Mazur-Ulam theorem for probabilistic 2-normed spaces

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Abstract

In this paper we prove the Mazur-Ulam theorem for probabilistic 2-normed spaces. Our study is a natural continuation of that of Cobzas [S. Cobzas, Aequationes Math., 77 (2009) 197–205]. ©2015 All rights reserved.

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1. Introduction

A mapping \( T \) from a metric space \( X \) into a metric space \( Y \) is called an isometry map if \( T \) satisfies \( d_Y(T(x), T(y)) = d_X(x, y) \) for all \( x, y \in X \), where \( d_X(\cdot, \cdot) \) and \( d_Y(\cdot, \cdot) \) denote the metrics in the spaces \( X \) and \( Y \), respectively. The map \( T \) is called affine if \( T \) is linear up to translation.

Mazur and Ulam [11], proved that every isometry \( T \) from a real normed space \( X \) onto another real normed space \( Y \) is affine, while Baker [6] proved that an isometry map from a real normed linear space \( X \) into a strictly convex real normed linear space \( Y \) is affine.

For related works on this subject, we refer the reader to Aleksandrov [1], Cobzas [6], Chu et al. [7, 8, 9], and Rassias et al. [13, 17, 18].

Probabilistic metric spaces are spaces on which there is a distance function taking as values distribution functions, the distance between two points \( a \) and \( b \) is a distribution function in the sense of probability theory \( \nu(a, b) \), whose values \( \nu(p, q)(x) \) can be interpreted as the probability that the distance between \( a \) and \( b \) is less than \( x \). The notion of probabilistic metric space was introduced by Menger [12]. The idea of Menger’s was to use distribution functions instead of nonnegative real numbers as values of the metric.

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Probabilistic normed spaces were introduced by Šerstnev in 1963 [19]. New definitions of probabilistic
normed spaces were studied by Alsina et al. [2, 3, 4]. It is remarkable that the probabilistic generalization
of metric spaces appears to be well adapted for the investigation of quantum particle physics, particularly
in connections with both string and \( \varepsilon^{\infty} \) theory, which where given and studied by El Naschie [14, 15].

The notion of the probabilistic \( n \)-normed space was introduced by A. Poumoslemi and M. Salimi [16],
while the notion of probabilistic 2-normed space was introduced by I. Golet [10]. In 2009, S. Cobzas studied
the Mazur-Ulam theorem for probabilistic normed spaces [6].

In this paper, we study the Mazur-Ulam theorem for probabilistic 2-normed spaces.

2. Basic Concepts

Denote by \( \triangle \) the set of distribution functions, meaning, nondecreasing, left continuous functions \( \nu: \mathbb{R} \to [0,1] \), with \( \nu(-\infty) = 0 \) and \( \nu(\infty) = 1 \). Let \( D \) be the subclass of \( \triangle \) formed by all functions \( \nu \in \triangle \) such that

\[
\lim_{x \to -\infty} \nu(x) = 0 \quad \text{and} \quad \lim_{x \to \infty} \nu(x) = 1.
\]

The set of distance functions are

\[
\triangle^+ = \{ \nu \in \triangle : \nu(0) = 0 \} \quad \text{and} \quad D^+ = D \cap \triangle^+.
\]

It follows that for \( \nu \in D^+ \), we have \( \nu(x) = 0 \) for all \( x \leq 0 \). Two important distance functions are

\[
\varepsilon_0(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 1 \end{cases}
\]

and

\[
\varepsilon_\infty(x) = \begin{cases} 0, & x < \infty; \\ 1, & x = \infty \end{cases}
\]

A triangle function \( T \) is a binary operation on \( \triangle^+ \) that is commutative and associative, nondecreasing in
each place and has \( \varepsilon_0 \) as identity, that is \( T(\nu, \varepsilon_0) = \nu \). A \( t \)-norm is a continuous binary operation on \([0,1]\),
that is commutative, associative, nondecreasing in each variable and has 1 as identity. The triangle function \( \tau_T \) associated to a \( t \)-norm \( T \) is defined by

\[
\tau_T(F,G)(x) = \sup\{T(F(s),G(t)) : s + t = x\}.
\]

In this paper we are interested in the definition of probabilistic \( n \)-normed spaces, specially in the case
of \( n = 2 \).

**Definition 2.1** ([10]). Let \( X \) be a real linear space with \( \text{dim } X \geq n \), let \( T \) be a triangle function, and let \( \nu \)
be a mapping from \( X \) into \( D^+ \). If the following conditions are satisfied:

1. \( \nu(x_1, \ldots, x_n) = \varepsilon_0 \) if \( x_1, \ldots, x_n \) are linearly dependent,
2. \( \nu(x_1, \ldots, x_n) \neq \varepsilon_0 \) if \( x_1, \ldots, x_n \) are linearly independent,
3. \( \nu(x_1, \ldots, x_n) = \nu(x_{j1}, \ldots, x_{jn}) \) for any permutation \( (j1, j2, \ldots, jn) \) of \((1,2,\ldots,n)\)
4. \( \nu(\beta x_1, \ldots, x_n) = \nu(x_1, \ldots, x_n)\left(\frac{\beta}{\varepsilon_0}\right) \) for every \( s > 0 \), and \( \beta \neq 0 \),
5. \( \nu(x_1, \ldots, x_{n-1}, x_n + y) \geq T(\nu(x_1, \ldots, x_{n-1}, x_n), \nu(x_1, \ldots, x_{n-1}, y)) \)

for \( y, x_1, \ldots, x_n \in X \), then \( \nu \) is called a probabilistic 2-norm on \( X \) and the triple \((X, \nu, T)\) is called a
probabilistic 2-normed space.

**Definition 2.2.** Let \( X \) be a real linear space and \( x, y, z \) mutually disjoint elements of \( X \). Then \( x, y \) and \( z \)
are said to be 2-collinear if

\[
y - z = t(x - z),
\]

for some real number \( t \).
3. Main Results

We start our work by giving the definition of probabilistic 2-normed space.

**Definition 3.1** ([10]). Let $X$ be a real linear space with dim $X \geq 2$, let $T$ be a triangle function, and let $\nu$ be a mapping from $X$ into $D^+$. If the following conditions are satisfied:

1. $\nu(x_1, x_2) = \varepsilon_0$ if $x_1$ and $x_2$ are linearly dependent,
2. $\nu(x_1, x_2) \neq \varepsilon_0$ if $x_1$ and $x_2$ are linearly independent,
3. $\nu(x_1, x_2) = \nu(x_2, x_1)$,
4. $\nu(\beta x_1, x_2) = \nu(x_1, x_2) \left(\frac{s}{|s|}\right)$, for every $s > 0$, and $\beta \neq 0$,
5. $\nu(x_1 + x_2, y) \geq T(\nu(x_1, y), \nu(x_2, y))$

for $y, x_1, x_2 \in X$, then $\nu$ is called a probabilistic 2-norm on $X$ and the triple $(X, \nu, T)$ is called a probabilistic 2-normed space.

From now on, unless otherwise stated, we let $(X, \nu, T)$ and $(Y, \nu, T)$ be probabilistic 2-normed spaces.

In our work, we assume that: If $x$ and $y$ are linearly independent elements in $X$ or in $Y$, then $\nu(x, y)$ is strictly increasing.

The following lemma due to A. Pourmoslemi and M. Salimi [10] is crucial in proving our next result.

**Lemma 3.2** ([10]). For $x_1, x_2 \in X$ and $\alpha \in \mathbb{R}$, we have

$$\nu(x_1, \alpha x_1 + x_2) = \nu(x_1, x_2).$$

The following result is essential for proving our main result.

**Lemma 3.3.** Let $x_1$ and $x_2$ be any two distinct elements in $X$, and let

$$u = \frac{x_1 + x_2}{2}.$$

Then $u$ is the unique element in $X$ satisfying for all $s > 0$ the following equalities:

$$\nu(x_1 - u, x_1 - c)(s) = \nu(x_2 - c, x_2 - u)(s) = \nu(x_1 - c, x_2 - c)(2s)$$

for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and $x_1, x_2, u$ are 2-collinear.

**Proof.** Choose $c \in X$ with $x_1 - c, x_2 - c$ being linearly independent. For $s > 0$ we have

$$\nu(x_1 - u, x_1 - c)(s) = \nu \left( x_1 - \frac{x_1 + x_2}{2}, x_1 - c \right)(s)$$

$$= \nu \left( \frac{x_1 - x_2}{2}, x_1 - c \right)(s)$$

$$= \nu(x_1 - x_2, x_1 - c)(2s)$$

$$= \nu(x_1 - c + c - x_2, x_1 - c)(2s)$$

$$= \nu(x_2 - c, x_1 - c)(2s)$$

$$= \nu(x_1 - c, x_2 - c)(2s).$$

Similarly, we can show that

$$\nu(x_2 - c, x_2 - u)(s) = \nu(x_1 - c, x_2 - c)(2s).$$

To prove the uniqueness, assume that $w$ is an element in $X$ satisfying for all $s > 0$ the equalities:

$$\nu(x_1 - w, x_1 - c)(s) = \nu(x_2 - c, x_2 - w)(s) = \nu(x_1 - c, x_2 - c)(2s)$$

(3.1)
for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and $x_1, x_2, w$ are 2-collinear. Since $x_1, x_2, w$ are 2-collinear, there is a scalar $t$ such that $w = (1-t)x_1 + tx_2$. Hence for $s > 0$, we have
\[
\nu(x_1 - w, x_1 - c)(s) = \nu(x_1 - (1-t)x_1 - tx_2), x_1 - c)(s) \\
= \nu(tx_1 - tx_2 - ct + ct), x_1 - c)(s) \\
= \nu(t(x_1 - c) - t(x_2 - c), x_1 - c)(s) \\
= \nu(-t(x_2 - c), x_1 - c)(s) \\
= \nu(x_1 - c, x_2 - c) \left( \frac{s}{|t|} \right)
\]
and
\[
\nu(x_2 - c, x_2 - w)(s) = \nu(x_2 - c, (1-t)x_2 - (1-t)x_1)(s) \\
= \nu(x_2 - c, (1-t)x_2 - (1-t)x_1 - (1-t)c + (1-t)c)(s) \\
= \nu(x_2 - c, (1-t)(x_2 - c) - (1-t)(x_1 - c))(s) \\
= \nu(x_2 - c, -(1-t)(x_1 - c))(s) \\
= \nu(x_2 - c, x_1 - c) \left( \frac{s}{|1-t|} \right) \\
= \nu(x_1 - c, x_2 - c) \left( \frac{s}{|1-t|} \right).
\]
Since $w$ satisfies Equation (3.1) and $\nu(x_1 - c, x_2 - c)$ is strictly increasing, we get that
\[
2 = \frac{1}{|1-t|} = \frac{1}{|t|}.
\]
So we conclude that $t = \frac{1}{2}$, and hence $w = u$.

Using similar arguments as in the proof of Lemma 3.3, we can prove the following result.

**Lemma 3.4.** Let $x_1$ and $x_2$ be any two distinct elements in $X$. Let
\[
u = \frac{x_1 + x_2}{2}.
\]
Then $u$ is the unique element in $X$ satisfying for all $s > 0$ the following equalities:
\[
u(u - x_1, x_2 - c)(s) = \nu(x_1 - c, u - x_2)(s) = \nu(x_1 - c, x_2 - c)(2s),
\]
for $c \in X$ where $x_1 - c$ and $x_2 - c$ are linearly independent and $x_1, x_2, u$ are 2-collinear.

To achieve our main result we introduce the following definition.

**Definition 3.5.** Let $(X, \nu, T)$ and $(Y, \nu, T)$ be probabilistic 2-normed spaces. We call the map $f: X \to Y$ probabilistic 2-isometry if
\[
\nu(f(x) - f(c), f(y) - f(c))(s) = \nu(x - c, y - c)(s)
\]
holds, for all $x, y, c \in X$ and all $s > 0$.

**Lemma 3.6.** Let $f: X \to Y$ be probabilistic 2-isometry from probabilistic 2-normed space $(X, \nu, T)$ into probabilistic 2-normed space $(Y, \nu, T)$. Define the map $g$ from $(X, \nu, T)$ into $(Y, \nu, T)$ by the rule $g(x) = f(x) - f(0)$. Then $f$ is probabilistic 2-isometry iff $g$ is probabilistic 2-isometry.
Proof. Assume that $f$ is probabilistic 2-isometry, then for $a, b, c \in X$ and $s > 0$ we have
\[
\nu(g(a) - g(c), g(b) - g(c))(s) = \nu(f(a) - f(0) - (f(c) - f(0)), f(b) - f(0) - (f(c) - f(0)))(s)
\]
\[
= \nu(f(a) - f(c), f(b) - f(c))(s)
\]
\[
= \nu(a - c, b - c)(s).
\]
So $g$ is probabilistic 2-isometry.
Similarly we may show that if $g$ is probabilistic 2-isometry, then $f$ is probabilistic 2-isometry.

We have furnished all necessary background to introduce and prove our main result.

**Theorem 3.7.** Let $f : X \to Y$ be probabilistic 2-isometry from probabilistic 2-normed space $(X, \nu, T)$ into probabilistic 2-normed space $(Y, \nu, T)$ with the property that if $a, b,$ and $c$ are 2-collinear in $X$, then $f(a), f(b),$ and $f(c)$ are 2-collinear in $Y$. Then $f$ is affine.

Proof. By Lemma 3.6, we may assume that $f(0) = 0$. So it suffices to prove that $f$ is linear. Let $x$ and $y$ be two distinct elements in $X$, and $u = \frac{x + y}{2}$. Since $\dim X \geq 2$, there is $c \in X$ such that $x - c$ and $y - c$ are linearly dependent. Now for $s > 0$, we have
\[
\nu(f(x) - f(u), f(x) - f(c))(s) = \nu(x - u, x - c)(s)
\]
\[
= \nu\left(x - \frac{x + y}{2}, x - c\right)
\]
\[
= \nu\left(-\frac{y}{2}, x - c\right)(s)
\]
\[
= \nu(x - c - (y - c), x - c)(2s)
\]
\[
= \nu(y - c, x - c)(2s)
\]
\[
= \nu(f(y) - f(c), f(x) - f(c))(2s)
\]
\[
= \nu(f(x) - f(c), f(y) - f(c))(2s).
\]

Similarly, we may prove that
\[
\nu(f(y) - f(u), f(y) - f(c))(s) = \nu(f(x) - f(c), f(y) - f(c))(2s).
\]

By Lemma 3.3, we conclude that
\[
f(u) = f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}. \quad (3.2)
\]

For $x \in X$, $s > 0$, and $\alpha \in \mathbb{R}^+ \setminus \{0\}$, we have
\[
\varepsilon_0(s) = \nu(\alpha x, x)(s) = \nu(\alpha x - 0, x - 0)(s) = \nu(f(\alpha x) - f(0), f(x) - f(0))(s) = \nu(f(\alpha x), f(x))(s).
\]

So $f(\alpha x)$ and $f(x)$ are linearly independent. Hence there is $k \in \mathbb{R}$ such that $f(\alpha x) = kf(x)$. Choose $y \in X$ such that $x$ and $y$ are linearly independent. Then for $s > 0$, we have
\[
\nu(x, y)\left(\frac{s}{\alpha}\right) = \nu(\alpha x, y)(s) = \nu(f(\alpha x), f(y))(s)
\]
\[
= \nu(kf(x), f(y))(s) = \nu(f(x), f(y))\left(\frac{s}{|k|}\right)
\]
\[
= \nu(x, y)\left(\frac{s}{|k|}\right),
\]
and hence $\alpha = |k|$. 


Claim: $k = \alpha$.
If $k = -\alpha$, then for $s > 0$, we have
\[
\nu(x, y) \left( \frac{s}{\alpha - 1} \right) = \nu((\alpha - 1)x, y)(s) = \nu(\alpha x - x, y - x)(s)
\]
\[
= \nu(f(\alpha x) - f(x), f(y) - f(x))(s) = \nu(-\alpha f(x) - f(x), f(y) - f(x))(s)
\]
\[
= \nu(f(x), f(y) - f(x)) \left( \frac{s}{\alpha + 1} \right) = \nu(f(x), f(y)) \left( \frac{s}{\alpha + 1} \right)
\]
\[
= \nu(x, y) \left( \frac{s}{\alpha + 1} \right).
\]
So $|\alpha - 1| = \alpha + 1$, and hence $\alpha = 0$ which is a contradiction. Therefore $k = \alpha$ and so that $f(\alpha x) = \alpha f(x)$, for all $\alpha \in \mathbb{R}^+ \setminus \{0\}$.
Similarly, we can show that $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}^- \setminus \{0\}$. Given two distinct elements $x$ and $y$ in $X$. Since
\[
f(x + y) = f \left( \frac{2x + 2y}{2} \right)
\]
by Equation (3.2), we get that
\[
f(x + y) = \frac{f(2x) + f(2y)}{2} = \frac{2f(x) + 2f(y)}{2} = f(x) + f(y).
\]
If $x = y$, then $f(x + y) = f(2x) = 2 f(x) = f(x) + f(x) = f(x) + f(y)$. So $f$ is affine.

References