Several complementary inequalities to inequalities of Hermite-Hadamard type for $s$-convex functions

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Abstract

In this paper, we establish some new Hermite-Hadamard inequalities for $s$-convex functions via fractional integrals. Some Hermite-Hadamard type inequalities for products of two convex and $s$-convex functions via Riemann-Liouville integrals are also established. ©2016 All rights reserved.

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1. Introduction and preliminaries

If $f : I \to R$ is a convex function on the interval $I$, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(t) dt \leq \frac{f(a) + f(b)}{2}.$$  \hspace{1cm} (1.1)

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

**Definition 1.1** ([7]). $f : I \subset [0, \infty) \to [0, \infty)$ is said to be $s$-convex in the second sense, or that $f$ belongs to the class $K^2_s$, if the inequality

$$f(\alpha x + (1 - \alpha)y) \leq \alpha^s f(x) + (1 - \alpha)^s f(y)$$  \hspace{1cm} (1.2)

holds for all $x, y \in I$, $\alpha \in [0, 1]$ and for some fixed $s \in (0, 1)$.

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It can be easily seen that for \( s = 1 \), \( s \)-convexity reduces to ordinary convexity of functions defined on \([0, \infty)\).

In \cite{6}, Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for the \( s \)-convex functions.

**Theorem 1.2** (\cite{6}). Suppose that \( f : [0, \infty) \to [0, \infty) \) is an \( s \)-convex function in the second sense, where \( s \in (0, 1] \) and let \( a, b \in [0, \infty) \), \( a < b \). If \( f \in L[a, b] \), then the following inequality holds

\[
2^{s-1} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(t) dt \leq \frac{f(a) + f(b)}{s + 1}.
\] (1.3)

In \cite{8}, İşcan gave definition of harmonically convexity as follows:

**Definition 1.3** (\cite{8}). Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \to \mathbb{R} \) is said to be harmonically convex, if

\[
f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x)
\] (1.4)

for all \( x, y \in I \) and \( t \in [0, 1] \). If the inequality in (1.4) is reversed, then \( f \) is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

**Theorem 1.4** (\cite{8}). Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) then we have

\[
f\left(\frac{2ab}{a + b}\right) \leq \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}.
\] (1.5)

The definition of harmonically \( s \)-convex functions is proposed by İşcan in \cite{9}.

**Definition 1.5** (\cite{9}). Let \( I \subseteq \mathbb{R} \setminus \{0\} \) be a real interval. A function \( f : I \subseteq (0, \infty) \to \mathbb{R} \) is said to be harmonically \( s \)-convex, if

\[
f\left(\frac{xy}{tx + (1-t)y}\right) \leq t^s f(y) + (1-t)^s f(x)
\] (1.6)

for all \( x, y \in I \), \( t \in [0, 1] \) and for some fixed \( s \in (0, 1] \). If the inequality in (1.6) is reversed, then \( f \) is said to be harmonically \( s \)-concave.

The following Hermite-Hadamard inequality for harmonically \( s \)-convex functions holds.

**Theorem 1.6** (\cite{9}). Let \( f : I \subseteq (0, \infty) \to \mathbb{R} \) be a harmonically \( s \)-convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) then we have

\[
2^{s-1} f\left(\frac{2ab}{a + b}\right) \leq \frac{ab}{b - a} \int_{a}^{b} \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{s + 1}.
\] (1.7)

In \cite{3}, Chen and Wu discussed Fejér and Hermite-Hadamard type inequalities for Harmonically convex functions and presented the following inequality:

**Theorem 1.7** (\cite{3}). Let \( f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R} \) be a harmonically convex function and \( a, b \in I \) with \( a < b \). If \( f \in L(a, b) \) then we have

\[
f\left(\frac{2ab}{a + b}\right) \int_{a}^{b} \frac{p(x)}{x^2} dx \leq \int_{a}^{b} \frac{f(x)}{x^2} p(x) dx \leq \frac{f(a) + f(b)}{2} \int_{a}^{b} \frac{p(x)}{x^2} dx,
\] (1.8)

where \( p : [a, b] \to \mathbb{R} \) is nonnegative, integrable and satisfies

\[
p\left(\frac{ab}{x}\right) = p\left(\frac{ab}{a + b - x}\right).
\] (1.9)
Some refinements of the Hermite-Hadamard inequality for convex functions have been extensively investigated by a number of authors (e.g., [1], [2], [3], [4] and [5]). In [11], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

**Theorem 1.8.** Let $f$ and $g$ be real-valued, nonnegative and convex functions on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b),$$

$$2f(\frac{a+b}{2})g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Some Hermite-Hadamard type inequalities for products of two convex and s-convex functions are proposed by Kirmaci et al. in [10].

**Theorem 1.9 (10).** Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0, \infty)$, $a < b$, be functions such that $g, fg \in L[a, b]$. If $f$ is convex and nonnegative on $[a, b]$ and $g$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s+2} M(a, b) + \frac{1}{(s+1)(s+2)} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

**Theorem 1.10 (10).** Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0, \infty)$, $a < b$, be functions such that $f, g$ and $fg \in L[a, b]$. If $f$ is $s_1$-convex and $g$ is $s_2$-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{s_1 + s_2 + 1} M(a, b) + \beta(s_2 + 1, s_1 + 1) N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

**Theorem 1.11 (10).** Let $f, g : [a, b] \to \mathbb{R}$, $a, b \in [0, \infty)$, $a < b$, be functions such that $fg \in L[a, b]$. If $f$ is convex and nonnegative on $[a, b]$ and $g$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then

$$2^s f(\frac{a+b}{2})g(\frac{a+b}{2}) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{(s+1)(s+2)} M(a, b) + \frac{1}{s+2} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(a)g(b)$.

In [5], Chen and Wu discussed Hermite-Hadamard type inequalities for Harmonically s-convex functions and obtained the following result:

**Theorem 1.12 (5).** Let $f, g : [a, b] \to [0, \infty)$, $a, b \in (0, \infty)$, $a < b$, be functions such that $f, g, fg \in L[a, b]$. If $f$ is harmonically $s_1$-convex and $g$ is harmonically $s_2$-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then

$$\frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx \leq \frac{1}{1 + s_1 + s_2} M(a, b) + \frac{\Gamma(1 + s_1)\Gamma(s_2 + 1)}{\Gamma(s_1 + s_2 + 2)} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$, $N(a, b) = f(a)g(b) + f(b)g(a)$.

**Theorem 1.13 (5).** Let $f, g : [a, b] \to [0, \infty)$, $a, b \in (0, \infty)$, $a < b$, be functions such that $f, g, fg \in L[a, b]$. If $f$ is harmonically $s_1$-convex and $g$ is harmonically $s_2$-convex on $[a, b]$ for some fixed $s_1, s_2 \in (0, 1]$, then
Riemann-Liouville fractional integrals.

\[2^{s_1+s_2-1} f\left(\frac{2ab}{a+b}\right) g\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)g(x)}{x^2} dx + M(a, b) \frac{\Gamma(1 + s_1)\Gamma(s_2 + 1)}{\Gamma(s_1 + s_2 + 2)}
\]

\[+ \frac{1}{s_2 + s_1 + 1} N(a, b),\]

where \(M(a, b) = f(a)g(a) + f(b)g(b),\) \(N(a, b) = f(a)g(b) + f(b)g(a)\).

Sarikaya et al. [12] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

**Theorem 1.14** ([12]). Let \(f : [a, b] \to R\) be a positive function with \(a < b\) and \(f \in L[a, b]\). If \(f\) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \leq \frac{f(a) + f(b)}{2},\]

with \(\alpha > 0\).

We remark that the symbol \(J_{a+}^\alpha\) and \(J_{b-}^\alpha\) denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order \(\alpha \geq 0\) with \(a \geq 0\) which are defined by

\[J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha - 1} f(t) dt, \quad x > a,\]

and

\[J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha - 1} f(t) dt, \quad x < b,\]

respectively. Here, \(\Gamma(\alpha)\) is the Gamma function defined by \(\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha - 1} dt\).

Chen and Wu [3] investigated the Hermite-Hadamard type inequalities for products of two \(h\)-convex functions and established the following inequality:

**Theorem 1.15.** Let \(f \in SX(h_1, I), \ g \in SX(h_2, I), \ a, b \in I, \ a < b,\) be functions such that \(fg \in L[a, b]\), and \(h_1h_2 \in L[0, 1]\), then the following inequality for fractional integrals holds:

\[\frac{\Gamma(\alpha)}{(b - a)^\alpha} [J_{a+}^\alpha f(b)g(b) + J_{b-}^\alpha f(a)g(a)] \leq M(a, b) \int_0^1 t^{\alpha - 1} [h_1(t)h_2(t) + h_1(1 - t)h_2(1 - t)] dt
\]

\[+ N(a, b) \int_0^1 t^{\alpha - 1} [h_1(t)h_2(1 - t) + h_1(1 - t)h_2(t)] dt,\]

where \(M(a, b) = f(a)g(a) + f(b)g(b),\) \(N(a, b) = f(a)g(b) + f(b)g(a)\).

**Theorem 1.16.** Let \(f \in SX(h_1, I), \ g \in SX(h_2, I), \ a, b \in I, \ a < b,\) be functions such that \(fg \in L[a, b]\), and \(h_1h_2 \in L[0, 1]\), then the following inequality for fractional integrals holds:

\[\frac{1}{ah_1(\frac{1}{2})h_2(\frac{1}{2})} f\left(\frac{a + b}{2}\right) g\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha)}{(b - a)^\alpha} [J_{a+}^\alpha f(b)g(b) + J_{b-}^\alpha f(a)g(a)]
\]

\[+ M(a, b) \int_0^1 t^{\alpha - 1} [h_1(t)h_2(t) + h_1(1 - t)h_2(t)] dt
\]

\[+ N(a, b) \int_0^1 t^{\alpha - 1} [h_1(t)h_2(1 - t) + h_1(1 - t)h_2(t)] dt,
\]

where \(M(a, b) = f(a)g(a) + f(b)g(b),\) \(N(a, b) = f(a)g(b) + f(b)g(a)\).

In [13], Set et al. established the following Hermite-Hadamard inequalities for \(s\)-convex functions in the second sense via fractional integrals.

[43x652]
Theorem 1.17 (13). Let \( f : [a, b] \rightarrow R \) be a positive function with \( 0 \leq a < b \) and \( f \in L[a,b] \). If \( f \) is s-convex in the second sense on \([a, b]\) for some fixed \( s \in (0,1] \), then the following inequalities for fractional integrals hold:

\[
2^{s-1}f\left(\frac{a + b}{2}\right) \leq \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] \leq \left[ \frac{1}{\alpha + s} + \beta(\alpha, s + 1) \right] \frac{f(a) + f(b)}{2},
\]

where \( \beta \) is Euler Beta function.

In this paper, we obtain some new Hermite-Hadamard type inequalities for s-convex functions via Riemann-Liouville fractional integrals. Several Hermite-Hadamard type inequalities for products of two convex and s-convex functions are also established.

2. Main results

In order to prove our main theorems, we need the following fundamental integral identity including the second order derivatives of a given function via Riemann-Liouville integrals which was established by Wang et al. in (13).

**Lemma 2.1 (13).** Let \( f : [a, b] \rightarrow R \) be a twice differentiable mapping on \((a, b)\) with \( a < b \). If \( f'' \in L[a,b] \), then the following equality for fractional integrals hold:

\[
\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] - \frac{f(a) + f(b)}{2} = \frac{(b-a)^2}{2} \int_0^1 (1-t)^{\alpha+1} + t^{\alpha+1} - 1 \frac{f''(ta + (1-t)b)}{\alpha + 1} dt.
\]

**Theorem 2.2.** Let \( f : [a, b] \rightarrow R \) be a twice differentiable mapping on \((a, b)\) with \( a < b \) such that \( f'' \in L[a,b] \). If \( |f''| \) is s-convex on \([a, b]\), then the following inequality holds:

\[
\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] - \frac{f(a) + f(b)}{2} \leq \frac{(b-a)^2}{2} \left[ \frac{1}{\alpha + s + 2} - \beta(\alpha + 2, s + 1) \right] \frac{|f''(a)| + |f''(b)|}{\alpha + 1}.
\]

**Proof.** From Lemma 2.1, we get

\[
\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] - \frac{f(a) + f(b)}{2} \leq \frac{(b-a)^2}{2} \int_0^1 \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha + 1} \left| f''(ta + (1-t)b) \right| dt.
\]

Because \((1-t)^{\alpha+1} + t^{\alpha+1} \leq 1\) for any \( t \in [0,1] \) and \( |f''| \) is s-convex on \([a, b]\), we get

\[
\int_0^1 \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha + 1} \left| f''(ta + (1-t)b) \right| dt \leq \int_0^1 \frac{1 - (1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha + 1} \left( t^{\alpha} |f''(a)| + (1-t)^{\alpha}|f''(b)| \right) dt
\]

\[
= \left( \frac{1}{\alpha + s + 2} - \beta(\alpha + 2, s + 1) \right) \frac{|f''(a)| + |f''(b)|}{\alpha + 1}.
\]

Now by (2.2), (2.3), we can obtain the desired result.

**Theorem 2.3.** Let \( f : [a, b] \rightarrow R \) be a twice differentiable mapping on \((a, b)\) with \( a < b \) such that \( f'' \in L[a,b] \). If \( |f''|^q \) is s-convex on \([a, b]\) with \( q \geq 1 \), then the following inequality holds:

\[
\frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a) \right] - \frac{f(a) + f(b)}{2} \leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \frac{1}{\alpha + 2} \right)^{1-q} \times \left( \frac{1}{\alpha + s + 2} - \beta(\alpha + 2, s + 1) - \frac{1}{\alpha + s + 2} \right)^{\frac{1}{2}} \left( |f''(a)| + |f''(b)| \right)^{\frac{1}{2}}.
\]
Proof. From Lemma 2.1 and the power mean inequality for $q$, we have

\[
\left| \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] - \frac{f(a) + f(b)}{2} \right| \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left[ \int_0^1 \left( 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right) dt \right]^{\frac{1}{q}} \\
\times \left( \int_0^1 (1 - t)\alpha+1 - (1 - t)\alpha+1 \right| f''(ta + (1-t)b)^q dt \right)^\frac{1}{q} \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left[ \int_0^1 (1 - (1-t)^{\alpha+1} - t^{\alpha+1}) dt \right]^{\frac{1}{q}} \\
\times \left( \int_0^1 (1 - t)^{\alpha+1} - t^{\alpha+1} \right| f''(ta + (1-t)b)^q dt \right)^\frac{1}{q} \\
= \frac{(b-a)^2}{2(\alpha + 1)} \left( \frac{1}{\alpha + 2} \right) \left( \frac{1}{s+1} - \frac{1}{\alpha + s + 2} \right)^\frac{1}{q} \left( f''(a)^q + f''(b)^q \right)^\frac{1}{q},
\]

which completes the proof. \[\square\]

**Theorem 2.4.** Let $f : [a, b] \to R$ be a twice differentiable mapping on $(a, b)$ with $a < b$ such that $f'' \in L[a, b]$. If $|f''|^q$ is s-convex on $[a, b]$ with $q > 1$, then the following inequality holds:

\[
\left| \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] - \frac{f(a) + f(b)}{2} \right| \leq \frac{(b-a)^2}{2(\alpha + 1)} \left( 1 - \frac{2}{p(\alpha + 1) + 1} \right)^\frac{1}{q} \left( f''(a)^q + f''(b)^q \right)^\frac{1}{q},
\]

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and using the Hölder inequality, we have

\[
\left| \frac{\Gamma(\alpha + 1)}{2(b-a)^{\alpha}} \left[ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \right] - \frac{f(a) + f(b)}{2} \right| \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \int_0^1 \left( 1 - (1-t)^{\alpha+1} - t^{\alpha+1} \right) dt \right)^\frac{1}{q} \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^\frac{1}{q} \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \int_0^1 \left( 1 - t^{\alpha+1} - t^{\alpha+1} \right) dt \right)^\frac{1}{q} \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^\frac{1}{q} \\
\leq \frac{(b-a)^2}{2(\alpha + 1)} \left( \int_0^1 \left( 1 - t^{p(\alpha+1)} - t^{p(\alpha+1)} \right) dt \right)^\frac{1}{q} \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^\frac{1}{q} \\
= \frac{(b-a)^2}{2(\alpha + 1)} \left( \frac{2}{p(\alpha + 1) + 1} \right) \left( \int_0^1 |f''(ta + (1-t)b)|^q dt \right)^\frac{1}{q}.
\]

Because $|f''|^q$ is s-convex, we have

\[
\int_0^1 |f''(ta + (1-t)b)|^q dt \leq \frac{|f''(a)|^q + |f''(b)|^q}{s+1}.
\]

Here we use

\[
(1 - (1-t)^{\alpha+1} - t^{\alpha+1})^p \leq 1 - (1-t)^{p(\alpha+1)} - t^{p(\alpha+1)}
\]

for any $t \in [0, 1]$, which follows from

\[
(A - B)^p \leq A^p - B^p
\]

for any $A > B > 0$ and $p \geq 1$. From (2.4) and (2.5), we complete the proof. \[\square\]
We note that the Beta and Gamma functions are defined, respectively, as follows

\[ \Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \quad x > 0, \]

\[ \beta(x, y) = \int_0^1 (1 - t)^{y-1}t^{x-1}dt, \quad x > 0, \ y > 0, \]

and

\[ \beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}. \]

**Theorem 2.5.** Let \( f, g : [a, b] \to \mathbb{R} \), \( a, b \in [0, \infty) \), \( a < b \), be functions such that \( g, fg \in L[a, b] \). If \( f \) is convex and nonnegative and \( g \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1]\), then the following inequality for fractional integrals holds:

\[
\frac{\Gamma(\alpha)}{(b - a)\alpha} [J_a^{\alpha} f(b)g(b) + J_b^{\alpha} f(a)g(a)] \\
\leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) M(a, b) + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) N(a, b),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b), N(a, b) = f(a)g(b) + f(b)g(a) \).

**Proof.** Since \( f \) is convex and \( g \) is \( s \)-convex on \([a, b]\), then for \( t \in [0, 1] \) we get

\[ f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b), \quad (2.6) \]

and

\[ g(ta + (1 - t)b) \leq t^s g(a) + (1 - t)^s g(b). \quad (2.7) \]

From (2.6) and (2.7), we get

\[ f(ta + (1 - t)b)g(ta + (1 - t)b) \leq t^{s+1}f(a)g(a) + (1 - t)^{s+1}f(b)g(b) + t(1 - t)^s f(a)g(b) + (1 - t)t^s f(b)g(a). \]

Similarly, we have

\[ f((1 - t)a + tb)g((1 - t)a + tb) \leq (1 - t)^{s+1}f(a)g(a) + t^{s+1}f(b)g(b) + (1 - t)t^s f(a)g(b) + t(1 - t)^s f(b)g(a). \]

So

\[ f(ta + (1 - t)b)g(ta + (1 - t)b) + f((1 - t)a + tb)g((1 - t)a + tb) \]

\[ \leq (t^{s+1} + (1 - t)^{s+1})[f(a)g(a) + f(b)g(b)] \]

\[ + (t(1 - t)^s + (1 - t)t^s)[f(a)g(b) + f(b)g(a)]. \]

Multiplying both sides of above inequality by \( t^{\alpha - 1} \), then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\int_0^1 t^{\alpha-1} f(ta + (1 - t)b)g(ta + (1 - t)b)dt + \int_0^1 t^{\alpha-1} f((1 - t)a + tb)g((1 - t)a + tb)dt \\
= \int_b^a \left( \frac{b - u}{b - a} \right)^{\alpha-1} f(u)g(u) \frac{du}{a - b} + \int_a^b \left( \frac{v - a}{b - a} \right)^{\alpha-1} f(v)g(v) \frac{dv}{b - a} \\
= \frac{\Gamma(\alpha)}{(b - a)\alpha} [J_a^{\alpha} f(b)g(b) + J_b^{\alpha} f(a)g(a)] \\
\leq [f(a)g(a) + f(b)g(b)] \int_0^1 t^{\alpha-1}(t^{s+1} + (1 - t)^{s+1})dt
\]
Proof. Since \( \frac{1}{\alpha + s + 1} \) holds:

\[
M(a, b) \leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) N(a, b).
\]

So

\[
\frac{\Gamma(\alpha)}{(b - a)^{\alpha}} J_a^\alpha \left[ \frac{1}{\alpha + s + 1} (f(a) + f(b)) \right] \leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) \right) M(a, b) + \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) N(a, b),
\]

which completes the proof. \( \square \)

Remark 2.6. Taking \( \alpha = 1 \) in Theorem 2.5, we have

\[
\frac{1}{b - a} \int_a^b f(x)g(x)dx \leq \left( \frac{1}{s + 2} + \beta(1, s + 2) \right) \frac{M(a, b)}{2} + \left( \beta(2, s + 1) + \frac{1}{(1 + s)(s + 2)} \right) \frac{N(a, b)}{2} = \frac{1}{s + 2} M(a, b) + \frac{1}{(1 + s)(s + 2)} N(a, b)
\]

which is the result of \( [1.12] \).

Remark 2.7. Choosing \( f(x) = 1 \) for all \( x \in [a, b] \) in Theorem 2.5 gives

\[
\frac{\Gamma(\alpha)}{(b - a)^{\alpha}} \left[ J_a^\alpha g(b) + J_b^\alpha f(a) \right] \leq \left( \frac{1}{\alpha + s + 1} + \beta(\alpha, s + 2) + \beta(\alpha + 1, s + 1) \right) [g(a) + g(b)] + \left( \frac{1}{\alpha + s} \right) [g(a) + g(b)],
\]

which is the right hand side of \( [1.16] \).

Theorem 2.8. Let \( f, g : [a, b] \to \mathbb{R}, \ a, b \in [0, \infty), \ a < b, \) be functions such that \( f, g, f g \in L[a, b]. \) If \( f \) is \( s_1 \)-convex and \( g \) is \( s_2 \)-convex on \( [a, b] \) for some fixed \( s_1, s_2 \in (0, 1], \) then the following inequality for fractional integrals holds:

\[
\frac{\Gamma(\alpha)}{(b - a)^{\alpha}} \left[ J_a^\alpha f(b)g(b) + J_b^\alpha f(a)g(a) \right] \leq \left( \frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1) \right) M(a, b) + \left( \beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1) \right) N(a, b),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b), \ N(a, b) = f(a)g(b) + f(b)g(a). \)

Proof. Since \( f \) is \( s_1 \)-convex and \( g \) is \( s_2 \)-convex on \( [a, b], \) then for \( t \in [0, 1] \) we get

\[
f(ta + (1 - t)b) \leq t^{s_1} f(a) + (1 - t)^{s_1} f(b), \tag{2.8}
\]

and

\[
g(ta + (1 - t)b) \leq t^{s_2} g(a) + (1 - t)^{s_2} g(b). \tag{2.9}
\]
From \((2.8)\) and \((2.9)\), we get
\[
f(ta + (1-t)b)g(ta + (1-t)b) \leq t^{s_1 + s_2} f(a)g(a) + (1-t)^{s_1 + s_2} f(b)g(b) + t^{s_1} (1-t)^{s_2} f(a)g(b) + (1-t)^{s_1} t^{s_2} f(b)g(a).
\]
Similarly, we have
\[
f((1-t)a + tb)g((1-t)a + tb) \leq (1-t)^{s_1 + s_2} f(a)g(a) + t^{s_1 + s_2} f(b)g(b) + (1-t)^{s_1} t^{s_2} f(a)g(b) + t^{s_1} (1-t)^{s_2} f(b)g(a).
\]
So
\[
f(ta + (1-t)b)g(ta + (1-t)b) + f((1-t)a + tb)g((1-t)a + tb)
\leq (t^{s_1 + s_2} + (1-t)^{s_1 + s_2})[f(a)g(a) + f(b)g(b)]
+ (t^{s_1} (1-t)^{s_2} + (1-t)^{s_1} t^{s_2})[f(a)g(b) + f(b)g(a)].
\]
Multiplying both sides of above inequality by \(t^{a-1}\), then integrating the resulting inequality with respect to \(t\) over \([0, 1]\), we obtain
\[
\int_0^1 t^{a-1} f(ta + (1-t)b)g(ta + (1-t)b)dt + \int_0^1 t^{a-1} f((1-t)a + tb)g((1-t)a + tb)dt
\leq [f(a)g(a) + f(b)g(b)] \int_0^1 t^{a-1} (t^{s_1 + s_2} + (1-t)^{s_1 + s_2})dt
+ [f(a)g(b) + f(b)g(a)] \int_0^1 t^{a-1} (t^{s_1} (1-t)^{s_2} + (1-t)^{s_1} t^{s_2})dt
= \left( \frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1) \right)[f(a)g(a) + f(b)g(b)]
+ \left( \beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1) \right)[f(a)g(b) + f(b)g(a)]
= \left( \frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1) \right)M(a, b) + \left( \beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1) \right)N(a, b).
\]
So
\[
\frac{\Gamma(\alpha)}{(b-a)^{\alpha}} [J_{a^+}^\alpha f(b)g(b) + J_{b^-}^\alpha f(a)g(a)] \leq \left( \frac{1}{\alpha + s_1 + s_2} + \beta(\alpha, s_1 + s_2 + 1) \right)M(a, b)
+ \left( \beta(\alpha + s_1, s_2 + 1) + \beta(\alpha + s_2, s_1 + 1) \right)N(a, b),
\]
which completes the proof. \(\square\)

Remark 2.9. Putting \(\alpha = 1\) in Theorem \((2.8)\) leads to
\[
\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \left( \frac{1}{s_1 + s_2 + 1} + \beta(1, s_1 + s_2 + 1) \right) \frac{M(a, b)}{2}
+ \left( \beta(1 + s_1, s_2 + 1) + \beta(1 + s_2, s_1 + 1) \right) \frac{N(a, b)}{2}
= \frac{1}{s_1 + s_2 + 1} M(a, b) + \beta(s_2 + 1, s_1 + 1) N(a, b),
\]
which is the result of \((1.13)\).
Theorem 2.10. Let \( f, g : [a, b] \rightarrow \mathbb{R}, a, b \in [0, \infty), a < b, \) be functions such that \( fg \in L[a, b] \). If \( f \) is convex and nonnegative on \([a, b]\) and \( g \) is \( s \)-convex on \([a, b]\) for some fixed \( s \in (0, 1] \), then

\[
2^s f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J^\alpha_a f(b)g(b) + J^\alpha_b f(a)g(a) \right] \\
+ \frac{1}{2} M(a,b) \left( \beta(\alpha+1, s+1) + \frac{1}{(\alpha+s)(\alpha+s+1)} \right) \\
+ \frac{1}{2} N(a,b) \left( \beta(\alpha, s+2) + \frac{1}{\alpha+s+1} \right),
\]

where \( M(a,b) = f(a)g(a) + f(b)g(b) \), \( N(a,b) = f(a)g(b) + f(a)g(b) \).

Proof. We can write

\[
\frac{a+b}{2} = \frac{(1-t)a + tb}{2} + \frac{ta + (1-t)b}{2},
\]

so

\[
f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \\
= f \left( \frac{ta + (1-t)b + (1-t)a + tb}{2} \right) g \left( \frac{ta + (1-t)b + (1-t)a + tb}{2} \right) \\
\leq \frac{1}{2^{s+1}} \left[ f(ta + (1-t)b) + f((1-t)a + tb) \right] g(ta + (1-t)b) + g(1-t)a + tb \\
+ f(ta + (1-t)b)g((1-t)a + tb) + f((1-t)a + tb)g((1-t)a + tb) \\
\leq \frac{1}{2^{s+1}} \left[ f(ta + (1-t)b)g(ta + (1-t)b) + f((1-t)a + tb)g((1-t)a + tb) \\
+ \frac{1}{2^{s+1}} \left\{ \begin{array}{l}
rf(a) + (1-t)rf(b) \\
(1-t)^s g(a) + t^s g(b)
\end{array} \right\} \\
+ \frac{1}{2^{s+1}} \left\{ \begin{array}{l}
(t(1-t)^s + (1-t)t^s)M(a,b) + (1-t)^{s+1} + t^{s+1} N(a,b)
\end{array} \right\}.
\]

Multiplying both sides of above inequality by \( t^\alpha - 1 \), then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we get

\[
f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \int_0^1 t^\alpha - 1 \, dt \leq \frac{1}{2^{s+1}} \left[ \int_0^1 t^\alpha - 1 f(ta + (1-t)b)g(ta + (1-t)b)dt \\
+ \int_0^1 t^\alpha - 1 f((1-t)a + tb)g((1-t)a + tb)dt \\
+ \frac{1}{2^{s+1}} \left\{ M(a,b) \int_0^1 t^\alpha - 1 [t(1-t)^s + (1-t)t^s]dt \\
+ N(a,b) \int_0^1 t^\alpha - 1 [(1-t)^{s+1} + t^{s+1}]dt \right\}.
\]
That is

\[ \frac{1}{\alpha} f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \leq \frac{1}{2s+1} \left[ \frac{\Gamma(\alpha)}{(b - a)^\alpha} \left[ J_{a^+}^\alpha f(b) g(b) + J_{b^-}^\alpha f(a) g(a) \right] + \frac{1}{2s+1} M(a, b) \int_0^1 t^{\alpha-1} \left[ t(1 - t)^s + (1 - t)t^s \right] dt \right. \\
+ \frac{1}{2s+1} \left. N(a, b) \int_0^1 t^{\alpha-1} \left[ (1 - t)^{s+1} + t^{s+1} \right] dt \right]. \]

From

\[ \int_0^1 t^{\alpha-1} \left[ t(1 - t)^s + (1 - t)t^s \right] dt = \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \]

and

\[ \int_0^1 t^{\alpha-1} \left[ (1 - t)^{s+1} + t^{s+1} \right] dt = \beta(\alpha, s + 2) + \frac{1}{\alpha + s + 1}, \]

we get

\[ 2s f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_{a^+}^\alpha f(b) g(b) + J_{b^-}^\alpha f(a) g(a) \right] + \frac{1}{2} M(a, b) \left( \beta(\alpha + 1, s + 1) + \frac{1}{(\alpha + s)(\alpha + s + 1)} \right) \]

\[ + \frac{1}{2} N(a, b) \left( \beta(\alpha, s + 2) + \frac{1}{\alpha + s + 1} \right), \]

which completes the proof.

Remark 2.11. Setting \( \alpha = 1 \) in Theorem [2.10] then

\[ 2s f \left( \frac{a + b}{2} \right) g \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(1 + 1)}{2(b - a)} \left[ J_{a^+} f(b) g(b) + J_{b^-} f(a) g(a) \right] + \frac{1}{2} M(a, b) \left( \beta(2, s + 1) + \frac{1}{(s + 1)(s + 2)} \right) \]

\[ + \frac{1}{2} N(a, b) \left( \beta(1, s + 2) + \frac{1}{s + 2} \right) = \frac{1}{b - a} \int_a^b f(x) g(x) dx \]

\[ + \frac{1}{b - a} \int_a^b f(x) g(x) dx + \frac{1}{(s + 1)(s + 2)} M(a, b) + \frac{1}{s + 2} N(a, b), \]

which is the result of (1.14).

Remark 2.12. Choosing \( f(x) = 1 \) for all \( x \in [a, b] \) in Theorem [2.10], we have

\[ 2s g \left( \frac{a + b}{2} \right) \leq \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ J_{a^+} g(b) + J_{b^-} g(a) \right] + \left( \beta(\alpha, s + 1) + \frac{1}{\alpha + s} \right) \frac{g(a) + g(b)}{2}. \]

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