Rectangular b-metric space and contraction principles

R. George\textsuperscript{a,}\textsuperscript{*}, S. Radenović\textsuperscript{b}, K. P. Reshma\textsuperscript{c}, S. Shukla\textsuperscript{d}

\textsuperscript{a}Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India.
\textsuperscript{b}Faculty of Mechanical Engineering, University of Belgrade, Kraljice Marije 16, 11120 Beograd, Serbia.
\textsuperscript{c}Department of Mathematics, Government VYT PG Autonomous College, Durg, Chhattisgarh, India.
\textsuperscript{d}Department of Applied Mathematics, S.V.I.T.S. Indore (M.P.), India.

Abstract

The concept of rectangular b-metric space is introduced as a generalization of metric space, rectangular metric space and b-metric space. An analogue of Banach contraction principle and Kannan’s fixed point theorem is proved in this space. Our result generalizes many known results in fixed point theory.

Keywords: Fixed points, b-metric space, rectangular metric space, rectangular b-metric space.


1. Introduction and Preliminaries

Since the introduction of Banach contraction principle in 1922, because of its wide applications, the study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Banach contraction Principle in various generalized metric spaces. In the sequel Branciari \cite{9} introduced the concept of rectangular metric space (RMS) by replacing the sum on the right hand side of the triangular inequality in the definition of a metric space by a three-term expression and proved an analogue of the Banach Contraction Principle in such space. Since then many fixed point theorems for various contractions on rectangular metric space appeared (see \cite{1,3,4,10,15,16,17,18,19,22,23,25,26}).

On the other hand, in \cite{5} Bakhtin introduced b-metric space as a generalization of metric space and proved analogue of Banach contraction principle in b-metric space. Since then, several papers have dealt

\textsuperscript{*}Corresponding author

Email addresses: renygeorge02@yahoo.com (R. George), radens@beotel.net (S. Radenović), b4reshma@yahoo.com (K. P. Reshma), satishmathematics@yahoo.co.in (S. Shukla)

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with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see \[2\],\[6\],\[7\],\[8\],\[11\],\[12\],\[13\],\[14\],\[20\] and the references therein).

In this paper we have introduced the concept of rectangular b-metric space, which is not necessarily Hausdorff and which generalizes the concept of metric space, rectangular metric space and b-metric space. Note that spaces with non Hausdorff topology plays an important role in Tarskian approach to programming language semantics used in computer science (For some details see \[24\]). Analog of the Banach contraction principle as well as the Kannan type fixed point theorem in rectangular b-metric space are proved. Some examples are included which shows that our generalizations are genuine.

**Definition 1.1** (\[9\]). Let \( X \) be a nonempty set and the mapping \( d: X \times X \to [0, \infty) \) satisfies:

(bM1) \( d(x, y) = 0 \) if and only if \( x = y \) for all \( x, y \in X \);

(bM2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(bM3) there exist a real number \( s \geq 1 \) such that \( d(x, y) \leq s[d(x, z) + d(z, y)] \) for all \( x, y, z \in X \).

Then \( d \) is called a b-metric on \( X \) and \((X, d)\) is called a b-metric space (in short bMS) with coefficient \( s \).

**Definition 1.2** (\[9\]). Let \( X \) be a nonempty set and the mapping \( d: X \times X \to [0, \infty) \) satisfies:

(RM1) \( d(x, y) = 0 \) if and only if \( x = y \) for all \( x, y \in X \);

(RM2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(RM3) \( d(x, y) \leq d(x, u) + d(u, v) + d(v, y) \) for all \( x, y \in X \) and all distinct points \( u, v \in X \setminus \{x, y\} \).

Then \( d \) is called a rectangular metric on \( X \) and \((X, d)\) is called a rectangular metric space (in short RMS).

We define a rectangular b-metric space as follows:

**Definition 1.3.** Let \( X \) be a nonempty set and the mapping \( d: X \times X \to [0, \infty) \) satisfies:

(RbM1) \( d(x, y) = 0 \) if and only if \( x = y \);

(RbM2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(RbM3) there exists a real number \( s \geq 1 \) such that \( d(x, y) \leq s[d(x, u) + d(u, v) + d(v, y)] \) for all \( x, y \in X \) and all distinct points \( u, v \in X \setminus \{x, y\} \).

Then \( d \) is called a rectangular b-metric on \( X \) and \((X, d)\) is called a rectangular b-metric space (in short RbMS) with coefficient \( s \).

Note that every metric space is a rectangular metric space and every rectangular metric space is a rectangular b-metric space (with coefficient \( s = 1 \)). However the converse of the above implication is not necessarily true.

**Example 1.4.** Let \( X = \mathbb{N} \), define \( d: X \times X \to X \) by

\[
d(x, y) = \begin{cases} 
0, & \text{if } x = y; \\
4\alpha, & \text{if } x, y \in \{1, 2\} \text{ and } x \neq y; \\
\alpha, & \text{if } x \text{ or } y \not\in \{1, 2\} \text{ and } x \neq y,
\end{cases}
\]

where \( \alpha > 0 \) is a constant. Then \((X, d)\) is a rectangular b-metric space with coefficient \( s = \frac{4}{3} > 1 \), but \((X, d)\) is not a rectangular metric space, as \( d(1, 2) = 4\alpha > 3\alpha = d(1, 3) + d(3, 4) + d(4, 2) \).

**Example 1.5.** Let \( X = \mathbb{N} \), define \( d: X \times X \to X \) such that \( d(x, y) = d(y, x) \) for all \( x, y \in X \) and

\[
d(x, y) = \begin{cases} 
0, & \text{if } x = y; \\
10\alpha, & \text{if } x = 1, y = 2; \\
\alpha, & \text{if } x \in \{1, 2\} \text{ and } y \in \{3\}; \\
2\alpha, & \text{if } x \in \{1, 2, 3\} \text{ and } y \in \{4\}; \\
3\alpha, & \text{if } x \text{ or } y \not\in \{1, 2, 3, 4\} \text{ and } x \neq y,
\end{cases}
\]

where \( \alpha > 0 \) is a constant. Then \((X, d)\) is a rectangular b-metric space with coefficient \( s = 2 > 1 \), but \((X, d)\) is not a rectangular metric space, as \( d(1, 2) = 10\alpha > 5\alpha = d(1, 3) + d(3, 4) + d(4, 2) \).
Note that every b-metric space with coefficient $s$ is a $RbMS$ with coefficient $s^2$ but the converse is not necessarily true. (See Example 1.7 below).

For any $x \in X$ we define the open ball with center $x$ and radius $r > 0$ by

$$B_r(x) = \{ y \in X : d(x, y) < r \}$$

The open balls in $RbMS$ are not necessarily open (See Example 1.7 below). Let $U$ be the collection of all subsets $A$ of $X$ satisfying the condition that for each $x \in A$ there exist $r > 0$ such that $B_r(x) \subseteq A$. Then $U$ defines a topology for the $RbMS$ $(X, d)$, which is not necessarily Hausdorff (See Example 1.7 below).

We define convergence and Cauchy sequence in rectangular b-metric space and completeness of rectangular b-metric space as follows:

**Definition 1.6.** Let $(X, d)$ be a rectangular b-metric space, $\{x_n\}$ be a sequence in $X$ and $x \in X$. Then

(a) The sequence $\{x_n\}$ is said to be convergent in $(X, d)$ and converges to $x$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in $(X, d)$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0, p > 0$ or equivalently, if $\lim_{n \to \infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) $(X, d)$ is said to be a complete rectangular b-metric space if every Cauchy sequence in $X$ converges to some $x \in X$.

Note that, limit of a sequence in a $RbMS$ is not necessarily unique and also every convergent sequence in a $RbMS$ is not necessarily a Cauchy sequence. The following example illustrates this fact.

**Example 1.7.** Let $X = A \cup B$, where $A = \{ \frac{1}{n} : n \in \mathbb{N} \}$ and $B$ is the set of all positive integers. Define $d: X \times X \to [0, \infty)$ such that $d(x, y) = d(y, x)$ for all $x, y \in X$ and

$$d(x, y) = \begin{cases} 
0, & \text{if } x = y; \\
2\alpha, & \text{if } x, y \in A; \\
\frac{1}{2n}, & \text{if } x \in A \text{ and } y \in \{2, 3\}; \\
\alpha, & \text{otherwise,}
\end{cases}$$

where $\alpha > 0$ is a constant. Then $(X, d)$ is a rectangular b-metric space with coefficient $s = 2 > 1$. However we have the following:

1) $(X, d)$ is not a rectangular metric space, as $d(\frac{1}{2}, \frac{1}{3}) = 2\alpha > \frac{17}{12} = d(\frac{1}{2}, 4) + d(4, 3) + d(3, \frac{1}{3})$ and hence not a metric space.

2) There does not exist $s > 0$ satisfying $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, and so $(X, d)$ is not a b-metric space.

3) $B_2(\frac{1}{2}) = \{2, 3, \frac{1}{2}\}$ and there does not exist any open ball with center 2 and contained in $B_2(\frac{1}{2})$. So $B_2(\frac{1}{2})$ is not an open set.

4) The sequence $\{\frac{n}{n}\}$ converges to 2 and 3 in $RbMS$ and so limit is not unique. Also $d(\frac{1}{n}, \frac{1}{n+p}) = 2\alpha \not\to 0$ as $n \to \infty$, therefore $\{\frac{1}{n}\}$ is not a Cauchy sequence in $RbMS$.

5) There does not exist any $r_1, r_2 > 0$ such that $B_{r_1}(2) \cap B_{r_2}(3) = \phi$ and so $(X, d)$ is not Hausdorff.

2. Main results

Following theorem is the analogue of Banach contraction principle in rectangular b-metric space.

**Theorem 2.1.** Let $(X, d)$ be a complete rectangular b-metric space with coefficient $s > 1$ and $T: X \to X$ be a mapping satisfying:

$$d(Tx, Ty) \leq \lambda d(x, y)$$

for all $x, y \in X$, where $\lambda \in [0, \frac{1}{s}]$. Then $T$ has a unique fixed point.
Proof. Let \( x_0 \in X \) be arbitrary. Define the sequence \( \{x_n\} \) by \( x_{n+1} = Tx_n \) for all \( n \geq 0 \). We shall show that \( \{x_n\} \) is Cauchy sequence. If \( x_n = x_{n+1} \) then \( x_n \) is fixed point of \( T \). So, suppose that \( x_n \neq x_{n+1} \) for all \( n \geq 0 \). Setting \( d(x_n, x_{n+1}) = d_n \), it follows from (2.1) that

\[
 d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda d(x_{n-1}, x_n)
\]

Repeating this process we obtain

\[
 d_n \leq \lambda^n d_0.
\]  

(2.2)

Also, we can assume that \( x_0 \) is not a periodic point of \( T \). Indeed, if \( x_0 = x_n \) then using (2.2), for any \( n \geq 2 \), we have

\[
 d(x_0, Tx_0) = d(x_0, TTx_0).
\]

\[
 d(x_0, x_1) = d(x_0, x_{n+1})
\]

\[
 d_0 = d_n
\]

\[
 d_0 \leq \lambda^n d_0,
\]

a contradiction. Therefore, we must have \( d_0 = 0 \), i.e., \( x_0 = x_1 \), and so \( x_0 \) is a fixed point of \( T \). Thus we assume that \( x_n \neq x_m \) for all distinct \( n, m \in \mathbb{N} \). Again setting \( d(x_n, x_{n+2}) = d_n^* \) and using (2.1) for any \( n \in \mathbb{N} \), we obtain

\[
 d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \leq \lambda d(x_{n-1}, x_{n+1})
\]

Repeating this process we obtain

\[
 d(x_n, x_{n+2}) \leq \lambda^n d_0^*.
\]  

(2.3)

For the sequence \( \{x_n\} \) we consider \( d(x_n, x_{n+p}) \) in two cases.

If \( p \) is odd say \( 2m + 1 \) then using (2.2) we obtain

\[
 d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})]
\]

\[
 \leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m+1})]
\]

\[
 \leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3} + d_{n+4} + d_{n+5}] + \cdots + s^{n}d_{n+2m}
\]

\[
 \leq s[\lambda^n d_0 + \lambda^{n+1} d_0] + s^2[\lambda^{n+2} d_0 + \lambda^{n+3} d_0] + s^3[\lambda^{n+4} d_0 + \lambda^{n+5} d_0] + \cdots + s^n\lambda^{n+2m} d_0
\]

\[
 \leq s\lambda^n [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0 + s\lambda^{n+1} [1 + s\lambda^2 + s^2\lambda^4 + \cdots] d_0
\]

\[
 = \frac{1 + \lambda}{1 - s\lambda^2} \lambda^n d_0 \quad \text{(note that } s\lambda^2 < 1). \]

Therefore,

\[
 d(x_n, x_{n+2m+1}) \leq \frac{1 + \lambda}{1 - s\lambda^2} \lambda^n d_0.
\]  

(2.4)

If \( p \) is even say \( 2m \) then using (2.2) and (2.3) we obtain

\[
 d(x_n, x_{n+2m}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m})]
\]

\[
 \leq s[d_n + d_{n+1}] + s^2[d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, x_{n+2m})]
\]

\[
 \leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3} + d_{n+4} + d_{n+5}] + \cdots + s^{n-1}[d_{2m-4} + d_{2m-3}] + s^{n-1}d(x_{n+2m-2}, x_{n+2m})
\]
Using (2.6) and (2.7) it follows from above inequality that
\[ d(\lambda d_0 + \lambda^{n+1}d_0) + s^2[\lambda^{n+2}d_0 + \lambda^{n+3}d_0] + s^3[\lambda^{n+4}d_0 + \lambda^{n+5}d_0] + \cdots + s^{m-1}[\lambda^{2m-4}d_0 + \lambda^{2m-3}d_0] + s^{m-1}\lambda^{2m-2}d_0^* \]
\[ \leq s\lambda^n[1 + \lambda^2 + s^2\lambda^4 + \cdots]d_0 + s\lambda^{n+1}[1 + \lambda^2 + s^2\lambda^4 + \cdots]d_0 + s^{m-1}\lambda^{2m-2}d_0^*, \]
i.e.
\[ d(x_n, x_{n+2m}) \leq \frac{1 + \lambda}{1 - s\lambda^2}s\lambda^n d_0 + s^{m-1}\lambda^{2m-2}d_0^* \]
\[ < \frac{1 + \lambda}{1 - s\lambda^2}s\lambda^n d_0 + (s\lambda)^{2m}\lambda^{n-2}d_0^* \quad \text{(as } 1 < s) \]
\[ \leq \frac{1 + \lambda}{1 - s\lambda^2}s\lambda^n d_0 + \lambda^{n-2}d_0^* \quad \text{(as } \lambda \leq \frac{1}{s}). \]
Therefore
\[ d(x_n, x_{n+2m}) \leq \frac{1 + \lambda}{1 - s\lambda^2}s\lambda^n d_0 + \lambda^{n-2}d_0^*. \tag{2.5} \]

It follows from (2.4) and (2.5) that
\[ \lim_{n \to \infty} d(x_n, x_{n+p}) = 0 \quad \text{for all } p > 0. \tag{2.6} \]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). By completeness of \( (X, d) \) there exists \( u \in X \) such that
\[ \lim_{n \to \infty} x_n = u. \tag{2.7} \]

We shall show that \( u \) is a fixed point of \( T \). Again, for any \( n \in \mathbb{N} \) we have
\[ d(u, Tu) \leq s[d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu)] = s[d(u, x_n) + d_n + d(Tx_n, Tu)] \leq s[d(u, x_n) + d_n + \lambda d(x_n, u)]. \]

Using (2.6) and (2.7) it follows from above inequality that \( d(u, Tu) = 0 \), i.e., \( Tu = u \). Thus \( u \) is a fixed point of \( T \).

For uniqueness, let \( v \) be another fixed point of \( T \). Then it follows from (2.1) that \( d(u, v) = d(Tu, Tv) \leq \lambda d(u, v) < d(u, v) \), a contradiction. Therefore, we must have \( d(u, v) = 0 \), i.e., \( u = v \). Thus fixed point is unique. \( \square \)

**Example 2.2.** Let \( X = A \cup B \), where \( A = \{ \frac{1}{n} : n \in \{2, 3, 4, 5\} \} \) and \( B = [1, 2] \). Define \( d \colon X \times X \to [0, \infty) \) such that \( d(x, y) = d(y, x) \) for all \( x, y \in X \) and

\[
\begin{align*}
&d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{3}, \frac{1}{2}) = 0.03 \\
&d(\frac{1}{7}, \frac{1}{5}) = d(\frac{1}{5}, \frac{1}{7}) = 0.02 \\
&d(\frac{1}{7}, \frac{1}{7}) = d(\frac{1}{7}, \frac{1}{7}) = 0.6 \\
&d(x, y) = |x - y|^2 \quad \text{otherwise}
\end{align*}
\]

Then \( (X, d) \) is a rectangular b-metric space with coefficient \( s = 4 > 1 \). But \( (X, d) \) is neither a metric space nor a rectangular metric space. Let \( T \colon X \to X \) be defined as:

\[ T x = \begin{cases} \frac{1}{3} & \text{if } x \in A \\ \frac{1}{5} & \text{if } x \in B \end{cases} \]

Then \( T \) satisfies the condition of Theorem 2.1 and has a unique fixed point \( x = \frac{1}{7} \).
Theorem 2.4. Let $T : X \to X$ be a mapping satisfying:
\[ d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)] \]
for all $x, y \in X$, where $\lambda \in \left[0, \frac{1}{s+1}\right]$. Then $T$ has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. We shall show that $\{x_n\}$ is Cauchy sequence. If $x_n = x_{n+1}$ then $x_n$ is fixed point of $T$. So, suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. Setting $d(x_n, x_{n+1}) = d_n$ it follows from (2.8) that
\[ d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \]
\[ d(x_n, x_{n+1}) = \lambda[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \]
\[ d_n = \lambda[d_{n-1} + d_n] \]
\[ d_n \leq \frac{\lambda}{1-\lambda} d_{n-1} = \beta d_{n-1}, \]
where $\beta = \frac{\lambda}{\lambda + 1} < \frac{1}{s}$ (as, $\lambda < \frac{1}{s+1}$). Repeating this process we obtain
\[ d_n \leq \beta^n d_0. \]
(2.9)

Also, we can assume that $x_0$ is not a periodic point of $T$. Indeed, if $x_0 = x_n$ then using (2.9), for any $n \geq 2$, we have
\[ d(x_n, Tx_0) = d(x_n, Tx_n) \]
\[ d(x_0, x_1) = d(x_n, x_{n+1}) \]
\[ d_0 = d_n \]
\[ d_0 \leq \beta^n d_0, \]
a contradiction. Therefore, we must have $d_0 = 0$, i.e., $x_0 = x_1$, and so $x_0$ is a fixed point of $T$. Thus we assume that $x_n \neq x_m$ for all distinct $n, m \in \mathbb{N}$. Again using (2.8) and (2.9) for any $n \in \mathbb{N}$, we obtain
\[ d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \leq \lambda[d(x_{n-1}, Tx_{n-1}) + d(x_{n+1}, Tx_{n+1})] \]
\[ = \lambda[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})] = \lambda[d_{n-1} + d_{n+1}] \]
\[ \leq \lambda[\beta^n d_0 + \beta^{n+1} d_0] \]
\[ = \lambda[1 + \beta^2] d_0 \]
\[ = \gamma \beta^n d_0. \]
(2.10)

Therefore,
\[ d(x_n, x_{n+2}) \leq \gamma \beta^n d_0, \]
where $\gamma = \lambda[1 + \beta^2] > 0$.

For the sequence $\{x_n\}$ we consider $d(x_n, x_{n+p})$ in two cases.

If $p$ is odd say $2m + 1$ then using (2.9) we obtain
\[ d(x_n, x_{n+2m+1}) \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2m+1})] \]
\[ \leq s[d_n + d_{n+1}] + s^2[d_{n+2} + d_{n+3}] + d_{n+4}\]
\[ + s^3[d_{n+2} + d_{n+3}] + s^5[d_{n+4} + d_{n+5}] + \cdots + s^{2m}d_{n+2m} \]
(2.11)
We shall show that
\[
\]
\[\lambda < \frac{1}{\beta}.\]
\[\therefore \quad d(x_n, x_{n+2m+1}) \leq \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0. \quad (2.11)\]

If \( p \) is even say \( 2m \) then using (2.9) and (2.10) we obtain
\[
d(x_n, x_{n+2m}) \leq \begin{cases} \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + s^{m-1} \gamma \beta^{n+2m-3} d_0 \\ < \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + \gamma (s\beta)^{2m} \beta^{n-3} d_0 \quad \text{(as } 1 < s) \\ \leq \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \quad \text{as } \beta \leq \frac{1}{s}. \end{cases}
\]

Therefore
\[
d(x_n, x_{n+2m}) \leq \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0. \quad (2.12)
\]

It follows from (2.11) and (2.12) that
\[
\lim_{n \to \infty} d(x_n, x_{n+p}) = 0 \quad \text{for all } p > 0. \quad (2.13)
\]

Thus \( \{x_n\} \) is a Cauchy sequence in \( X \). By completeness of \( (X, d) \) there exists \( u \in X \) such that
\[
\lim_{n \to \infty} x_n = u. \quad (2.14)
\]

We shall show that \( u \) is a fixed point of \( T \). Again, for any \( n \in \mathbb{N} \) we have
\[
d(u, Tu) \leq \begin{cases} \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + \gamma \beta^{n-3} d_0 \\ \leq \frac{1 + \beta}{1 - s\beta^2} s\beta^n d_0 + \lambda d(x_n, x_{n+1}) \quad (\lambda < \frac{1}{\beta}). \end{cases}
\]

Using (2.13) and (2.14) and the fact that \( \lambda < \frac{1}{\beta^2} \), it follows from above inequality that \( d(u, Tu) = 0 \), i.e., \( Tu = u \). Thus \( u \) is a fixed point of \( T \).
For uniqueness, let \( v \) be another fixed point of \( T \). Then it follows from (2.8) that 
\[
d(u, v) = d(Tu, Tv) \leq \lambda [d(u, Tu) + d(v, Tv)] = \lambda [d(u, u) + d(v, v)] = 0.
\]
Therefore, we have \( d(u, v) = 0 \), i.e., \( u = v \). Thus fixed point is unique.

**Remark 2.5.** On the basis of discussion contained in this paper, we have the following:
1) The open ball defined in b-metric space, RMS and RbMS are not necessarily open set.
2) The collection of open balls in RbMS, RMS and b-metric space do not necessarily form a basis for a topology.
3) RbMS, RMS and b-metric space are not necessarily Hausdorff.

**Open Problems**:

1) In Theorem 2.1, can we extend the range of \( \lambda \) to the case \( \frac{1}{2} < \lambda < 1 \).

2) Prove analogue of Chatterjee contraction, Reich contraction, Ciric contraction and Hardy-Rogers contraction in RbMS.

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**References**


