Approximate ternary quadratic derivation on ternary Banach algebras and $C^*$-ternary rings: revisited

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Abstract

Recently, Shagholi et al. [S. Shagholi, M. Eshaghi Gordji, M. B. Savadkouhi, J. Comput. Anal. Appl., 13 (2011), 1097–1105] defined ternary quadratic derivations on ternary Banach algebras and proved the Hyers-Ulam stability of ternary quadratic derivations on ternary Banach algebras. But the definition was not well-defined. Using the fixed point method, Bodaghi and Alias [A. Bodaghi, I. A. Alias, Adv. Difference Equ., 2012 (2012), 9 pages] proved the Hyers-Ulam stability and the superstability of ternary quadratic derivations on ternary Banach algebras and $C^*$-ternary rings. There are approximate $C$-quadraticity conditions in the statements of the theorems and the corollaries, but the proofs for the $C$-quadraticity were not completed. In this paper, we correct the definition of ternary quadratic derivation and complete the proofs of the theorems and the corollaries. ©2015 All rights reserved.

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1. Introduction


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The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \] (1.1)
is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.1) is said to be a quadratic mapping. The Hyers-Ulam stability problem for the quadratic functional equation (1.1) was proved by Skof for mappings \( f : A \to B \), where \( A \) is a normed space and \( B \) is a Banach space (see [8]).

In [7], Shagholi et al. defined a ternary quadratic derivation \( D \) from a ternary Banach algebra \( A \) into a ternary Banach algebra \( B \) such that
\[ D[x, y, z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)] \]
for all \( x, y, z \in A \). But \( x^2, y^2, z^2 \) are not defined and the brackets of the right side are not defined, since \( A \) is not an algebra and \( D(x) \in B \) and \( y^2, z^2 \in A \). So we correct them as follows.

**Definition 1.1.** Let \( A \) be a complex algebra-ternary Banach algebra with norm \( \| \cdot \| \) or a complex algebra-\( C^* \)-ternary ring with norm \( \| \cdot \| \). A \( C \)-linear mapping \( D : A \to A \) is called a ternary quadratic derivation if

1. \( D \) is a quadratic mapping,
2. \( D[x, y, z] = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)] \) for all \( x, y, z \in A \).

There are approximate \( C \)-quadraticity conditions in the statements of the theorems and the corollaries in [2], but the proofs for the \( C \)-quadraticity were not completed.

In this paper, we complete the proofs of the theorems and the corollaries given in [2]. Throughout this paper, let \( A \) be a complex algebra-ternary Banach algebra with norm \( \| \cdot \| \) or a complex algebra-\( C^* \)-ternary ring with norm \( \| \cdot \| \).

### 2. Stability of ternary quadratic derivations

We need the following lemma to obtain the main results.

**Lemma 2.1.** Let \( f : A \to A \) be a quadratic mapping such that \( f(\mu x) = \mu^2 f(x) \) for all \( x \in A \) and \( \mu \in \mathbb{T}^1 = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \} \). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then the mapping \( f : A \to A \) satisfies \( f(\mu x) = \mu^2 f(x) \) for all \( x \in A \) and all \( \mu \in \mathbb{C} \).

The proof is similar to the proof of the corresponding lemma given in [5].

**Proof.** Let \( r \) be a rational number. It is easy to show that \( f(rx) = r^2 f(x) \) for all \( x \in A \).

By the same reasoning as in the proof of main theorem of [6], one can show that \( f(rx) = r^2 f(x) \) for all \( x \in A \) and all \( r \in \mathbb{R} \). So
\[ f(\mu x) = f \left( |\mu| \frac{\mu}{|\mu|} x \right) = |\mu|^2 f \left( \frac{\mu}{|\mu|} x \right) = |\mu|^2 \cdot \frac{\mu^2}{|\mu|^2} f(x) = \mu^2 f(x) \]
for all \( \mu \in \mathbb{C} \setminus \{0\} \) and all \( x \in A \). Since \( f(0) = 0, f(\mu x) = \mu^2 f(x) \) for all \( x \in A \) and all \( \mu \in \mathbb{C} \).

We recall a fundamental result in fixed point theory.

**Theorem 2.2.** ([3]) Let \( (X, d) \) be a complete generalized metric space and let \( J : X \to X \) be a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then, for each given element \( x \in X \), either
\[ d(J^n x, J^{n+1} x) = \infty \]
for all nonnegative integers \( n \) or there exists a positive integer \( n_0 \) such that
1. \( d(J^n x, J^{n+1} x) < \infty \) for all \( n \geq n_0 \);
2. the sequence \( \{ J^n x \} \) converges to a fixed point \( y^* \) of \( J \);
3. \( y^* \) is the unique fixed point of \( J \) in the set \( Y = \{ y \in X \mid d(J^{n_0} x, y) < \infty \} \);
4. \( d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \) for all \( y \in Y \).
Theorem 2.3. Let $A$ be a complex algebra-$C^*$-ternary ring. Let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\varphi : A^5 \to [0, \infty)$ be a function such that

\[
\left\|2f\left(\frac{a + b}{2}\right) + 2f\left(\frac{a - b}{2}\right) - \mu^2(f(a) + f(b))\right\| \leq \varphi(a, b, 0, 0, 0),
\]

(2.1)

\[
\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)]\| \leq \varphi(0, 0, x, y, z)
\]

(2.2)

for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. Assume that there exists a constant $M \in (0, 1)$ such that

\[
\varphi(2a, 2b, 2x, 2y, 2z) \leq 4M \varphi(a, b, x, y, z)
\]

(2.3)

for all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that

\[
\|f(a) - D(a)\| \leq \frac{M}{1 - M} \varphi(a, 0, 0, 0)
\]

(2.4)

for all $a \in A$.

Proof. It follows from (2.3) that

\[
\lim_{j \to \infty} \frac{\varphi(2^ja, 2^jb, 2^jx, 2^jy, 2^jz)}{4^j} = 0
\]

for all $a, b, x, y, z \in A$.

Putting $b = 0$ and $\mu = 1$ and replacing $a$ by $2a$ in (2.1), we get

\[
\|4f(a) - f(2a)\| \leq \varphi(2a, 0, 0, 0, 0) \leq 4M \varphi(a, 0, 0, 0)
\]

and so

\[
\left\|f(a) - \frac{1}{4}f(2a)\right\| \leq M \varphi(a, 0, 0, 0)
\]

(2.5)

for all $a \in A$.

We consider the set $\Omega := \{h : A \to A \mid h(0) = 0\}$ and introduce the generalized metric $d$ on $\Omega$ as follows:

\[
d(h_1, h_2) := \inf \{K \in [0, \infty) : \|h_1(a) - h_2(a)\| \leq K \varphi(a, 0, 0, 0, 0), \forall a \in A\}
\]

if there exists such constant $K$, and $d(h_1, h_2) = \infty$, otherwise. One can easily show that $(\Omega, d)$ is complete. We define the linear mapping $J : \Omega \to \Omega$ by

\[
J(h)(a) = \frac{1}{4}h(2a)
\]

(2.6)

for all $a \in A$.

Given $h_1, h_2 \in \Omega$, let $K \in \mathbb{R}_+$ be an arbitrary constant with $d(h_1, h_2) \leq K$, that is

\[
\|h_1(a) - h_2(a)\| \leq K \varphi(a, 0, 0, 0, 0)
\]

(2.7)

for all $a \in A$. Replacing $a$ by $2a$ in (2.7) and using (2.3) and (2.6), we have

\[
\|(Jh_1)(a) - (Jh_2)(a)\| = \frac{1}{4}\|h_1(2a) - h_2(2a)\| \leq \frac{1}{4}K \varphi(2a, 0, 0, 0, 0) \leq KM \varphi(a, 0, 0, 0, 0)
\]

for all $a \in A$ and so $d(Jh_1, Jh_2) \leq KM$. Thus we conclude that $d(Jh_1, Jh_2) \leq Md(h_1, h_2)$ for all $h_1, h_2 \in \Omega$. It follows from (2.5) that

\[
d(Jf, f) \leq M.
\]

(2.8)
By Theorem 2.2, the sequence \( \{ J^n f \} \) converges to a unique fixed point \( D : A \to A \) in the set \( \Omega_1 := \{ h \in \Omega, d(f, h) < \infty \} \), i.e.,
\[
\lim_{n \to \infty} \frac{f(2^n a)}{4^n} = D(a)
\]
for all \( a \in A \). By Theorem 2.3 and (2.8), we have
\[
d(f, D) \leq \frac{d(Jf, f)}{1-M} \leq \frac{M}{1-M}.
\]

The last inequality shows that (2.4) holds for all \( a \in A \). Replacing \( a, b \) by \( 2^na, 2^nb \) in (2.1), respectively, and dividing both sides of the resulting inequality by \( 4^n \), and letting \( n \) tend to infinity, we obtain
\[
2D \left( \frac{a+b}{2} \right) + 2D \left( \frac{a-b}{2} \right) = \mu^2 D(a) + \mu^2 D(b)
\]
for all \( a, b \in A \) and all \( \mu \in T^1 \). Putting \( \mu = 1 \) in (2.9), we have
\[
2D \left( \frac{a+b}{2} \right) + 2D \left( \frac{a-b}{2} \right) = D(a) + D(b)
\]
for all \( a, b \in A \). Hence \( D \) is a quadratic mapping. It follows from (2.9) that \( D(\mu a) = \mu^2 D(a) \) for all \( a \in A \) and \( \mu \in T^1 \). By Lemma 2.1 and the same reasoning as in the proof of main theorem of [6], one can show that \( D(\mu a) = \mu^2 D(a) \) for all \( a \in A \) and \( \mu \in C \).

Replacing \( x, y, z \) by \( 2^nx, 2^ny, 2^nz \) in (2.2), respectively, and dividing by \( 4^{3n} \), we obtain
\[
\| f([2^n x, 2^n y, 2^n z]) - [f(2^n x), 4^n y^2, 4^n z^2] - [4^n x^2, f(2^n y), 4^n z^2] - [4^n x^2, A^n y^2, f(2^n z)] \|
\leq \varphi(0, 0, 2^n x, 2^n y, 2^n z) \leq \varphi(0, 0, 2^n x, 2^n y, 2^n z) \leq \varphi(0, 0, 2^n x, 2^n y, 2^n z)
\]
which tends to zero as \( n \to \infty \). So
\[
D([x, y, z]) = [D(x), y^2, z^2] + [x^2, D(y), z^2] + [x^2, y^2, D(z)]
\]
for all \( x, y, z \in A \). So \( D \) is a ternary quadratic derivation.

Corollary 2.4. Let \( p, \theta \) be nonnegative real numbers with \( p < 2 \) and let \( A \) be a complex algebra-\( C^* \)-ternary ring. Let \( f : A \to A \) be a mapping such that
\[
\left\| 2f \left( \frac{a+b}{2} \right) + 2f \left( \frac{a-b}{2} \right) - \mu^2 (f(a) + f(b)) \right\| \leq \theta(\|a\|^p + \|b\|^p),
\]
(2.10)
\[
\|f([x, y, z]) - [f(x), y^2, z^2] - [x^2, f(y), z^2] - [x^2, y^2, f(z)]\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p)
\]
(2.11)
for all \( \mu \in T^1 \) and all \( a, b, x, y, z \in A \). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique ternary quadratic derivation \( D : A \to A \) such that
\[
\|f(a) - D(a)\| \leq \frac{2^{p+1}}{4-2p} \|a\|^p
\]
for all \( a \in A \).

Proof. The result follows from Theorem 2.3 by putting \( \varphi(a, b, x, y, z) = \theta(\|a\|^p + \|b\|^p + \|x\|^p + \|y\|^p + \|z\|^p) \).

Now we prove the superstability of ternary quadratic derivations on complex algebra-\( C^* \)-ternary rings.
Corollary 2.5. Let $p, \theta$ be nonnegative real numbers with $p < \frac{3}{2}$ and let $A$ be a complex algebra-$C^*$-ternary ring. Let $f : A \to A$ be a mapping such that
\[
\left\|2f \left( \frac{a+b}{2} \right) + 2f \left( \frac{a-b}{2} \right) - \mu^2(f(a) + f(b)) \right\| \leq \theta(\|a\|^p \cdot \|b\|^p),
\]
(2.12)
for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then $f : A \to A$ is a ternary quadratic derivation.

Proof. Putting $a = b = 0$ in (2.12), we get $f(0) = 0$. Letting $b = 0$, $\mu = 1$ and replacing $a$ by $2a$ in (2.13), we get $f(2a) = 4f(a)$ for all $a \in A$. It is easy to show that $f(2^na) = 4^nf(a)$ and so $f(a) = \frac{f(2^na)}{2^n}$ for all $a \in A$. It follows from Theorem 2.3 that $f : A \to A$ is a quadratic mapping. The result follows from Theorem 2.3 by putting $\varphi(a, b, x, y, z) = \theta(\|a\|^p \cdot \|b\|^p + \|x\|^p \cdot \|y\|^p \cdot \|z\|^p)$.

Theorem 2.6. Let $A$ be a complex algebra-ternary Banach algebra. Let $f : A \to A$ be a mapping with $f(0) = 0$ and let $\varphi : A^3 \to [0, \infty)$ be a function satisfying (2.2) and
\[
\left\|f(\mu(a+b)) + f(\mu(a-b)) - 2\mu^2(f(a) + f(b)) \right\| \leq \varphi(a, b, 0, 0, 0)
\]
(2.14)
for all $\mu \in \mathbb{T}^1$ and all $a, b, x, y, z \in A$. Assume that there exists a constant $M \in (0, 1)$ such that
\[
\varphi(2a, 2b, 2x, 2y, 2z) \leq 4M \varphi(a, b, x, y, z)
\]
(2.15)
for all $a, b, x, y, z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that
\[
\|f(a) - D(a)\| \leq \frac{1}{4(1-M)}\varphi(a, a, 0, 0, 0)
\]
for all $a \in A$.

Proof. It follows from (2.15) that
\[
\lim_{j \to \infty} \frac{\varphi(2^ja, 2^jb, 2^jx, 2^jy, 2^jz)}{4^j} = 0
\]
for all $a, b, x, y, z \in A$.

Putting $b = a$ and $\mu = 1$ in (2.14), we get
\[
\left\|4f(a) - f(2a) \right\| \leq \varphi(a, a, 0, 0, 0)
\]
and so
\[
\left\|f(a) - \frac{1}{4}f(2a) \right\| \leq \frac{1}{4}\varphi(a, a, 0, 0, 0)
\]
for all $a \in A$.

We consider the set $\Omega := \{ h : A \to A \mid h(0) = 0 \}$ and introduce the generalized metric $d$ on $\Omega$ as follows:
\[
d(h_1, h_2) := \inf \{ K \in [0, \infty) : \|h_1(a) - h_2(a)\| \leq K\varphi(a, a, 0, 0, 0), \forall a \in A \}
\]
if there exists such constant $K$, and $d(h_1, h_2) = \infty$, otherwise. One can easily show that $(\Omega, d)$ is complete. We define the linear mapping $J : \Omega \to \Omega$ by
\[
J(h)(a) = \frac{1}{4} h(2a)
\]
for all $a \in A$.

The rest of the proof is similar to the proof of Theorem 2.3.

Corollary 2.7. Let $p, \theta$ be nonnegative real numbers with $p < 2$ and let $A$ be a complex algebra-ternary Banach algebra. Let $f : A \to A$ be a mapping satisfying (2.11) and
\[
\| f(\mu(a+b)) + f(\mu(a-b)) - 2\mu^2(f(a) + f(b)) \| \leq \theta(\|a\|^p + \|b\|^p)
\]
(2.16)
for all $\mu \in T_1$ and all $a,b,x,y,z \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique ternary quadratic derivation $D : A \to A$ such that
\[
\|f(a) - D(a)\| \leq \frac{2\theta}{4 - 2p}\|a\|^p
\]
for all $a \in A$.

Proof. The result follows from Theorem 2.6 by putting $\varphi(a,b,x,y,z) = \theta(\|a\|^p + \|b\|^p + \|x\|^p + \|y\|^p + \|z\|^p)$.

Remark 2.8. Bodaghi and Alias [2] provided the conditions (2.1), (2.10), (2.12), (2.14) and (2.16), which are approximate $C$-quadraticity conditions. But they only proved the quadraticity of the resulting mappings. In this paper, the $C$-quadraticity has been proved for each case.

References