Existence and nonexistence of solutions for nonlinear second order $q$-integro-difference equations with non-separated boundary conditions

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Abstract

In this paper, we investigate a nonlinear second order boundary value problem of $q$-integro-difference equations supplemented with non-separated boundary conditions. Sufficient conditions for the existence and nonexistence of solutions are presented. Examples are provided for explanation of the obtained work. ©2015 All rights reserved.

Keywords: $q$-integro-difference equations, non-separated boundary conditions, existence, nonexistence.


1. Introduction

Consider the following nonlinear second order $q$-integro-difference equation with non-separated boundary conditions:

$$
\begin{aligned}
D_q^2 u(t) &= f(t, u(t)) + I_q g(t, u(t)), \quad t \in I_q, \\
u(0) &= \eta u(T), \\
D_q u(0) &= \eta D_q u(T),
\end{aligned}
$$

(1.1)

where $f, g \in C(I_q \times \mathbb{R}, \mathbb{R})$, $I_q = [0, T] \cap q\mathbb{N}$, $q\mathbb{N} := \{q^n : n \in \mathbb{N}\} \cup \{0\}$, $T \in q\mathbb{N}$ is a fixed constant and $\eta \neq 1$ is a fixed real number.

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The study of $q$-difference equations, initiated with the works of Jackson [18, 19], Carmichael [14], Mason [22] and Adams [1], has recently gained a considerable interest. The subject of $q$-calculus is also known as quantum calculus and distinguishes itself from the classical calculus in the sense that the notion of $q$-derivative is independent of the concept of limit and that $q$-difference equations are always completely controllable. The tools of $q$-calculus are found to be of a great value in studying $q$-optimal control problems [10]. The $q$-analogue of continuous variational calculus is variational $q$-calculus, where the extra-parameter $q$ accounts for a physical or economical situation. In fact, the variational calculus on $q$-calculus are found to be of a great value in studying $q$-optimal control problems rather than solving the Euler-Lagrange equation itself [11]. The $q$-difference equations have potential applications in several fields such as special functions, super-symmetry, operator theory, combinatorics, etc. For examples and details, see a series of books (7, 8, 16, 20) and papers (2, 12, 23) and the references cited therein. Concerning the theory of initial and boundary value problems of $q$-difference equations, we refer the reader to the works obtained in papers (3, 4, 5, 6, 9, 13, 15, 17, 24).

In the sequel, we use the following conditions and notation:

\[(H_1) \lim_{|u| \to \infty} \frac{f(t, u)}{|u|} = a(t) \quad \text{and} \quad \lim_{|u| \to \infty} \frac{g(t, u)}{|u|} = b(t) \text{ uniformly on } I_q.\]

\[(H_2) \lim_{|u| \to 0^+} \frac{f(t, u)}{|u|} = 0 \quad \text{and} \quad \lim_{|u| \to \infty} \frac{f(t, u)}{|u|} = 0 \text{ uniformly on } I_q.\]

\[(H_3) \lim_{|u| \to 0^+} \frac{g(t, u)}{|u|} = 0 \quad \text{and} \quad \lim_{|u| \to \infty} \frac{g(t, u)}{|u|} = 0 \text{ uniformly on } I_q.\]

\[\Lambda = \sup_{t \in I_q} \left\{ \int_0^t |t - qs|a(s)|d_qs + \int_0^t \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} |b(r)|d_qr \right.\]

\[+ \left. \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)a(s)|d_qs + \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2 - qT^2}{1 + q} |b(r)|d_qr \right\},\]

\[\mathcal{M} = \frac{T^2}{1 + q} + \frac{T^3}{(1 + q)(1 + q + q^2)} + \frac{|\eta|T^2}{(\eta - 1)^2} + \frac{|\eta|T^2}{|\eta|} + \frac{|\eta|T^2}{|\eta|} + \frac{|\eta|q}{|\eta| - 1(1 + q + q^2)}.\]

The main objective of the present paper is to establish the following results which deal with the existence and nonexistence of solutions for the problem \([1.1]\).

**Theorem 1.1.** Assume that the condition \((H_1)\) holds. Then the problem \([1.1]\) has at least one solution provided that \(0 < \Lambda < 1\).

**Theorem 1.2.** Assume that the conditions \((H_2)\) and \((H_3)\) hold. If there exists a constant \(B > 0\) such that \(\mathcal{M} < B\), then the non-separated boundary value problem \([1.1]\) has no solution.

### 2. Preliminaries

Let us describe some basic concepts of $q$-calculus [7, 16].

For $0 < q < 1$, the $q$-derivative of a real valued function $f$ is defined as

\[D_qf(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad D_qf(0) = \lim_{t \to 0} D_qf(t).\]
The $q$-integral of a function $f$ is defined as
\[
\int_a^x f(t)d_qt := \sum_{n=0}^{\infty} x(1-q^n)f(xq^n) - a(1-q^n)f(aq^n), \quad x \in [a, b],
\]
and for $a = 0$, we denote
\[
I_qf(x) = \int_0^x f(t)d_qt = \sum_{n=0}^{\infty} x(1-q^n)f(xq^n),
\]
provided the series converges. If $a \in [0, b]$ and $f$ is defined on the interval $[0, b]$, then
\[
\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt.
\]
Similarly, we have
\[
I_q^n f(t) = f(t), \quad I_q^n f(t) = I_qI_q^{n-1} f(t), \quad n \in \mathbb{N}.
\]
Observe that
\[
D_qI_qf(x) = f(x), \quad (2.1)
\]
and if $f$ is continuous at $x = 0$, then $I_qD_qf(x) = f(x) - f(0)$. In $q$-calculus, the product rule and integration by parts formula are
\[
D_q(gh)(t) = D_qg(t)h(t) + g(qt)D_qh(t), \quad (2.2)
\]
\[
\int_0^x f(t)D_qg(t)dqt = \left[ f(t)g(t) \right]_0^x - \int_0^x D_qf(t)g(qt)dqt. \quad (2.3)
\]

We introduce the Banach space $X = C(I_q, \mathbb{R}) = \{ u : I_q \to \mathbb{R} \mid u \in C(I_q) \}$ equipped with a topology of uniform convergence with respect to the norm $\|u\| = \sup_{t \in I_q}|u(t)|$.

**Lemma 2.1** ([4]). The linear problem of a second order $q$-difference equation supplemented with non-separated boundary conditions:
\[
\begin{cases}
D_q^2u(t) = y(t), & t \in I_q, \\
u(0) = \eta u(T), & D_qu(0) = \eta D_qu(T),
\end{cases} \quad (2.4)
\]
has a unique solution given by
\[
u(t) = \int_0^T (t-qs)y(s)d_qs + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t-qs)]y(s)d_qs. \quad (2.5)
\]

To transform the problem (1.1) into a fixed point problem we use Lemma 2.1 to define an operator $T : X \to X$ as
\[
(Tu)(t) = \int_0^t (t-qs)[f(s,u(s)) + I_qg(s,u(s))]d_qs + \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t-qs)]f(s,u(s))d_qs
\]
\[
+ \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t-qs)]I_qg(s,u(s))d_qs
\]
\[
= \int_0^t (t-qs)f(s,u(s))d_qs + \int_0^t (t-qs) \int_0^s g(r,u(r))d_rd_qs
\]
\[
+ \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t-qs)]f(s,u(s))d_qs
\]
\[
+ \frac{\eta}{(\eta - 1)^2} \int_0^T [T + (1 - \eta)(t-qs)] \int_0^s g(r,u(r))d_rd_qs,
\]
which can alternatively be written as
\[(Tu)(t) = \int_0^t (t-qs)f(s,u(s))ds + \int_0^t \left( \frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q} \right) g(r,u(r))dt \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T [T + (1-\eta)(t-qs)]f(s,u(s))ds \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T \left[ (T + (1-\eta)t)(T-qr) + (1-\eta)\frac{q^3r^2 - qT^2}{1+q} \right]g(r,u(r))dt \]  \hspace{1cm} (2.7)

Observe that the problem (1.1) has a solution if and only if the operator \(T\) has a fixed point.

### 3. The Proof of Main Results

In order to obtain the proof of Theorem 1.1 we need the following fixed point theorem.

**Theorem 3.1** (\cite{21}). Let \(X\) be a Banach Space. Let \(T : X \to X\) be a completely continuous mapping and let \(L : X \to X\) be a bounded linear mapping such that \(1\) is not an eigenvalue of \(L\). Suppose that

\[ \lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0. \]

Then \(T\) has a fixed point in \(X\).

**Proof of Theorem 1.1**

In the first step, we show that \(T\) is a completely continuous operator. Obviously, the operator \(T\) is continuous in view of continuity of functions \(f\) and \(g\).

Let \(\Omega \subset C(I_q, \mathbb{R})\) be bounded. Then, for any \(t \in I_q\), there exist positive constants \(L_1\) and \(L_2\) such that \(|f(t,u)| \leq L_1\) and \(|g(t,u)| \leq L_2, \forall u \in \Omega\). Then, we have

\[ \| (Tu)(t) \| = \left| \int_0^t (t-qs)f(s,u(s))ds + \int_0^t \left( \frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q} \right) g(r,u(r))dt \right| \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T [T + (1-\eta)(t-qs)]f(s,u(s))ds \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T \left[ (T + (1-\eta)t)(T-qr) + (1-\eta)\frac{q^3r^2 - qT^2}{1+q} \right]g(r,u(r))dt \]
\[\leq \int_0^t |t-qs||f(s,u(s))|ds + \int_0^t \left| \frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q} \right| |g(r,u(r))|dr \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T |T + (1-\eta)(t-qs)||f(s,u(s))|ds \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T \left[ |T + (1-\eta)t|(T-qr) + (1-\eta)\frac{q^3r^2 - qT^2}{1+q} \right]|g(r,u(r))|dr \]
\[\leq \int_0^t |t-qs|L_1ds + \int_0^t \left| \frac{t^2}{1+q} - qrt + \frac{q^3r^2}{1+q} \right| L_2dr \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T |T + (1-\eta)(t-qs)|L_1ds \]
\[+ \frac{\eta}{(\eta-1)^2} \int_0^T \left[ |T + (1-\eta)t|(T-qr) + (1-\eta)\frac{q^3r^2 - qT^2}{1+q} \right]L_2dr \]
\[\leq \sup_{t \in I_q} \left\{ \frac{t^2}{1+q}L_1 + \frac{t^3}{(1+q)(1+q + q^2)}L_2 + \frac{|\eta|T^2}{(\eta-1)^2L_1} + \frac{|\eta|q}{|\eta-1|(1+q + q^2)L_2} \right\} \]
Thus, which implies that \( \|Tu\| \leq L \). Moreover, for \( \forall t_1, t_2 \in I_q, \ t_1 < t_2 \), we obtain

\[
\|(Tu)(t_2) - (Tu)(t_1)\| \\
= \left| \int_0^{t_2} (t_2 - qs) f(s, u(s)) \, ds \right| - \left| \int_0^{t_1} (t_1 - qs) f(s, u(s)) \, ds \right| \\
+ \int_0^{t_2} \left( \frac{t_2^2}{1 + q} - qrt_2 + \frac{q^3r^2}{1 + q} g(r, u(r)) \right) \, dr \\
+ \int_0^{t_1} \left( \frac{t_1^2}{1 + q} - qrt_1 + \frac{q^3r^2}{1 + q} g(r, u(r)) \right) \, dr \\
+ \eta(t_2 - t_1) \int_0^T f(s, u(s)) \, ds + \frac{\eta(t_2 - t_1)}{(1 - \eta)} \int_0^T (T - qr) g(r, u(r)) \, dr,
\]

which tends to zero independent of \( u \) as \( t_2 - t_1 \to 0 \). Thus, the operator \( T \) is relatively compact on \( \Omega \). Hence, by the Arzela-Ascoli Theorem, the operator \( T \) is compact on \( \Omega \). Hence, the operator \( T \) is completely continuous.

Next, we show that \( T \) has a fixed point. Define an operator \( L : X \to X \) as

\[
(Lu)(t) = \int_0^t (t - qs)a(s)|u(s)| \, ds + \int_0^t \left( \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} b(r) \right) |u(r)| \, dr \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T \left[ T + (1 - \eta)(t - qs) \right] a(s) |u(s)| \, ds \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T \left[ T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2}{1 + q} T^2 \right] b(r) |u(r)| \, dr.
\]

Thus \( L \) is a bounded linear operator. In addition, we have

\[
|(Lu)(t)| = \left| \int_0^t (t - qs)a(s)|u(s)| \, ds + \int_0^t \left( \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} b(r) \right) |u(r)| \, dr \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T \left[ T + (1 - \eta)(t - qs) \right] a(s) |u(s)| \, ds \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T \left[ T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2}{1 + q} T^2 \right] b(r) |u(r)| \, dr \right| \leq \int_0^t |t - qs||a(s)| \, ds + \int_0^t \left| \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} b(r) \right| |u(r)| \, dr \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)||a(s)| \, ds \\
+ \frac{\eta}{(\eta - 1)^2} \int_0^T \left[ T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2}{1 + q} T^2 \right] |b(r)| \, dr
\]

(3.1)
3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, there exists a fixed point $u$ such that for any $t \in I_q$, we have
\[
|f(t, u) - a(t)| < \varepsilon |u| \quad \text{and} \quad |g(t, u) - b(t)| < \varepsilon |u|.
\]
Thus, for $t > 0$ and $|u| > M$, we have
\[
\|Tu(t) - Lu(t)\| \leq \left\{ \int_0^t |t - qs| d_q s + \int_0^t \left| \frac{t^2}{1 + q} - qrt + \frac{q^3 r^2}{1 + q} \right| \|b(r)\| d_q r \right. \]
\[
+ \left. \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)| a(s) d_q s \right. \]
\[
+ \left. \frac{|\eta|}{(\eta - 1)^2} \int_0^T \left| (T + (1 - \eta)t)(T - qr) + (1 - \eta) \frac{q^3 r^2 - qT^2}{1 + q} \right| \|b(r)\| d_q r \right\} \varepsilon \|u\|.
\]
This yields $\|Tu - Lu\| \leq M \varepsilon \|u\|$, that is, \[
\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0.
\]
Therefore, all the conditions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, there exists a fixed point $u$ of the operator $T$. Thus, for any $\varepsilon > 0$, there exists a positive constant $M$ such that for any $|u| > M$, we have $|f(t, u) - a(t)| < \varepsilon |u|$ and $|g(t, u) - b(t)| < \varepsilon |u|$ for any $t \in I_q$. Therefore, all the conditions of Theorem 3.1 are satisfied.
which corresponds to a solution the problem (1.1). This completes the proof.

The Proof of Theorem 1.2

Proof. In view of (H2), we have

\[
\begin{align*}
\forall \varepsilon_1 > 0 \text{ there is } \delta_1 > 0 \text{ such that if } 0 < |u| < \delta_1 \text{ then } \left| \frac{f(t, u)}{|u|} - 0 \right| < \varepsilon_1, \text{ i.e. } |f(t, u)| < \varepsilon_1 |u|, \\
\forall \varepsilon_2 > 0 \text{ there is } N_1 > 0 \text{ such that if } |u| > N_1 \text{ then } \left| \frac{f(t, u)}{|u|} - 0 \right| < \varepsilon_2, \text{ i.e. } |f(t, u)| < \varepsilon_2 |u|.
\end{align*}
\]

Without loss of generality, let \( \delta_1 < N_1 \) and

\[ \varepsilon' = \max \left\{ \varepsilon_1, \varepsilon_2, \max \left\{ \frac{f(t, u)}{|u|} : \delta_1 \leq |u| \leq N_1, t \in I_q \right\} \right\}. \]

Then, for all \( t \in I_q \) and \( u \in \mathbb{R} \), we have \( |f(t, u)| < \varepsilon'|u| \).

Similarly, using (H3), we get

\[
\begin{align*}
\forall \varepsilon_3 > 0 \text{ there is } \delta_2 > 0 \text{ such that if } 0 < |u| < \delta_2 \text{ then } \left| \frac{g(t, u)}{|u|} - 0 \right| < \varepsilon_3, \text{ i.e. } |g(t, u)| < \varepsilon_3 |u|, \\
\forall \varepsilon_4 > 0 \text{ there is } N_2 > 0 \text{ such that if } |u| > N_2 \text{ then } \left| \frac{g(t, u)}{|u|} - 0 \right| < \varepsilon_4, \text{ i.e. } |g(t, u)| < \varepsilon_4 |u|.
\end{align*}
\]

Let \( \delta_2 < N_2 \) and

\[ \varepsilon'' = \max \left\{ \varepsilon_3, \varepsilon_4, \max \left\{ \frac{g(t, u)}{|u|} : \delta_2 \leq |u| \leq N_2, t \in I_q \right\} \right\}. \]

Thus, for all \( t \in I_q \) and \( u \in \mathbb{R} \), we have \( |g(t, u)| < \varepsilon''|u| \).

Let us pick \( \varepsilon = \max \{ \varepsilon', \varepsilon'' \} \). Then, for any \( t \in I_q \) and \( u \in \mathbb{R} \), we obtain \( |f(t, u)| < \varepsilon|u|, \ |g(t, u)| < \varepsilon|u| \)

and

\[
|(Tu)(t)| \leq \int_0^t |t - qs| |f(s, u(s))| \, dq \, s + \int_0^t \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} |g(r, u(r))| \, dq \, r
\]

\[
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)| |f(s, u(s))| \, dq \, s
\]

\[
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T \left[ |T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2}{1 + q} \right] |g(r, u(r))| \, dq \, r
\]

\[
\leq \int_0^t |t - qs| \varepsilon|u(s)| \, dq \, s + \int_0^t \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} \varepsilon|u(r)| \, dq \, r
\]

\[
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)| \varepsilon|u(s)| \, dq \, s
\]

\[
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T \left[ |T + (1 - \eta)t(T - qr) + (1 - \eta) \frac{q^3r^2}{1 + q} \right] |g(r, u(r))| \, dq \, r
\]

\[
\leq \left\{ \int_0^t |t - qs| \, dq \, s + \int_0^t \frac{t^2}{1 + q} - qrt + \frac{q^3r^2}{1 + q} \, dq \, r
\right\} \varepsilon \|u\|.
\]
This completes the proof.

which is a contradiction. That is,

\[ M \varepsilon \|u\|, \]

which implies that

\[ \|Tu\| \leq M \varepsilon \|u\|. \] (3.5)

Take \( B = \frac{1}{\varepsilon} (B > M) \). If \( u \) is a solution of the non-separated boundary value problem (1.1), then \( u \) is a fixed point of operator \( T \). Thus, \( \|u\| = \|Tu\| \). This, together with (3.5) and \( M < B \), yields

\[ \|u\| = \|Tu\| \leq M \varepsilon \|u\| < \|u\|, \]

which is a contradiction. That is, \( u \) is not a solution of the non-separated boundary value problem (1.1). This completes the proof. \( \square \)

4. Examples

**Example 4.1.** Consider the following non-separated boundary value problem given by

\[
\begin{cases}
D_\frac{1}{2} u(t) = 3t^3 + t^2 \sin u(t) + \frac{t^2}{7} |u(t)| + \int_0^t (2s^3 + \frac{s^5}{2u^2(s)}) + \frac{s^2}{4} |u(s)| d\frac{1}{2} s, & t \in [0, 1], \\
u(0) = -2u(1), & D_\frac{1}{2} u(0) = -2D_\frac{1}{2} u(1),
\end{cases}
\] (4.1)

Here, \( q = \frac{1}{2}, \eta = -2, T = 1, f(t, u) = 3t^3 + t^2 \sin u + \frac{t^2}{7} |u| \) and \( g(t, u) = 2t^3 + \frac{t^5}{u^2} + \frac{t^2}{4} |u| \).

Obviously,

\[
\lim_{|u| \to \infty} \frac{f(t, u)}{|u|} = \lim_{|u| \to \infty} \frac{3t^3 + t^2 \sin u + \frac{t^2}{7} |u|}{|u|} = \frac{t^2}{7},
\]

\[
\lim_{|u| \to \infty} \frac{g(t, u)}{|u|} = \lim_{|u| \to \infty} \frac{2t^3 + \frac{t^5}{u^2} + \frac{t^2}{4} |u|}{|u|} = \frac{t^2}{4}.
\]

For \( a(t) = \frac{t^2}{7}, b(t) = \frac{t^2}{4} \), we have

\[
\Lambda = \sup_{t \in I} \left\{ \int_0^t |t - qs| \frac{s^2}{7} d_q s + \int_0^t \frac{t^2}{1 + q} - qrt + \frac{q^3 r^2}{1 + q} \frac{r^2}{4} d_q r \right. \\
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)(t - qs)| \frac{s^2}{7} d_q s \\
+ \frac{|\eta|}{(\eta - 1)^2} \int_0^T |T + (1 - \eta)t| |T - qr| + (1 - \eta) \frac{q^3 r^2 - qT^2}{4} \frac{r^2}{1 + q} \frac{r^2}{4} d_q r \left. \right\} \\
\leq \sup_{t \in [0, 1]} \left\{ \frac{1}{7} \int_0^t |t - \frac{1}{2} s| s^2 d_\frac{1}{2} s + \frac{1}{4} \int_0^t \frac{2t^2}{3} - \frac{1}{2} r t + \frac{r^2}{12} r^2 d_\frac{1}{2} r \\
+ \frac{2}{63} \int_0^1 |1 + 3(t - \frac{1}{2} s)| s^2 d_\frac{1}{2} s + \frac{1}{18} \int_0^1 3(1 - \frac{r}{2}) t + \frac{r}{2} |T - \frac{1}{2} | r^2 d_\frac{1}{2} r \right\} \\
\leq \frac{2}{21} + \frac{1}{7} + \frac{2}{21} + \frac{8}{21} = \frac{15}{21}.
\]
Thus, all the conditions of Theorem 1.1 are satisfied. Hence, we conclude that the problem (4.1) has at least one solution.

**Example 4.2.** Consider the following non-separated boundary value problem

\[
\begin{aligned}
D_{\frac{3}{5}} u(t) &= t^3 \sin u(t)(1 - \cos u(t)) + \int_{0}^{t} (s^2 u^2(s)e^{-|u(s)|})D_{\frac{3}{5}} s, \quad t \in [0,1], \\
u(0) &= \frac{1}{5} u(1), \quad D_{\frac{3}{5}} u(0) = \frac{1}{5} D_{\frac{3}{5}} u(1).
\end{aligned}
\] (4.2)

Here, \( q = \frac{1}{3}, \eta = \frac{5}{1}, \ T = 1, \) \( f(t, u) = t^3(1 - \cos u) \sin u \) and \( g(t, u) = t^2 u^2 e^{-|u|} \).

Note that

\[
\begin{aligned}
\lim_{|u| \to 0^+} \frac{f(t, u)}{|u|} &= \lim_{|u| \to 0^+} \frac{t^3(1 - \cos u) \sin u}{|u|} = 0, \\
\lim_{|u| \to \infty} \frac{f(t, u)}{|u|} &= \lim_{|u| \to \infty} \frac{t^3(1 - \cos u) \sin u}{|u|} = 0, \\
\lim_{|u| \to 0^+} \frac{g(t, u)}{|u|} &= \lim_{|u| \to 0^+} \frac{t^2 |u|}{e^{|u|}} = 0, \\
\lim_{|u| \to \infty} \frac{g(t, u)}{|u|} &= \lim_{|u| \to \infty} \frac{t^2 |u|}{e^{|u|}} = 0.
\end{aligned}
\] (4.3)

Clearly all the conditions of Theorem 1.2 hold. Consequently, the non-separated boundary value problem (4.2) has no solution.

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**References**


