t_i)f(b))}{\sum_{i=1}^n q_i} \]
On the other hand, letting $x_i = a + \left(\frac{b-a}{n}\right)i$, $i = 0, 1, \ldots, n$, we have

$$\triangle x_i = x_i - x_{i-1} = \frac{b-a}{n}, \quad i = 1, \ldots, n.$$ 

Let $u_i := \phi(x_i)$ and $q_i := p(x_i) \triangle x_i$, $i = \ldots, n$, we obtain

$$f\left(\frac{\sum_{i=1}^{n} p(x_i) \phi(x_i) \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i}\right) \geq E_f(a,b) \frac{\sum_{i=1}^{n} p(x_i) f(\phi(x_i)) \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i} ,$$

where

$$E_f(a,b) = f\left(\frac{\sum_{i=1}^{n} p(x_i) \lambda a_{x_i} \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i}\right) + b \left(\frac{1 - \sum_{i=1}^{n} p(x_i) \lambda a_{x_i} \triangle x_i}{\sum_{i=1}^{n} p(x_i) \triangle x_i}\right),$$

By taking limits as $n \to \infty$, we get

$$f\left(\frac{\int_{a}^{b} p(x) \phi(x) dx}{\int_{a}^{b} p(x) dx}\right) \geq E_f(a,b) \frac{\int_{a}^{b} p(x) f(\phi(x)) dx}{\int_{a}^{b} p(x) dx} ,$$

where

$$E_f(a,b) = \frac{f(ma + nb)}{mf(a) + nf(b)}$$

for some $0 \leq m, n \leq 1$ with $m + n = 1$.

If $m = \lambda$ and $n = 1 - \lambda$, (2.1) easily follows. \hfill \Box

Lemma 2.1 was also proved in [19] by the author, but there’s a little mistake in that proof. A complete and correct proof has shown here. 

**Proof of Theorem 1.6**

Proof. Let $y(t) = \int_{\alpha}^{t} |x'(s)| ds$, $t \in [\alpha, \tau]$ so that $y'(t) = |x'(t)|$ and in view of

$$|x(t)| \leq \int_{\alpha}^{t} |x'(s)| ds,$$

we have

$$y(t) \geq |x(t)|.$$

From the hypotheses and in view of the reverse Jensen’s inequality in Lemma 2.1, we obtain for $0 \leq \lambda \leq 1$

$$g \left(\frac{|x(t)|}{p(t)}\right) \geq g \left(\frac{y(t)}{p(t)}\right)$$

$$= g \left(\frac{\int_{\alpha}^{t} p'(s) \frac{|x'(s)|}{p(s)} ds}{\int_{\alpha}^{t} p'(s) ds}\right)$$

$$\geq \left(\frac{g(\lambda \alpha + (1 - \lambda)t)}{\lambda g(\alpha) + (1 - \lambda)g(t)}\right) \frac{1}{p(t)} \int_{\alpha}^{t} p'(s) g \left(\frac{|x'(s)|}{p'(x)}\right) ds.$$
On the other hand, from the hypotheses and by using Inequality (2.3), we have
\[
\int_0^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right) f'(\frac{\lambda g(\alpha) + (1 - \lambda)g(t)}{g(\lambda \alpha + (1 - \lambda)t)} \cdot p(t)g\left(\frac{|x(t)|}{p(t)}\right))dt
\geq \int_0^\tau p'(t)g\left(\frac{y'(t)}{p'(t)}\right) f'(\int_\alpha^t p'(s)g\left(\frac{y'(s)}{p'(s)}\right)ds)dt
\]
\[
= \int_\alpha^\tau \frac{d}{dt} f\left(\int_\alpha^t p'(s)g\left(\frac{y'(s)}{p'(s)}\right)ds\right)dt
\]
\[
= f\left(\int_\alpha^\tau p'(t)g\left(\frac{y'(t)}{p'(t)}\right)dt\right)
\]
\[
= f\left(\int_\alpha^\tau p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)dt\right).
\]

This completes the proof. \qed

**Proof of Theorem 1.8**

**Proof.** From the reverse Jensen’s inequality, we obtain
\[
p(t)g\left(\frac{|x(t)|}{p(t)}\right) \geq E_{g,\lambda}(0,t)y(t),
\]
where \(E_{g,\lambda}(0,t)\) is as in (2.2). Because \(g\) and \(h\) are convex and concave functions, respectively, so there exists \(0 \leq \lambda, \mu \leq 1\), so that
\[
E_{g,\lambda}^{-1}(0,t) = \frac{\lambda g(0) + (1 - \lambda)g(t)}{g((1 - \lambda)t)} \geq 1,
\]
and
\[
E_{h,\mu}(0,\tau) = \frac{h((1 - \mu)\tau)}{\mu h(0) + (1 - \mu)h(\tau)} \geq 1.
\]

Hence
\[
E_{h,\mu}(0,\tau) \int_0^\tau f'\left(E_{g,\lambda}^{-1}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right) \cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \geq E_{h,\mu}(0,\tau) \int_0^\tau f'(y(t)) \cdot v(y'(t))dt.
\]

From (2.4) and (2.4), we have
\[
E_{h,\mu}(0,\tau) \int_0^\tau f'\left(E_{g,\lambda}^{-1}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right) \cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \geq E_{h,\mu}(0,\tau) \int_0^\tau f'(y(t)) y'(t)h\left(\phi\left(\frac{1}{y'(t)}\right)\right)dt.
\]

From (2.1), (2.5) and in view of \(h\) is concave function, we obtain
\[
E_{h,\mu}(0,\tau) \int_0^\tau f'\left(E_{g,\lambda}^{-1}(0,t)p(t)g\left(\frac{|x(t)|}{p(t)}\right)\right) \cdot v\left(p'(t)g\left(\frac{|x'(t)|}{p'(t)}\right)\right)dt \geq E_{h,\mu}(0,\tau) \frac{\int_0^\tau f'(y(t)) y'(t)dt}{\int_0^\tau f'(y(t)) y'(t)dt} \int_0^\tau f'(y(t)) y'(t)dt
\]
\[
\geq h\left(\frac{\int_0^\tau f'(y(t)) y'(t)dt}{\int_0^\tau f'(y(t)) y'(t)dt}\right) f(y(\tau)).
\]
From (1.7), (1.10), (2.6) and in view of $h$ is increasing function, we obtain

$$E_{h,u}(0, \tau) \int_0^\tau f'( \frac{E_{0,\lambda}(0, t)p(t)g \left( \frac{|x(t)|}{p(t)} \right)}{p'(t)} ) \cdot v \left( p'(t)g \left( \frac{|x'(t)|}{p'(t)} \right) \right) dt$$

$$\geq h \left( \frac{\int_0^\tau f(y(\tau)) \cdot \phi' \left( \frac{t}{y(\tau)} \right) dt}{\int_0^\tau f'(y(t)) \cdot \phi' \left( \frac{t}{y(\tau)} \right) dt} \right) f(y(\tau))$$

$$= h \left( \frac{\int_0^\tau \phi' \left( \frac{t}{y(\tau)} \right) dt}{\int_0^\tau f'(y(t)) \cdot \phi' \left( \frac{t}{y(\tau)} \right) dt} \right) f(y(\tau))$$

$$= h \left( \phi \left( \frac{\tau}{y(\tau)} \right) \right) f(y(\tau))$$

$$= \omega(y(\tau)) = \omega \left( \int_0^\tau p'(t) \left( \frac{|x'(t)|}{p'(t)} \right) dt \right).$$

This completes the proof. \qed

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**References**


