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Bifurcation techniques for a class of boundary value problems of fractional impulsive differential equations

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Abstract

This paper investigates the existence of positive solutions for a class of boundary value problems (BVP) of fractional impulsive differential equations and presents a number of new results. First, by constructing a novel transformation, the considered impulsive system is convert into a continuous system. Second, using a specially constructed cone, the Krein-Rutman theorem, topological degree theory, and bifurcation techniques, some sufficient conditions are obtained for the existence of positive solutions to the considered BVP. Finally, an example is worked out to demonstrate the main result. ©2015 All rights reserved.

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1. Introduction

During the last decades, fractional calculus and fractional differential equations have been studied extensively. As a matter of fact, fractional derivatives provide a more excellent tool for the description of memory and hereditary properties of various materials and processes than integer derivatives. Engineers and scientists have developed new models that involve fractional differential equations. These models have been applied successfully, e.g., in mechanics (theory of viscoelasticity and viscoplasticity), (bio-)chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), etc. For details, see [5, 12, 13, 19, 20] and references therein. As an important issue for the theory of fractional differential equations, the existence of solutions to kinds of boundary value problems (BVPs) has attracted many scholars attention, and lots of excellent

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results have been obtained [1, 2, 3, 10, 11, 23] by means of fixed point theorems, upper and lower solutions technique, and so forth.

For example, in [3], Bai and Lv investigated the following nonlinear fractional differential equation Dirichlet-type BVP

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1); \\ u(0) = u(1) = 0 \end{cases} \tag{1.1}$$

where $1 < \alpha \le 2$, $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville differentiation. The corresponding Green function is deduced. By using fixed-point theorems on cone, the existence and multiplicity of positive solutions for BVP (1.1) were obtained.

In [11], Jiang and Yuan further investigated BVP (1.1). Comparing with [3], they deduced some new properties of the Green function, which extended the results of integer-order Dirichlet boundary value problems. Based on these new properties and Krasnoselskii fixed point theorem, the existence and multiplicity of positive solutions for BVP (1.1) were considered.

In this paper, we consider the following boundary value problem of fractional impulsive differential equation

$$\begin{cases} D^{\alpha}_{0^+}u(t) + f(t, u(t)) = 0, \quad t \in (0, 1),\ t \neq t_k; \\ u(t_k^+) = u(t_k^-) - c_k u(t_k^-); \\ u(0) = u(1) = 0 \end{cases} \tag{1.2}$$

where $k = 1, 2, \cdots, m$, $1 < \alpha \le 2$, $D^{\alpha}_{0^+}$ is is the standard Riemann-Liouville differentiation, $c_k \in (0,\ \frac{1}{2})$, and $f : [0,\ 1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a given continuous function satisfying some assumptions that will be specified later.

Impulsive differential equations has received a lot of attention recently since such equations arise in many mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics (see for example [4, 6, 8, 14, 27] and references therein). Also there are some papers concerned with boundary or initial value problems of fractional differential equations with impulse (see, for instance, [2, 7, 24, 25] and references therein). It is remarkable that the method used in these references are fixed point theorems. As we know, the bifurcation technique is widely used in solving boundary value problems (see, for instance, [15, 16, 17, 26] and references therein). Unfortunately, there is almost no paper except [16, 18] studying impulsive differential equations using bifurcation ideas. To the best of our knowledge, there is no paper studying such fractional impulsive differential equations using bifurcation techniques. The purpose of present paper is to fill this gap. The main features of this paper are as follows. First, by constructing a novel transformation, the considered impulsive system is convert into a continuous system. Second, using a specially constructed cone, the Krein-Rutman theorem, topological degree theory, and bifurcation techniques, some sufficient conditions are obtained for the existence of positive solutions to the considered BVP, which is firstly studied in this paper by using bifurcation techniques.

The paper is organized as follows. Section 2 contains background materials and preliminaries. In Section 3, some transformations are introduced to convert BVP (1.2) to solvable form. In Section 4, by using bifurcation techniques, and topological degree theory, bifurcation results from infinity and trivial solution are established. Then the main results of present paper are given and proved. Finally, in Section 5, an example is worked out to demonstrate the main result.

2. Background materials and preliminaries

We first recall some well known results about Riemann-Liouville derivative. For details, please refer to [20] and references therein.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds,$$

provided the right side is pointwise defined on $(0, \infty)$.

Lemma 2.2. *Let $\alpha > 0$, then the differential equation*

$$D_{0+}^{\alpha} u(t) = 0.$$

has solutions $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Notice that $D_{0+}^{\alpha} I^{\alpha} h(t) = h(t)$ for all $h \in C(0,1) \cap L(0,1)$. From Lemma 2.2, we deduce the following result.

Lemma 2.3. *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a derivative of order n that belongs to $C(0,1) \cap L^1[0,1]$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}.$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$, where n is the smallest integer greater than or equal to α.

Next, we list the following theorems on topological degree and bifurcation results of completely operators.

Lemma 2.4. (K. Schmitt, R. C. Thompson [22]). *Let V be a real reflexive Banach space, $G : \mathbb{R} \times V \to V$ be completely continuous such that $G(\lambda, 0) = 0$ for each $\lambda \in \mathbb{R}$. Let $a, b \in \mathbb{R} (a < b)$ be such that $u = 0$ is an isolated solution of the equation*

$$u - G(\lambda, u) = 0, \ \ u \in V, \tag{2.1}$$

for $\lambda = a$ and $\lambda = b$, where $(a, 0)$, $(b, 0)$ are not bifurcation points of (2.1). Furthermore, assume that

$$deg(I - G(a, \cdot), B_r(0), 0) \neq deg(I - G(b, \cdot), B_r(0), 0),$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$\mathscr{T} = \overline{\{(\lambda, u) : \ (\lambda, u) \text{ is a solution of } (2.1) \text{ with } u \neq 0\}} \cup ([a, b] \times 0).$$

Then there exists a connected component $\mathcal{C}$ of $\mathscr{T}$ containing $[a, b] \times 0$ in $\mathbb{R} \times V$, and either
(i) $\mathcal{C}$ is unbounded in $\mathbb{R} \times V$, or
(ii) $\mathcal{C} \cap [(\mathbb{R} \setminus [a, b]) \times 0] \neq \emptyset$.

Lemma 2.5. (K. Schmitt [21]). *Let V be a real reflexive Banach space, $G : \mathbb{R} \times V \to V$ be completely continuous. Let $a, b \in \mathbb{R} (a < b)$ be such that the solutions of (2.1) are, a priori, bounded in V for $\lambda = a$ and $\lambda = b$, i.e., there exists an $R > 0$ such that*

$$G(a, u) \neq u \neq G(b, u)$$

for all u with $\|u\| \geq R$. Furthermore, assume that

$$deg(I - G(a, \cdot), B_R(0), 0) \neq deg(I - G(b, \cdot), B_R(0), 0),$$

for $R > 0$ large. Then there exists a closed connected set $\mathcal{C}$ of solutions of (2.1) that is unbounded in $[a, b] \times V$, and either
(i) $\mathcal{C}$ is unbounded in λ direction, or
(ii) there exists an interval $[c, d]$ such that $(a, b) \cap (c, d) = \emptyset$ and $\mathcal{C}$ bifurcates from infinity in $[c, d] \times V$.

Lemma 2.6. (D. Guo [9]). *Let Ω be a bounded open set of real Banach space E, $A : \bar{\Omega} \to E$ be completely continuous. If there exists $y_0 \in E$, $y_0 \neq \theta$ such that*

$$x \in \partial\Omega, \ \ \tau \geq 0 \Rightarrow x - Ax \neq \tau y_0.$$

Then

$$deg(I - A, \Omega, \theta) = 0.$$

3. Conversion of BVP(1.2)

The basic space used in this paper is $E = C[0,1]$. Obviously, E is a Banach space with norm $\|u\| = \max_{t\in J}|u(t)|$ $(\forall u \in E)$, where $J = [0,\ 1]$.

Let

$PC(J) = \{u:\ u$ is a map from J into $\mathbb{R}$ such that $u(t)$ is continuous at $t \neq t_k$, and right continuous at $t = t_k$, and the left limit $u(t_k^-)$ exists for $k = 1, 2, \ldots, m\}$.

Evidently, $PC(J)$ is also a Banach space with the norm $\|x\|_{pc} = \sup\limits_{t\in J}|x(t)|$. It is noted that $PC(J)$ is not the same as usual we used.

To convert BVP(1.2) into a continuous system, we first define an operator $A:\ PC(J) \to PC(J)$ by

$$Au(t) = \int_0^1 G(t,s)f(s,u(s)))\mathrm{d}s + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}u(t_k),\ \forall u \in PC(J). \tag{3.1}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1;\\ \\ [t(1-s)]^{\alpha-1}, & 0 \le t \le s \le 1.\end{cases} \tag{3.2}$$

Lemma 3.1. *If $u \in PC(J)$ is a fixed point of the operator A defined by (3.1), then u is a solution of $BVP(1.2)$.*

Proof. Suppose $u \in PC(J)$ is a fixed point of the operator A. Then by (3.1), we know

$$u(t) = \int_0^1 G(t,s)f(s,u(s)))\mathrm{d}s + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}u(t_k), \quad t \in J.$$

From Lemma 2.2- 2.3 and a process similar to the proof of Lemma 2.3 in [3], it follows that $u(t)$ satisfies

$$\begin{cases}D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad t \in (0,1),\ t \neq t_k;\\ \\ u(0) = u(1) = 0.\end{cases}$$

Now it remains to show $u(t_k^+) = u(t_k^-) - c_k u(t_k^-)$. In fact, by (3.1) and $u \in PC(J)$, we know $u(t_k^+) - u(t_k^-) = \dfrac{-c_k}{1-c_k}u(t_k) = \dfrac{-c_k}{1-c_k}u(t_k^+)$, which means $u(t_k^+) = (1-c_k)u(t_k^-)$. □

For $u \in PC(J)$, let

$$v(t) = u(t) - t^{\alpha-1}\sum_{t<t_k<1}t_k^{1-\alpha}\frac{c_k}{1-c_k}u(t_k), t \in (0,1).$$

Then

$$u(t) = v(t) + t^{\alpha-1}\sum_{t<t_k<1}t_k^{1-\alpha}\frac{c_k}{1-c_k}u(t_k),$$

that is,

$$u(t) = \begin{cases}v(t) + t^{\alpha-1}\sum\limits_{k=1}^{m}\frac{c_k}{1-c_k}t_k^{1-\alpha}u(t_k), & t \in (0,t_1),\\ v(t) + t^{\alpha-1}\sum\limits_{k=2}^{m}\frac{c_k}{1-c_k}t_k^{1-\alpha}u(t_k), & t \in [t_1,t_2),\\ \ldots & \\ v(t) + \frac{c_m}{1-c_m}t_m^{1-\alpha}u(t_m)t^{\alpha-1}, & t \in [t_{m-1},t_m),\\ v(t), & t \in [t_m,1).\end{cases} \tag{3.3}$$

From this one can define an operator T on Banach space E by

$$Tv(t) = \begin{cases} v(t) + t^{\alpha-1} \sum_{k=1}^{m} \frac{c_k}{1-c_k} t_k^{1-\alpha} Tv(t_k), & t \in (0, t_1), \\ v(t) + t^{\alpha-1} \sum_{k=2}^{m} \frac{c_k}{1-c_k} t_k^{1-\alpha} Tv(t_k), & t \in [t_1, t_2), \\ \dots \\ v(t) + \frac{c_m}{1-c_m} t_m^{1-\alpha} Tv(t_m) t^{\alpha-1}, & t \in [t_{m-1}, t_m), \\ v(t), & t \in [t_m, 1), \end{cases} \tag{3.4}$$

for each $v \in E$. Therefore,

$$Tv(t) = v(t) + t^{\alpha-1} \sum_{t<t_k<1} \frac{c_k}{1-c_k} t_k^{1-\alpha} Tv(t_k), \quad \forall v \in E.$$

Then from (3.1), the operator equation $u(t) = Au(t)$ is converted into

$$v(t) = \int_0^1 G(t,s) f(s, Tv(s)) \mathrm{d}s. \tag{3.5}$$

Therefore, $u = Tv$ satisfies $u(t) = Au(t)$ if v is a solution of (3.5), which means that the BVP(1.2) is transformed into the continuous one (3.5).

We also need the following lemmas and some further transformations.

Lemma 3.2. ([11]) *The function $G(t,s)$ defined by (3.2) has the following properties:*
(i) $G(t,s) > 0, \quad \forall\ t, s \in (0,1)$.
(ii) The function $G^*(t,s) =:\ t^{2-\alpha} G(t,s)$ has the following properties:

$$\frac{\alpha-1}{\Gamma(\alpha)} t(1-t)s(1-s)^{\alpha-1} \le G^*(t,s) \le \frac{1}{\Gamma(\alpha)} s(1-s)^{\alpha-1} \quad \text{for} \quad t, s \in [0,1].$$

Let

$$Q := \{y \in E : y(t) \ge (\alpha-1)t(1-t)y(s) \ge 0,\ \forall s, t \in (0,1)\}. \tag{3.6}$$

It is easy to see Q is a cone of E. Moreover, from (3.6), we have for all $y \in Q$,

$$y(t) \ge (\alpha-1)t(1-t)\|y\|, \quad \forall t \in [0,1]. \tag{3.7}$$

For convenience, let

$$\bar{y}(t) =:\ t^{\alpha-2} y(t) \text{ and } (Ly)(t) =:\ T\bar{y}(t), \quad \forall y \in C(J),\ t \in (0,1), \tag{3.8}$$

where T defined by (3.4).

Lemma 3.3. *The operator T defined by (3.4) is a linear operator from E to $PC(J)$. In addition,*

$$\|Tv\|_{pc} \le 2^m \|v\|, \quad \forall v \in Q$$

Proof. First, it is not difficult to see T is a linear operator from E to $PC(J)$. Next for each $v \in Q$, from (3.4), we know $Tv(t) = v(t)$ for $t \in [t_m,\ 1)$. From $c_k \in (0,\ \frac{1}{2})$, it follows that

$$0 < \frac{c_k}{1-c_k} < 1, \quad k = 1,\ 2, \cdots,\ m.$$

Then $Tv(t_m) = v(t_m)$ and

$$Tv(t) \le v(t) + \frac{c_m}{1-c_m} v(t_m) \le v(t) + v(t_m), \quad t \in [t_{m-1},\ t_m).$$

So $Tv(t_{m-1}) \le v(t_{m-1}) + \frac{c_m}{1-c_m} v(t_m) \le v(t_{m-1}) + v(t_m)$ and

$$Tv(t) \le v(t) + c_{m-1} Tv(t_{m-1}) + \frac{c_m}{1-c_m} v(t_m) \le v(t) + v(t_{m-1}) + 2v(t_m), \quad t \in [t_{m-2},\ t_{m-1}).$$

By induction, one can obtain that

$$Tv(t_{i+1}) \le v(t_{i+1}) + v(t_{i+2}) + 2v(t_{i+3}) + \cdots + 2^{m-i-2} v(t_m)$$

and

$$Tv(t) \le v(t) + v(t_{i+1}) + 2v(t_{i+2}) + 4v(t_{i+3}) + \cdots + 2^{m-i-1} v(t_m), \quad t \in [t_i,\ t_{i+1}).$$

Consequently,

$$Tv(t) \le v(t) + v(t_1) + 2v(t_2) + 4v(t_3) + \cdots + 2^{m-1} v(t_m), \quad t \in (0,\ t_1).$$

On the other hand, by induction it is easy to see that $Tv(t) > 0$ for $t \in (0,\ 1)$.

From above, we know that

$$\|Tv\|_{pc} \le 2^m \|v\|, \quad \forall v \in E,$$

which implies that T is a bounded operator from Q to $PC(J)$. □

Lemma 3.4. *The operator L defined by* (3.8) *is a linear operator from E to $C(0,\ 1)$. In addition,*

$$(Ly)(t) \le t^{\alpha-2} Ty(t), \quad \forall y \in Q,\ t \in (0,1).$$

Proof. Firstly, it is easy to see L is a linear operator from E to $C(0,\ 1)$ since T is linear.

Secondly, for each $y \in Q$, from (3.4) and (3.8) we know

$$(Ly)(t) = T\bar{y}(t) = \bar{y}(t) = t^{\alpha-2} y(t) = t^{\alpha-2} Ty(t), \quad t \in [t_m,\ 1).$$

Then

$$T\bar{y}(t_m) = \bar{y}(t_m) = t_m^{\alpha-2} y(t_m) = t_m^{\alpha-2} Ty(t_m).$$

This together with $t_m^{\alpha-2} \le t^{\alpha-2}$ for $t \in [t_{m-1},\ t_m)$ and $0 < \dfrac{c_k}{1-c_k} < 1$ $(k = 1,\ 2, \cdots,\ m)$ guarantees that

$$\begin{aligned}
(Ly)(t) &= T\bar{y}(t) = \bar{y}(t) + \frac{c_m}{1-c_m} t_m^{1-\alpha} T\bar{y}(t_m) t^{\alpha-1} \\
&= t^{\alpha-2} y(t) + \frac{c_m}{1-c_m} t_m^{1-\alpha} t_m^{\alpha-2} Ty(t_m) t^{\alpha-1} \\
&\le t^{\alpha-2} Ty(t), \quad t \in [t_{m-1},\ t_m).
\end{aligned}$$

By induction, one can obtain that $(Ly)(t) \le t^{\alpha-2} Ty(t)$ for $t \in (0,1)$. □

Now let's list the following assumption satisfied throughout the paper.

(H1) There exist functions $a_0,\ a^0,\ b_\infty,\ a^0 \in C(J,\ \mathbb{R}^+)$ with $a_0(t),\ a^0(t),\ b_\infty(t),\ a^0(t) \not\equiv 0$ in any subinterval of $[0,1]$ such that

$$f(t,u) \in \Big[a_0(t)\big(u - \xi_1(t,u)\big),\ a^0(t)\big(u + \xi_2(t,u)\big)\Big] \cap \Big[b_\infty(t)\big(u - \zeta_1(t,u)\big),\ b^\infty(t)\big(u + \zeta_2(t,u)\big)\Big],$$

for $\forall(t,u) \in J \times \mathbb{R}^+$, where $\xi_i, \eta_i \in C(J \times \mathbb{R}^+)$ with $\xi_i(t, t^{\alpha-2}u) = o(u)$ as $u \to 0$ uniformly with respect to $t \in (0,1)$, $(i = 1,2)$, and $\zeta_i(t, t^{\alpha-2}u) = o(u)$ as $u \to +\infty$ uniformly with respect to $t \in (0,1)$, $(i = 1,2)$.

For the sake of using bifurcation technique to investigate BVP (1.2), we study the following fractional boundary value problem with parameters:

$$\leq \begin{cases} D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad t \in (0,1),\ t \neq t_k; \\ u(t_k^+) = u(t_k^-) - c_k u(t_k^-); \\ u(0) = u(1) = 0 \end{cases} \tag{3.9}$$

A function (λ, u) is said to be a solution of BVP(3.9) if (λ, u) satisfies (3.9). In addition, if $\lambda > 0,\ u(t) > 0$ for $t \in (0,1)$, then (λ, u) is said to be a positive solution of BVP(3.9).

Define

$$\bar{f}(t,u) = \begin{cases} f(t,u), & (t,u) \in J \times \mathbb{R}^+; \\ f(t,0), & (t,u) \in J \times (-\infty, 0). \end{cases}$$

Then $\bar{f}(t,u) \geq 0$ on $J \times \mathbb{R}$. Now we define an operator Φ_λ on $C[0,1]$ as follows:

$$\Phi_\lambda y(t) =: \lambda \int_0^1 G^*(t,s) \bar{f}(s, (Ly)(s)) \mathrm{d}s, \quad \forall\ y \in C[0,1], \tag{3.10}$$

where $\lambda \in \mathbb{R}$ is a parameter. By assumption (H1) and using a similar process of the proof of Lemma 4.1 in [11], we know $\Phi_\lambda : C[0,1] \to Q$ is completely continuous.

From (3.10), if $y \in C[0,1]$ is the the fixed point of operator Φ_λ, that is,

$$y(t) = \lambda \int_0^1 G^*(t,s) \bar{f}(s, (Ly)(s)) \mathrm{d}s, \tag{3.11}$$

then $v(t) = t^{\alpha-2} y(t)$ is the solution of

$$v(t) = \lambda \int_0^1 G(t,s) \bar{f}(s, Tv(s)) \mathrm{d}s. \tag{3.12}$$

Let

$$\Sigma =: \overline{\{(\lambda, y) \in \mathbb{R}^+ \times C[0,1] :\ y = \Phi_\lambda y,\ y \neq \theta\}} \tag{3.13}$$

where θ is the zero element of $C[0,1]$. From Lemma 3.2, the definitions of $\bar{f}$, and the cone Q, it is easy to see $\Sigma \subset Q$. Moreover, we have the following conclusion.

Lemma 3.5. *For $\lambda > 0$, if y is a nontrivial fixed point of operator Φ_λ, then $\bar{y}$ is a positive solution of the operator equation* (3.12). *Furthermore, $(\lambda, T\bar{y})$ is a positive solution of BVP* (3.9)*, where $\bar{y}(t) = t^{\alpha-2} y(t)$ for $t \in (0,1)$.*

For $a \in C(J,\ \mathbb{R}^+)$ with $a(t) \not\equiv 0$ in any subinterval of J, define the linear operator $\mathscr{L}_a : C(J) \to C(J)$ by

$$\mathscr{L}_a y(t) = \int_0^1 G^*(t,s) a(s) (Ly)(s) ds,\ \forall y \in C(J), \tag{3.14}$$

where $G^*(t,s)$ is defined by Lemma 3.2 and the operator L is given by (3.8).

From (3.4), Lemma 3.2, and the well known Krein-Rutman Theorem, one can obtain the following Lemma.

Lemma 3.6. *The operator $\mathscr{L}_a : C(J) \to C(J)$ defined by* (3.14) *is completely continuous and has a unique characteristic value $\lambda_1(a)$, which is positive, real, simple and the corresponding eigenfunction $\phi(t)$ is of one sign in $(0,1)$, i.e., we have $\phi(t) = \lambda_1(a) \mathscr{L}_a \phi(t)$.*

Notice that the operator $\mathscr{L}_a$ can be regarded as $\mathscr{L}_a: L^2[0,1] \to L^2[0,1]$. This together with Lemma 3.6 guarantees that $\lambda_1(a)$ is also the characteristic value of $\mathscr{L}_a^*$, where $\mathscr{L}_a^*$ is the conjugate operator of $\mathscr{L}_a$. Let φ^* denote the nonnegative eigenfunction of $\mathscr{L}_a^*$ corresponding to $\lambda_1(a)$. Then we have

$$\varphi^*(t) = \lambda_1(a)\mathscr{L}_a^*\varphi^*(t), \quad \forall t \in J.$$

4. Main Results

The main results of present paper are the following two theorems.

Theorem 4.1. *Suppose either*
(i) $\lambda_1(a_0) < 1 < \lambda_1(b^\infty)$ *or*
(ii) $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$.
Then $BVP(1.2)$ has at least one positive solution.

Theorem 4.2. *Suppose*
(H2) There exist $R > 0$ and $h \in L[0,\ 1]$ such that

$$f(t,u) \le h(t)u \quad \forall (t,\ u) \in [0,\ 1) \times (0, 2^m t^{\alpha-2}R]$$

and

$$\frac{2^m}{\Gamma(\alpha)} \int_0^1 [s(1-s)]^{\alpha-1} h(s) ds < 1.$$

In addition, suppose

$$\lambda_1(a_0) < 1 \quad \text{and} \quad \lambda_1(b_\infty) < 1.$$

Then $BVP(1.2)$ has at least two positive solutions.

To prove Theorem 4.1 and Theorem 4.2, we first prove the following lemmas.

Lemma 4.3. *For any $[c,\ d] \subset \mathbb{R}^+$ satisfying $[\lambda_1(a^0),\ \lambda_1(a_0)] \cap [c,\ d] = \emptyset$, there exists $\delta_1 > 0$ such that*

$$y \ne \Phi_\lambda y,\ \forall \lambda \in [c,\ d],\ \forall y \in E \text{ with } 0 < \|y\| \le \delta_1.$$

Proof. If this is false, then there exist $\{(\mu_n, y_n)\} \subset [c,\ d] \times C[0,1]$ with $\|y_n\| \to 0 (n \to +\infty)$ such that $y_n = \Phi_{\mu_n} y_n$. No loss of generality, assume $\mu_n \to \mu \in [c,\ d]$. Notice that $y_n \in Q$. By Lemma 3.5 and (3.6), we have $y_n(t) > 0$ in $(0,1)$. Set $w_n = \dfrac{y_n}{\|y_n\|}$. Then $w_n = \dfrac{\Phi_{\mu_n} y_n}{\|y_n\|}$. From the definition of $\bar{f}(t,u)$, condition (H1), and Ascoli-Arzela theorem, it is easy to see that $\{w_n\}$ is relatively compact in $C[0,1]$. Taking a subsequence and relabeling if necessary, suppose $w_n \to w$ in $C[0,\ 1]$. Then $\|w\| = 1$ and $w \in Q$.

On the other hand, from (H1) we know

$$f(t,u) \in \Big[a_0(t)\big(u - \xi_1(t,u)\big),\ a^0(t)\big(u + \xi_2(t,u)\big)\Big],\ \forall (t,u) \in J \times \mathbb{R}^+. \tag{4.1}$$

Therefore, by virtue of (3.10), we know

$$\begin{aligned} w_n(t) &= \frac{\mu_n}{\|y_n\|} \int_0^1 G^*(t,s)\bar{f}(s,(Ly_n)(s))\mathrm{d}s \\ &\le \mu_n \int_0^1 G^*(t,s)a^0(s)\Big((Lw_n)(s) + \frac{\xi_2(s,(Ly_n)(s))}{\|y_n\|}\Big)ds \end{aligned} \tag{4.2}$$

and

$$w_n(t) \geq \mu_n \int_0^1 G^*(t,s)a_0(s)\Big((Lw_n)(s) - \frac{\xi_1(s,(Ly_n)(s))}{\|y_n\|}\Big)ds. \tag{4.3}$$

Let ψ^* and ψ_* be the positive eigenfunctions of $\mathscr{L}^*_{a^0}$, $\mathscr{L}^*_{a_0}$ corresponding to $\lambda_1(a^0)$ and $\lambda_1(a_0)$, respectively. Then from (4.2) it follows that

$$\langle w_n,\ \psi^*\rangle \leq \mu_n\langle \mathscr{L}_{a^0}w_n,\ \psi^*\rangle + \mu_n \int_0^1 \psi^*(t)\int_0^1 G^*(t,s)a_0(s)\frac{\xi_2(s,(Ly_n)(s))}{\|y_n\|}dsdt. \tag{4.4}$$

Notice that

$$\frac{\xi_2(s,(Ly_n)(s))}{\|y_n\|} = \frac{\xi_2(s,s^{\alpha-2}s^{2-\alpha}(Ly_n)(s))}{s^{2-\alpha}(Ly_n)(s))}\cdot\frac{s^{2-\alpha}(Ly_n)(s))}{\|y_n\|},\ s\in(0,\ 1).$$

Using condition (H1), Lemma 3.3, and Lemma 3.4, we have $\dfrac{\xi_2(s,(Ly_n)(s))}{\|y_n\|} \to 0$ as $n\to+\infty$ uniformly with respect to $s\in(0,1)$.

Letting $n\to+\infty$ in (4.4)

$$\langle w,\ \psi^*\rangle \leq \mu\langle \mathscr{L}_{a^0}w,\ \psi^*\rangle = \mu\langle w,\ \mathscr{L}^*_{a^0}\psi^*\rangle = \mu\langle w,\ \frac{\psi^*}{\lambda_1(a^0)}\rangle,$$

which implies $\mu\geq\lambda_1(a^0)$. Similarly, one can deduce from (4.3) that $\mu\leq\lambda_1(a_0)$.

To sum up, $\lambda_1(a^0)\leq\mu\leq\lambda_1(a_0)$, which contradicts with $\mu\in[c,\ d]$. The conclusion of this Lemma follows. □

Lemma 4.4. *For $\mu\in(0,\ \lambda_1(a^0))$, there exists $\delta_1>0$ such that*

$$deg(I-\Phi_\mu,\ B_\delta,\ 0) = 1,\quad \forall\delta\in(0,\delta_1],$$

where $B_\delta = \{u\in E:\ \|u\|<\delta\}$.

Proof. Notice that $[0,\ \mu]\cap[\lambda_1(a^0),\ \lambda_1(a_0)]=\emptyset$. By virtue of Lemma 4.3, there exists $\delta_1>0$ such that

$$y\neq\Phi_\lambda y,\ \forall\lambda\in[0,\ \mu],\ \forall y\in C[0,1] \text{ with } 0<\|y\|\leq\delta_1,$$

which means

$$y\neq\tau\Phi_\mu y,\ \forall\tau\in[0,\ 1],\ \forall y\in C[0,1] \text{ with } 0<\|y\|\leq\delta_1.$$

It follows from the homotopy invariance of topological degree that

$$deg(I-\Phi_\mu,B_\delta,0) = deg(I,\ B_\delta,0) = 1,\quad \forall\delta\in(0,\delta_1].$$

□

Lemma 4.5. *For $\lambda>\lambda_1(a_0)$, there exists $\delta_2>0$ such that*

$$deg(I-\Phi_\lambda,\ B_\delta,\ 0) = 0,\quad \forall\delta\in(0,\delta_2].$$

Proof. Let φ_0 be the positive eigenfunctions of $\mathscr{L}_{a_0}$ corresponding to $\lambda_1(a_0)$. First we show that for $\lambda>\lambda_1(a_0)$, there exists $\delta_2>0$ such that

$$y-\Phi_\lambda y\neq\tau\varphi_0,\quad \forall\tau\geq0,\ \forall y\in C[0,1] \text{ with } 0<\|y\|\leq\delta_2. \tag{4.5}$$

Suppose, on the contrary, that there exist $y_n \in C[0,1]$ with $\|y_n\| \to 0+ (n \to +\infty)$ and $\tau_n \geq 0$ such that

$$y_n - \Phi_\lambda y_n = \tau_n \varphi_0.$$

Set $w_n = \dfrac{y_n}{\|y_n\|}$. Then

$$w_n = \frac{\Phi_\lambda y_n}{\|y_n\|} + \frac{\tau_n}{\|y_n\|}\varphi_0. \tag{4.6}$$

By virtue of $\Phi_\lambda y_n \in Q$, we know $w_n \geq \dfrac{\tau_n}{\|y_n\|}\varphi_0$. As a result, $\dfrac{\tau_n}{\|y_n\|}$ is bounded. On the other hand, from (3.10), condition (H1), and Ascoli-Arzela theorem, it is easy to see $\left\{\dfrac{\Phi_\lambda y_n}{\|y_n\|}\right\}$ is relatively compact. This together with (4.6) guarantees that $\{w_n\}$ is also relatively compact. No loss of generality, suppose $w_n \to w$ as $n \to +\infty$.

Consequently, it follows from (3.10) and (4.6) that

$$w_n(t) \geq \lambda \int_0^1 G^*(t,s)a_0(s)\Big((Lw_n)(s) - \frac{\xi_1(s,(Ly_n)(s))}{\|y_n\|}\Big)ds. \tag{4.7}$$

Also let ψ_* be the positive eigenfunction of $\mathscr{L}^*_{a_0}$ corresponding to $\lambda_1(a_0)$. Then by (4.7), we know

$$\begin{aligned}\langle w_n,\ \psi_*\rangle &\geq \lambda\langle \mathscr{L}_{a_0} w_n,\ \psi_*\rangle - \lambda\int_0^1 \psi_*(t)\int_0^1 G^*(t,s)a_0(s)\frac{\xi_1(s,(Ly_n)(s))}{\|y_n\|}dsdt \\ &= \lambda\langle w_n,\ \mathscr{L}^*_{a_0}\psi_*\rangle - \lambda\int_0^1 \psi_*(t)\int_0^1 G^*(t,s)a_0(s)\frac{\xi_1(s,(Ly_n)(s))}{\|y_n\|}dsdt.\end{aligned} \tag{4.8}$$

Similar as in the proof of Lemma 4.3, we have $\dfrac{\xi_1(s,(Ly_n)(s))}{\|y_n\|} \to 0$ as $n \to +\infty$ uniformly with respect to $s \in (0,1)$. Letting $n \to \infty$ in (4.8), we obtain that

$$\langle w,\ \psi_*\rangle \geq \lambda\langle w,\ \mathscr{L}^*_{a_0}\psi_*\rangle = \lambda\langle w,\ \frac{\psi_*}{\lambda_1(a_0)}\rangle.$$

This means $\lambda \leq \lambda_1(a_0)$, which is a contradiction. Consequently, (4.5) holds. By virtue of Lemma 2.6, for each $\lambda > \lambda_1(a_0)$, there exists $\delta_2 > 0$ such that

$$deg(I - \Phi_\lambda,\ B_\delta,\ 0) = 0, \quad \forall \delta \in (0, \delta_2].$$

The conclusion of this Lemma follows. □

Theorem 4.6. $[\lambda_1(a^0),\ \lambda_1(a_0)]$ *is a bifurcation interval of positive solutions from the trivial solution for* (3.11) *, that is, there exists an unbounded component* $\mathcal{C}_0$ *of positive solutions of* (3.11) *, which meets* $[\lambda_1(a^0),\ \lambda_1(a_0)] \times \{0\}$*. Moreover, there exists no bifurcation interval of positive solutions from the trivial solution which is disjointed with* $[\lambda_1(a^0),\ \lambda_1(a_0)]$.

Proof. For $n \in \mathbb{N}$ with $\lambda_1(a^0) - \frac{1}{n} > 0$, by Lemma 4.4-4.5, and their proof, there exists $r > 0$ such that the conditions of Lemma 2.4 are satisfied with $G(\lambda, u) = \Phi_\lambda u$, $a = \lambda_1(a^0) - \frac{1}{n}$, and $b = \lambda_1(a_0) + \frac{1}{n}$. By Lemma 3.5, there exists a closed connected set $\mathcal{C}_n$ of solutions of (3.11) containing $[\lambda_1(a^0) - \dfrac{1}{n},\ \lambda_1(a_0) + \dfrac{1}{n}] \times 0$ in $\mathbb{R}^+ \times C[0,1]$. From Lemma 4.3, the case (ii) of Lemma 2.4 can not occur. Therefore, $\mathcal{C}_n$ bifurcates from $[\lambda_1(a^0) - \dfrac{1}{n},\ \lambda_1(a_0) + \dfrac{1}{n}] \times 0$ and is unbounded in $\mathbb{R}^+ \times C[0,1]$. In addition, for any $[c,d] \subset [\lambda_1(a^0) - \frac{1}{n},\ \lambda_1(a_0) + \frac{1}{n}] \setminus [\lambda_1(a^0),\ \lambda_1(a_0)]$, it follows from Lemma 4.3 that $\delta_1 > 0$ such that the set $\{v \in C[0,1] :\ (\lambda, v) \in \Sigma, 0 < \|v\| \leq \delta_1, \lambda \in [c,d]\} = \emptyset$. Thus, $\mathcal{C}_n$ must be bifurcated from $[\lambda_1(a^0),\ \lambda_1(a_0)] \times \{0\}$, which implies $\mathcal{C}_n$ can be regarded as $\mathcal{C}_0$. Furthermore, using Lemma 4.3 again, there exists no bifurcation interval of positive solutions from the trivial solution which is disjointed with $[\lambda_1(a^0),\ \lambda_1(a_0)]$. □

From a process similar to the above, the following conclusions can be obtained.

Lemma 4.7. *For any $[c,\ d] \subset \mathbb{R}^+$ satisfying $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)] \cap [c,\ d] = \emptyset$, there exists $R_1 > 0$ such that*

$$u \neq \Phi_\lambda u,\ \forall \lambda \in [c,\ d],\ \forall u \in C[0,1] \text{ with } \|u\| \geq R_1.$$

Lemma 4.8. *For $\mu \in (0,\ \lambda_1(b^\infty))$, there exists $R_1 > 0$ such that*

$$deg(I - \Phi_\mu, B_R, 0) = 1, \quad \forall R \geq R_1.$$

Lemma 4.9. *For $\lambda > \lambda_1(b_\infty)$, there exists $R_2 > 0$ such that*

$$deg(I - \Phi_\lambda,\ B_R, 0) = 0, \quad \forall R \geq R_2.$$

Theorem 4.10. $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)]$ *is a bifurcation interval of positive solutions from infinity for* (3.11)*, and there exists no bifurcation interval of positive solutions from infinity which is disjoint with* $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)]$*. More precisely, there exists an unbounded component* $\mathcal{C}^\infty$ *of solutions of* (3.11) *which meets* $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)] \times \infty$*, and is unbounded in λ direction.*

Now we are ready to prove Theorem 4.1 and Theorem 4.2.

Proof of Theorem 4.1. Obviously, the solution of the form $(1, u)$ $(u \neq \theta)$ for BVP (3.9) is a positive solution of BVP(1.2). By virtue of Lemma 3.5 it is sufficient to prove that there is a component $\mathcal{C}$ of Σ crosses the hyperplane $\{1\} \times C[0,1]$, where $\Sigma \subset \mathbb{R}^+ \times C[0,1]$ is defined by (3.13).

Case (i). $\lambda_1(a_0) < 1 < \lambda_1(b^\infty)$.

By Theorem 4.6, there exists an unbounded component $\mathcal{C}_0$ of positive solutions of (3.11), which meets $[\lambda_1(a^0),\ \lambda_1(a_0)] \times \{\theta\}$. From unboundedness of $\mathcal{C}_0$, there exists $(\mu_n, y_n) \in \mathcal{C}_0$ such that

$$\mu_n + \|y_n\| \to +\infty \text{ as } n \to +\infty. \tag{4.9}$$

If $\mu_n \geq 1$ for some $n \in \mathbb{N}$, then the conclusion follows. On the contrary, suppose $\mu_n < 1$ for all $n \in \mathbb{N}$. Since $(0, \theta)$ is the only solution of (3.11) with $\lambda = 0$. By Lemma 4.3 and 4.7, we know $\mathcal{C}_0 \cap (\{0\} \times C[0,1]) = \emptyset$. Therefore, $\mu_n \in (0,1)$ for all $n \in \mathbb{N}$. Taking a subsequence and relabeling if necessary, assume $\mu_n \to \mu^*$ as $n \to +\infty$. Then $\mu^* \in [0,1]$. This together with (4.9) guarantees that $\|y_n\| \to +\infty$.

Letting $[c,d] = [0, \lambda_1(b^\infty) - \frac{1}{m}]$ $(m \in \mathbb{N})$ in Lemma 4.7, we have $\mu^* > \lambda_1(b^\infty) - \frac{1}{m}$ for each $m \in \mathbb{N}$, which means $\mu^* \geq \lambda_1(b^\infty) > 1$. This is a contradiction.

Case (ii). $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$.

From Theorem 4.10, it follows that there exists an unbounded component $\mathcal{C}^\infty$ of solutions of (3.9) which bifurcates from $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)] \times \infty$, and is unbounded in λ direction.

If $\mathcal{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) = \emptyset$, by using the fact that $\mathcal{C}^\infty \cap (\{0\} \times C[0,1]) = \emptyset$ and $\mathcal{C}^\infty$ is unbounded in λ direction, we know $\mathcal{C}^\infty$ must crosses the hyperplane $\{1\} \times C[0,1]$.

If $\mathcal{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) \neq \emptyset$, from $\mathcal{C}^\infty \cap (\{0\} \times C[0,1]) = \emptyset$ and Theorem 4.6, it follows $\mathcal{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) \in [\lambda_1(a^0),\ \lambda_1(a_0)] \times \{0\}$. Therefore, $\mathcal{C}^\infty$ joins $[\lambda_1(a^0),\ \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty),\ \lambda_1(b_\infty)] \times \infty$. Noticing $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$, we know that $\mathcal{C}^\infty$ crosses the hyperplane $\{1\} \times C[0,1]$. □

Proof of Theorem 4.2. First we show

$$\Sigma \cap ([0, 1+\varepsilon] \times \partial B_R) = \emptyset \tag{4.10}$$

for some $\varepsilon > 0$, where $B_R = \{y \in C[0,1] : \ \|y\| < R\}$, $\Sigma \subset \mathbb{R}^+ \times C[0,1]$ is defined by (3.13).

In fact, from (H2) it follows that there exists $\varepsilon > 0$ such that

$$\frac{2^m(1+\varepsilon)}{\Gamma(\alpha)} \int_0^1 [s(1-s)]^{\alpha-1} h(s)ds < 1.$$

If there is a solution (λ, y) of (3.11) such that $0 \le \lambda \le 1+\varepsilon$ and $\|y\| = R$, then it follows from Lemma 3.3 and Lemma 3.4 that

$$0 \le (Ly)(t) \le t^{\alpha-2}Ty(t) \le t^{\alpha-2}\|Ty\|_{pc} \le 2^m t^{\alpha-2}\|y\| \quad \text{for} \quad t \in (0,1).$$

Using (3.10) and Lemma 3.2, we have

$$\begin{aligned} R &= \|y\| = \max_{t\in J} \lambda \int_0^1 G^*(t,s)\bar{f}(s,(Ly)(s))\mathrm{d}s \\ &\le 2^m(1+\varepsilon)R \max_{t\in J} \int_0^1 G^*(t,s)s^{\alpha-2}h(s)ds \\ &\le \frac{2^m(1+\varepsilon)R}{\Gamma(\alpha)} \int_0^1 [s(1-s)]^{\alpha-1}h(s)ds < R. \end{aligned}$$

This is a contradiction. Thus, $\Sigma \cap ([0, 1+\varepsilon] \times \partial B_R) = \emptyset$.

Next, it follows from Theorem 4.6 that there exists an unbounded components $\mathcal{C}_0$ of solutions of (3.9), which meet $[\lambda_1(a^0), \ \lambda_1(a_0)] \times \{0\}$. By virtue of (4.10) we know $\mathcal{C}_0 \cap ([0, 1+\varepsilon] \times \partial B_R) = \emptyset$. Notice the fact that $\mathcal{C}_0$ is unbounded, $\lambda_1(a_0) < 1$, and $\mathcal{C}_0 \cap (\{0\} \times C[0,1]) = \emptyset$, which guarantee that $\mathcal{C}_0$ crosses the hyperplane $\{1\} \times C[0,1]$. Then (3.11) has a positive solution $(1, y_1) \in \mathcal{C}_0$ with $\|y_1\| < R$.

Very similarly, by Theorem 4.10 and (4.10), (3.11) has a positive solution $(1, y_2) \in \mathcal{C}^\infty$ with $\|y_2\| > R$. By Lemma 3.1, the conclusion follows. □

Immediately, from the proof of Theorem 4.2, we have the following result.

Corollary 4.11. *Assume that (H2) holds. In addition, assume one of the following two conditions holds:*
(i) $\lambda_1(a_0) < 1$*;*
(ii) $\lambda_1(b_\infty) < 1$.
Then BVP(1.2) *has at least one positive solution.*

5. An Example

Let

$$(Ly)(t) = \begin{cases} t^{\alpha-2}y(t) + \frac{2}{3}t^{\alpha-1}y(\frac{1}{2}), & t \in (0, \ \frac{1}{2}); \\ t^{\alpha-2}y(t), & t \in [\frac{1}{2}, \ 1) \end{cases} \tag{5.1}$$

for each $y \in C(J)$.

Let ρ be the unique characteristic value of $\mathscr{L}_a$ corresponding to positive eigenfunctions with $a(t) \equiv t$ and $(Ly)(t)$ defined by (5.1) in (3.14). From Lemma 3.6, it follows that ρ exists. Now we are ready to give the following example.

Example 5.1. Consider the following boundary value problem of fractional impulsive differential equations

$$\begin{cases} D_{0^+}^{1.8}u(t)+f(t,u(t))=0, \quad t\in(0,1),\ t\neq\frac{1}{2}; \\ u(\frac{1}{2}+0)=u(\frac{1}{2}-0)-\frac{1}{4}u(\frac{1}{2}-0); \\ u(0)=u(1)=0 \end{cases} \tag{5.2}$$

where

$$f(t,u)=\rho tu[h(u)+\frac{1}{4}\sin\frac{1}{u}+t\sin(tu)], \tag{5.3}$$

$$h(u)=\begin{cases} \frac{1}{2}, & u\in(0,\frac{1}{2}]; \\ u, & u\in[\frac{1}{2},3); \\ 3, & u\in[3,+\infty). \end{cases} \tag{5.4}$$

Then BVP(5.2) has at least one positive solution.

Proof. BVP(5.2) can be regarded as the form (1.2) with $\alpha=1.8$, where there is only one impulsive point $t_1=\frac{1}{2}$ with $c_1=\frac{1}{4}$. Let $f(t,u)=0$ for $u=0$, then $f(t,u)$ is continuous.

From (5.2)-(5.4), choose $a_0(t)=\frac{\rho t}{4},\ a^0(t)=\frac{3}{4}\rho t,\ b_\infty(t)=2\rho t,\ b^\infty(t)=4\rho t,$

$$\xi_1(t,u)=-4tu\sin(tu),$$

$$\xi_2(t,u)=\begin{cases} \frac{4}{3}tu\sin(tu), & (t,u)\in J\times[0,\ \frac{1}{2}]; \\ \frac{4}{3}u(u-\frac{1}{2})+\frac{4}{3}tu\sin(tu), & (t,u)\in J\times(\frac{1}{2},\ +\infty), \end{cases}$$

$$\zeta_1(t,u)=\begin{cases} -\frac{u}{2}(h(u)-3)-\frac{u}{8}\sin\frac{1}{u}, & (t,u)\in J\times(0,\ 3]; \\ -\frac{u}{8}\sin\frac{1}{u}, & (t,u)\in J\times(3,\ +\infty), \end{cases}$$

$$\zeta_2(t,u)=\begin{cases} \frac{u}{4}(h(u)-3)+\frac{u}{16}\sin\frac{1}{u}, & (t,u)\in J\times(0,\ 3]; \\ \frac{u}{16}\sin\frac{1}{u}, & (t,u)\in J\times(3,\ +\infty). \end{cases}$$

It is easy to see $\xi_i(t,t^{\alpha-2}u)=o(u)$ as $u\to0$ and $\zeta_i(t,t^{\alpha-2}u)=o(u)$ as $u\to+\infty$ both uniformly with respect to $t\in(0,1)$, $(i=1,2)$.

Therefore, (H1) is satisfied.

By computation, it is easy to see $(Ly)(t)=T\bar{y}(t)$, where L is defined by (5.1). Therefore, from the definition of ρ, it is easy to see $\lambda_1(a^0)=\frac{4}{3}$, $\lambda_1(b_\infty)=\frac{1}{2}$.

As a result, by Theorem 4.1, BVP(5.2) has at least one positive solution. □

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