Hybrid projection algorithms for approximating fixed points of asymptotically quasi-pseudocontractive mappings

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

The purpose of this paper is to modify Ishikawa iterative process to have strong convergence without any compact assumptions for asymptotically quasi-pseudocontractive mappings in the framework of real Hilbert spaces.

Keywords: Asymptotically pseudocontractive mapping; asymptotically nonexpansive mapping; fixed point; hybrid projection algorithm.

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1. Introduction and Preliminaries

Throughout this paper, we always assume that $H$ is a real Hilbert space with inner product $\langle \cdot , \cdot \rangle$, and norm $\| \cdot \|$. Assume that $C$ is a nonempty closed convex subset of $H$ and $T : C \to C$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of $T$.

$T$ is said to be nonexpansive if

$$\| Tx - Ty \| \leq \| x - y \|, \quad \forall x, y \in C.$$
For expansive mappings as a subclass, which can be seen from the following example. The class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings. We remark here that the class of asymptotically pseudocontractive mappings was introduced by Schu; see Schu for more details. They proved that, if $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $E$, then every asymptotically nonexpansive self-mapping $T$ on $C$ has a fixed point. Further, the set $F(T)$ of fixed points of $T$ is closed and convex.

$T$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[
\|T^n x - T^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C, \ n \geq 1.
\]

$T$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and (1.1) holds for every $x \in C$ but $y \in F(T)$. We remark here that the class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk; see Goebel and Kirk for more details. They proved that, if $C$ is a nonempty bounded closed convex subset of a uniformly convex Banach space $E$, then every asymptotically nonexpansive self-mapping $T$ on $C$ has a fixed point. Further, the set $F(T)$ of fixed points of $T$ is closed and convex.

$T$ is said to be pseudocontractive if
\[
\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C.
\]

$T$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[
\langle T^n x - T^n y, x - y \rangle \leq k_n \|x - y\|^2, \quad \forall x, y \in C.
\]

We remark here that the class of asymptotically pseudocontractive mappings was introduced by Schu; see Schu for more details.

It is clear that (1.2) is equivalent to
\[
\|T^n x - T^n y\|^2 \leq (2k_n - 1)\|x - y\|^2 + \|(I - T^n)x - (I - T^n)y\|^2, \quad \forall x, y \in C.
\]

The class of asymptotically pseudocontractive mappings contains properly the class of asymptotically nonexpansive mappings as a subclass, which can be seen from the following example.

**Example.** (15) For $x \in [0, 1]$, define a mapping $T : [0, 1] \to [0, 1]$ by
\[
Tx = (1 - x^2)^3.
\]

Then $T$ is asymptotically pseudocontractive but it is not asymptotically nonexpansive.

$T : C \to C$ is said to be asymptotically quasi-pseudocontractive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[
\langle T^n x - p, x - p \rangle \leq k_n \|x - p\|^2, \quad \forall x \in C, \ p \in F(T).
\]

It is clear that (1.4) is equivalent to
\[
\|T^n x - p\|^2 \leq (2k_n - 1)\|x - p\|^2 + \|x - T^n x\|^2, \quad \forall x \in C, \ p \in F(T).
\]

In 1991, Schu proved the following results for asymptotically pseudocontractive mappings in the framework of Hilbert spaces.

**Theorem Schu.** Let $C$ be a nonempty closed bounded convex subset of a Hilbert space $H$. Let $L > 0$ and $T : C \to C$ be completely continuous, uniformly $L$-Lipschitzian and asymptotically pseudo-contractive with sequence $\{k_n\} \subset [1, \infty)$, $q_n = 2k_n - 1$ for all $n \geq 1$, $\sum_{n=1}^{\infty} (q_n^2 - 1) < \infty$, $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$, $\epsilon \leq \alpha_n \leq \beta_n \leq b$ for all $n \geq 1$ and for some $\epsilon > 0$ and some $b \in (0, L^{-2\sqrt{1 + L^2} - 1})$. For given $x_1 \in K$, define a sequence $\{x_n\}$ in $C$ by the following algorithm:

\[
\begin{aligned}
y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad \forall n \geq 1.
\end{aligned}
\]

Then $\{x_n\}$ converges strongly to some fixed point of $T$. 


Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping and its extensions. The first one was introduced by Mann [7], which is defined as follows:

\[
\begin{aligned}
  x_0 &\in C \quad \text{arbitrary chosen,} \\
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTx_n, \quad \forall n \geq 0,
\end{aligned}
\]  

(1.6)

where \(\{\alpha_n\}\) is a sequence in the interval \((0, 1)\).

The second one was referred to as Ishikawa iteration process [4], which is defined recursively as follows:

\[
\begin{aligned}
  x_0 &\in C \quad \text{arbitrary chosen,} \\
  y_n &= (1 - \beta_n)x_n + \beta_nTx_n, \\
  z_n &= (1 - \alpha_n)x_n + \alpha_nTy_n, \\
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_nTz_n, \quad \forall n \geq 0,
\end{aligned}
\]  

(1.7)

where \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in the interval \((0, 1)\).

But both (1.6) and (1.7) have only weak convergence, in general; see [2] and [19]. Reich [14] shows that, if \(E\) is a uniformly convex and has a Fréchet differentiable norm, and the sequence \(\{\alpha_n\}\) is such that \(\sum_{n=0}^{\infty} \alpha_n(1 - \alpha_n) = \infty\), then the sequence \(\{x_n\}\) generated by the process (1.6) converges weakly to a point in \(F(T)\) (an extension of the results to the process (1.7) can be found in [19]). Therefore, many authors have attempted to modify (1.6) and (1.7) to have strong convergence.

In 2006, Martinez-Yanes and Xu [9] modified (1.7) to have strong convergence by hybrid projection algorithms in Hilbert spaces. To be more precise, They proved the following result.

**Theorem MYX.** Let \(C\) be a closed convex subset of a Hilbert space \(H\) and \(T : C \to C\) be a nonexpansive mapping such that \(F(T) \neq \emptyset\). Assume that \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in \([0, 1]\) such that \(\alpha_n \leq 1 - \delta\) for some \(\delta \in (0, 1]\) and \(\beta_n \to 1\). Define a sequence \(\{x_n\}\) in \(C\) by the following algorithm:

\[
\begin{aligned}
  x_0 &\in C \quad \text{chosen arbitrarily,} \\
  z_n &= \beta_nx_n + (1 - \beta_n)Tx_n, \\
  y_n &= \alpha_nx_n + (1 - \alpha_n)Tz_n, \\
  C_n &= \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1 - \alpha_n)(\|z_n\|^2 - \|x_n\|^2 + 2\langle x_n - z_n, v \rangle)\}, \\
  Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v \rangle \geq 0\}, \\
  x_{n+1} &= P_{C_n \cap Q_n}x_0.
\end{aligned}
\]

Then \(\{x_n\}\) converges in norm to \(P_{F(T)}x_0\).

Recently, Qin, Su and Shang [13] improved the results of Martinez-Yanes and Xu [9] from nonexpansive mappings to asymptotically nonexpansive mappings. More precisely, They proved the following theorem.

**Theorem QSS.** Let \(C\) be a bounded closed convex subset of a Hilbert space \(H\) and \(T : C \to C\) be an asymptotically nonexpansive mapping with a sequence \(\{k_n\}\) such that \(k_n \to 1\) as \(n \to \infty\). Assume that \(\{\alpha_n\}\) is a sequence in \((0, 1)\) such that \(\alpha_n \leq 1 - \delta\) for all \(n\) and for some \(\delta \in (0, 1]\) and \(\beta_n \to 1\). Define a sequence \(\{x_n\}\) in \(C\) by the following algorithm:

\[
\begin{aligned}
  x_0 &\in C \quad \text{chosen arbitrarily,} \\
  z_n &= \beta_nx_n + (1 - \beta_n)T^nx_n, \\
  y_n &= \alpha_nx_n + (1 - \alpha_n)T^nz_n, \\
  C_n &= \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1 - \alpha_n)(k_n^2\|z_n\|^2 - \|x_n\|^2 \\
  &\quad + (k_n^2 - 1)M + 2\langle x_n - k_n^2z_n, v \rangle)\}, \\
  Q_n &= \{v \in C : \langle x_0 - x_n, x_n - v \rangle \geq 0\}, \\
  x_{n+1} &= P_{C_n \cap Q_n}x_0,
\end{aligned}
\]

where \(M\) is a appropriate constant such that \(M > \|v\|^2\) for each \(v \in C_n\), then \(\{x_n\}\) converges to \(P_{F(T)}x_0\).
Very recently, Zhou [20] improved the results of Martinez-Yanes and Xu [9] from nonexpansive mappings to Lipschitz pseudo-contractions. To be more precise, he proved the following theorem.

**Theorem Zhou.** Let $C$ be a closed convex subset of a real Hilbert space $H$ and $T : C \to C$ be a Lipschitz pseudo-contraction such that $F(T) \neq \emptyset$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0, 1)$ satisfying the conditions:

(a) $\beta_n \leq \alpha_n, \forall n \geq 0$;
(b) $\liminf_{n \to \infty} \alpha_n > 0$;
(c) $\limsup_{n \to \infty} \alpha_n \leq \alpha \leq \frac{1}{\sqrt{1+L^2}+1}, \forall n \geq 0$, where $L \geq 1$ is the Lipschitzian constant of $T$. 

Let a sequence $\{x_n\}$ generated by

$$
\begin{align*}
&x_0 \in C, \\
y_n = (1 - \alpha_n)x_n + \alpha_nTx_n, \\
z_n = (1 - \beta_n)x_n + \beta_nTy_n, \\
C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 - \alpha_n\beta_n(1 - 2\alpha_n - L^2\alpha_n^2)\|x_n - T^nx_n\|^2\}, \\
Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}x_n.
\end{align*}
$$

Then $\{x_n\}$ converges strongly to a fixed point $v$ of $T$, where $v = P_{F(T)}x_0$.

In this paper, motivated by Acedo and Xu [1], Kim and Xu [5, 6], Marino and Xu [8], Martinez-Yanes and Xu [9], Nakajo and Takahashi [10], Qin et al. [11], Qin, Cho and Zhou [12], Qin, Su and Shang [13], Su and Qin [17, 18] and Zhou [20, 21], we modify Ishikawa iterative process (1.7) to have strong convergence for asymptotically quasi-pseudocontractive mappings in the framework of Hilbert spaces without any compact assumption.

In order to prove our main results, we need the following lemmas.

**Lemma 1.1.** Let $H$ be a real Hilbert space. Then the following equations hold:

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ for all $x, y \in H$.
(b) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ for all $t \in [0, 1]$ and $x, y \in H$.

**Lemma 1.2.** Let $C$ be a closed convex subset of real Hilbert space $H$ and $P_C$ be the metric projection from $H$ onto $C$ (i.e., for $x \in H$, $P_Cx$ is the only point in $C$ such that $\|x - P_Cx\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in H$ and $z \in C$, $z = P_Cx$ if and only if there holds the relations: $\langle x - z, y - z \rangle \leq 0$ for any $y \in C$.

The following lemma can be found in Zhou and Su [22], we still give the proof for the completeness of the paper.

**Lemma 1.3.** Let $C$ be a nonempty bounded closed convex subset of $H$ and $T : C \to C$ be a uniformly $L$-Lipschitzian and asymptotically quasi-pseudocontractive mapping. Then $F(T)$ is a closed convex subset of $C$.

**Proof.** From the continuity of $T$, we can conclude that $F(T)$ is closed.

Next, we show that $F(T)$ is convex. If $F(T) = \emptyset$, then the conclusion is always true. Let $p_1, p_2 \in F(T)$. We prove $p \in F(T)$, where $p = tp_1 + (1-t)p_2$, for $t \in (0, 1)$. Put $y_{(\alpha, n)} = (1-\alpha)p + \alpha T^np$, where $\alpha \in (0, \frac{1}{1+L})$. Then...
For all \( w \in F(T) \), we see that

\[
\|p - T^n p\|^2 = \langle p - T^n p, p - T^n p \rangle = \frac{1}{\alpha} \langle p - y_{(\alpha,n)}, p - T^n p \rangle
\]

\[
= \frac{1}{\alpha} \langle p - y_{(\alpha,n)}, p - T^n p \rangle - \langle y_{(\alpha,n)} - T^n y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle
\]

\[
= \frac{1}{\alpha} \langle p - y_{(\alpha,n)}, p - T^n p \rangle - \langle y_{(\alpha,n)} - T^n y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle
\]

\[
\leq \frac{1}{\alpha} \langle p - y_{(\alpha,n)}, p - T^n p \rangle - \langle y_{(\alpha,n)} - T^n y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle
\]

\[
\leq (1 + L) \alpha \|p - T^n p\|^2 + \frac{1}{\alpha} \langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle + \frac{1}{\alpha} \langle w - y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle
\]

\[
= (1 + L) \alpha \|p - T^n p\|^2 + \frac{1}{\alpha} \langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle + \frac{1}{\alpha} \langle w - y_{(\alpha,n)}, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle
\]

This implies that

\[
\alpha [1 - (1 + L)\alpha] \|p - T^n p\|^2 \leq \langle p - w, y_{(\alpha,n)} - T^n y_{(\alpha,n)} \rangle + (k_n - 1)\|w - y_{(\alpha,n)}\|^2, \quad \forall w \in F(T). \tag{1.8}
\]

Taking \( w = p_i, i = 1, 2 \) in (1.8), multiplying \( t \) and \((1 - t)\) on the both sides of (1.8), respectively and adding up, we see that

\[
\alpha [1 - (1 + L)\alpha] \|p - T^n p\|^2 \leq (k_n - 1)\|w - y_{(\alpha,n)}\|^2.
\]

This shows that \( T^n p - p \to 0 \) as \( n \to \infty \). Note that \( T \) is uniformly \( L \)-Lipschitzian. It follows that \( T^{n+1} p - Tp \to 0 \) as \( n \to \infty \). This is, \( p \in F(T) \). This completes the proof. \( \square \)

### 2. Main Results

**Theorem 2.1.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( T : C \to C \) be a uniformly \( L \)-Lipschitz and asymptotically quasi-pseudocontractive mapping such that \( F(T) \) is nonempty and bounded. Let \( \{x_n\} \) be a sequence generated in the following algorithm:

\[
\begin{aligned}
& x_0 \in H \text{ chosen arbitrarily,} \\
& C_1 = C, \\
& x_1 = P_{C_1} x_0, \\
& y_n = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \\
& z_n = (1 - \beta_n)x_n + \beta_n T^n y_n, \\
& C_{n+1} = \{ z \in C_n : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n \theta_n - \alpha_n \beta_n (1 - 2\alpha_n - L^2 \alpha_n^2) \|x_n - T^n x_n\|^2 \}, \\
& x_{n+1} = P_{C_{n+1}} x_0,
\end{aligned}
\]

where

\[
\theta_n = 2(k_n - 1)[2k_n + 1 + (1 + L)^2] \left( \sup_{z \in F(T)} \|x_n - z\| \right)^2 \to 0.
\]

Assume that the control sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \((0,1)\) satisfy the restrictions:

(a) \( \beta_n \leq \alpha_n, \forall n \geq 1; \)

(b) \( \lim \inf_{n \to \infty} \alpha_n > 1; \)

(c) \( \lim \sup_{n \to \infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1 + L^2}}, \forall n \geq 0. \)

Then the sequence \( \{x_n\} \) converges strongly to \( P_{F(T)} x_0 \).
Proof. We divide the proof into five parts.

Step 1. Show that $C_n$ is closed and convex for all $n \geq 1$.

It is obvious that $C_1$ is closed and convex. Assume that $C_m$ is closed and convex. Next, we show that $C_{m+1}$ is closed and convex for the same $m$. For all $z \in C_m$, we see that

$$\|z_m - z\|^2 \leq \|x_m - z\|^2 + \beta_m \theta_m - \alpha_m \beta_m (1 - 2\alpha_m - L^2 \alpha_m^2) \|x_m - T^m x_m\|^2$$

is equivalent to the following inequality

$$2 \langle x_m - z_m, z \rangle \leq \|x_m\|^2 - \|z_m\|^2 + \beta_m \theta_m - \alpha_m \beta_m (1 - 2\alpha_m - L^2 \alpha_m^2) \|x_m - T^m x_m\|^2.$$

This shows that $C_{m+1}$ is closed and convex. We, therefore, obtain that $C_n$ is convex for every $n \geq 1$.

Step 2. Show that $F(T) \subseteq C_n, \forall n \geq 1$.

It is obvious that $F(T) \subseteq C_1$. Assume that $F(T) \subseteq C_m$ for some $m$. Next, we show that $F(T) \subseteq C_{m+1}$ for the same $m$. In view of Lemma 1.1, for all $u \in F(T) \subseteq C_m$, we see from (1.3) that

$$\|z_m - u\|^2 = \|(1 - \beta_m)(x_m - u) + \beta_m (T^m y_m - u)\|^2$$

$$= (1 - \beta_m)\|x_m - u\|^2 + \beta_m \|T^m y_m - u\|^2 - \beta_n (1 - \beta_m)\|x_m - T^m y_m\|^2$$

$$\leq (1 - \beta_m)\|x_m - u\|^2 + \beta_m \|(2k_m - 1)\|y_m - u\|^2 + \|y_m - T^m y_m\|^2$$

$$- \beta_m (1 - \beta_m)\|x_m - T^m y_m\|^2$$

and

$$\|y_m - T^m y_m\|^2$$

$$= \|(1 - \alpha_m)(x_m - T^m y_m) + \alpha_m (T^m x_m - T^m y_m)\|^2$$

$$= (1 - \alpha_m)\|x_m - T^m y_m\|^2 + \alpha_m \|T^m x_m - T^m y_m\|^2 - \alpha_m (1 - \alpha_m)\|x_m - T^m x_m\|^2$$

$$\leq (1 - \alpha_m)\|x_m - T^m y_m\|^2 + \alpha_m (L^2 \alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2.$$

Note that

$$\|y_m - u\|^2 = (1 - \alpha_m)\|x_m - u\|^2 + \alpha_m \|T^m x_m - u\|^2 - \alpha_m (1 - \alpha_m)\|x_m - T^m x_m\|^2$$

$$\leq (1 - \alpha_m)\|x_m - u\|^2 + \alpha_m (2k_m - 1)\|x_m - u\|^2 + \alpha_m \|x_m - T^m x_m\|^2$$

$$- \alpha_m (1 - \alpha_m)\|x_m - T^m x_m\|^2$$

$$\leq [1 + 2\alpha_m (k_m - 1)]\|x_m - u\|^2 + \alpha_m \|x_m - T^m x_m\|^2.$$

Substituting (2.2) and (2.3) into (2.1), we arrive at

$$\|z_m - u\|^2 \leq (1 - \beta_m)\|x_m - u\|^2 + \beta_m (2k_m - 1)\|1 + 2\alpha_m (k_m - 1)\|x_m - u\|^2$$

$$+ (2k_m - 1)\alpha_m \beta_m \|x_m - T^m x_m\|^2 + \alpha_m \beta_m (L^2 \alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2$$

$$+ \beta_m (\beta_m - \alpha_m)\|x_m - T^m y_m\|^2$$

$$\leq (1 - \beta_m)\|x_m - u\|^2 + \beta_m (2k_m - 1)\|1 + 2\alpha_m (k_m - 1)\|x_m - u\|^2$$

$$+ 2(k_m - 1)\alpha_m \beta_m \|x_m - T^m x_m\|^2 + \alpha_m \beta_m (L^2 \alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2$$

$$+ \beta_m (\beta_m - \alpha_m)\|x_m - T^m y_m\|^2$$

$$\leq \|x_m - u\|^2 + 2(k_m - 1)\beta_m [2\alpha_m k_m + 1 - \alpha_m + \alpha_m^2 (L^2)]\|x_m - u\|^2$$

$$+ \alpha_m \beta_m (L^2 \alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2 + \beta_m (\beta_m - \alpha_m)\|x_m - T^m y_m\|^2$$

$$\leq \|x_m - u\|^2 + 2(k_m - 1)\beta_m [2k_m + 1 + (L^2)]\|x_m - u\|^2$$

$$+ \alpha_m \beta_m (L^2 \alpha_m^2 + \alpha_m - 1)\|x_m - T^m x_m\|^2 + \beta_m (\beta_m - \alpha_m)\|x_m - T^m y_m\|^2.$$
From the condition (a), we obtain that
\[ \|z_m - u\|^2 \leq \|x_m - u\|^2 + \beta_m \theta_m - \alpha_m \beta_m (1 - 2\alpha_m - L^2 \alpha_m^2) \|x_m - T^m x_m\|^2. \]

Therefore, we obtain that \( u \in C_{m+1} \). This concludes that \( F(T) \subset C_n, \forall n \geq 1 \).

Step 3. Show that \( \{x_n\} \) is a Cauchy sequence in \( C \).

In view of \( x_n = P_{C_n} x_0 \) and \( P_{F(T)} x_0 \in F(T) \subset C_n \) for each \( n \geq 1 \), we see that
\[ \|x_n - x_{n+1}\| \leq \|x_0 - P_{F(T)} x_0\|. \]

This proves that the sequence \( \{x_n\} \) is bounded. From \( x_n = P_{C_n} x_0 \), we see that
\[ \langle x_0 - x_n, x_n - y \rangle \geq 0, \forall y \in C_n. \quad (2.4) \]

In view of \( x_{n+1} \in C_{n+1} \subset C_n \), we see that
\[ 0 \leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \]
\[ = \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \]
\[ \leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\| \|x_0 - x_{n+1}\|, \]

that is, \( \|x_0 - x_n\| \leq \|x_0 - x_{n+1}\| \). This together with the boundedness of \( \{x_n\} \) implies that \( \lim_{n \to \infty} \|x_0 - x_n\| \) exists. By the construction of \( C_n \), we see that \( C_n \subset C_n \) and \( x_m = P_{C_n} x_0 \in C_n \) for any positive integer \( m \geq n \). From \( x_n = P_{C_n} x_0 \), we see that
\[ \langle x_0 - x_n, x_n - x_m \rangle \geq 0. \quad (2.5) \]

It follows that
\[ \|x_m - x_n\|^2 = \|x_m - x_0 + x_0 - x_n\|^2 \]
\[ = \|x_m - x_0\|^2 + \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_0 - x_m\rangle \]
\[ \leq \|x_m - x_0\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_0 - x_m\rangle \]
\[ \leq \|x_m - x_0\|^2 - \|x_0 - x_n\|^2. \quad (2.6) \]

Letting \( m, n \to \infty \) in (2.6), we have \( \lim_{m,n \to \infty} \|x_n - x_m\| = 0 \). Hence, \( \{x_n\} \) is a Cauchy sequence.

Step 4. Show that \( TX_n - x_n \to 0 \) as \( n \to \infty \).

Since \( H \) is a Hilbert space and \( C \) is closed and convex, we may assume that
\[ x_n \to q \in C \quad \text{as} \ n \to \infty. \quad (2.7) \]

Next, we show that \( q = P_{F(T)} x_0 \). To end this, we first show that \( q \in F(T) \). By taking \( m = n + 1 \) in (2.6), we arrive at
\[ \lim_{n \to \infty} \|x_n - x_{n+1}\| = 0, \quad (2.8) \]

In view of \( x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \), we obtain that
\[ \|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \beta_n \theta_n - \alpha_n \beta_n (1 - 2\alpha_n - L^2 \alpha_n^2) \|x_n - T^m x_n\|^2. \quad (2.9) \]

On the other hand, we have
\[ \|z_n - x_{n+1}\|^2 = \|z_n - x_n + x_n - x_{n+1}\|^2 \]
\[ = \|z_n - x_n\|^2 + 2\langle x_n - z_n, x_{n+1} - x_n\rangle + \|x_n - x_{n+1}\|^2. \quad (2.10) \]

Combining (2.9) with (2.10) and noting that \( z_n = (1 - \beta_n) x_n + \beta_n T^m y_n \), we see that
\[ \beta_n^2 \|x_n - T^m y_n\|^2 + 2\beta_n \langle x_n - T^m y_n, x_{n+1} - x_n\rangle \leq \beta_n \theta_n - \alpha_n \beta_n (1 - 2\alpha_n - L^2 \alpha_n^2) \|x_n - T^m x_n\|^2. \]
That is,
\[ \beta_n \| x_n - T^n y_n \|^2 + 2 \langle x_n - T^n y_n, x_{n+1} - x_n \rangle \leq \theta_n - \alpha_n (1 - 2\alpha_n - L^2 \alpha_n^2) \| x_n - T^n x_n \|^2. \]

It follows that
\[ \alpha_n (1 - 2\alpha_n - L^2 \alpha_n^2) \| x_n - T^n x_n \|^2 \leq \theta_n - 2 \langle x_n - T^n y_n, x_{n+1} - x_n \rangle. \]

From the assumptions on \( \{\alpha_n\} \), we can choose \( a \in (\alpha, \frac{1}{1+L^2a^2}) \). For such chosen \( a \), there exists a positive integer \( N \geq 1 \) such that \( \alpha_n < a \) for all \( n \geq N \). It follows that \( 1 - 2a - L^2a^2 > 0 \). On the other hand, one can choose \( b \in (0, c) \), where \( c = \lim \inf_{n \to \infty} \alpha_n \). we obtain that \( \alpha_n > b \) for \( n \) large enough. It follows that
\[ b(1 - 2a - L^2a^2) \| x_n - T^n x_n \|^2 \leq \theta_n + M \| x_{n+1} - x_n \| \]
for \( n \geq 0 \) large enough, where \( M = 2 \sup_{n \geq 0} \{ \| x_n - T^n y_n \| \} \). From (2.8), we obtain that
\[ \lim_{n \to \infty} \| x_n - T^n x_n \| = 0. \] (2.11)

On the other hand, we have
\[ \| x_n - Tx_n \| = \| x_n - x_{n+1} \| + \| x_{n+1} - T^{n+1} x_{n+1} \| + \| T^{n+1} x_{n+1} - T^{n+1} x_n \| \]
\[ \leq \| x_n - x_{n+1} \| + \| x_{n+1} - T^{n+1} x_{n+1} \| + M \| x_{n+1} - x_n \| + L \| T^n x_n - x_n \|. \]

From (2.8) and (2.11), we arrive at
\[ \lim_{n \to \infty} \| x_n - Tx_n \| = 0. \] (2.12)

Step 5. Show that \( x_n \to q = P_{F(T)}x_0 \) as \( n \to \infty \).

Notice that
\[ \| q - Tq \| \leq \| q - x_n \| + \| x_n - Tx_n \| + \| Tx_n - Tq \| \]
\[ \leq (1 + L) \| q - x_n \| + \| x_n - Tx_n \|. \]

It follows from (2.7) and (2.12) that \( q \in F(T) \). From (2.4), we see that
\[ \langle x_0 - x_n, x_n - y \rangle \geq 0, \quad \forall y \in F(T) \subset C_n. \] (2.13)

Taking the limit in (2.13), we obtain that \( \langle x_0 - q, q - y \rangle \geq 0, \quad \forall y \in F(T) \). In view of Lemma 1.2, we see that \( q = P_{F(T)}x_0 \). This completes the proof. \( \square \)

**Remark 2.2.** Theorem 2.1 includes Theorem 4.1 of Kim and Xu [6] as a special case. It also improves the results of Kim and Xu [3] and Qin, Su and Shang [13] from asymptotically nonexpansive mappings to asymptotically quasi-pseudocontractive mappings.

For the class of Lipschitz quasi-pseudocontractive mappings, we have from Theorem 2.1 the following result.

**Corollary 2.3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( T : C \to C \) be a \( L \)-Lipschitz and quasi-pseudocontractive mapping such that \( F(T) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated in the following algorithm:

\[
\begin{align*}
  x_0 & \in H \quad \text{chosen arbitrarily,} \\
  C_1 & = C, \\
  x_1 & = P_{C_1}x_0, \\
  y_n & = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\
  z_n & = (1 - \beta_n)x_n + \beta_n Ty_n, \\
  C_n & = \{ z \in C_n : \| z - x \|^2 \leq \| x_n - z \|^2 - \alpha_n \beta_n (1 - 2\alpha_n - L^2 \alpha_n^2) \| x_n - Tx_n \|^2 \}, \\
  x_{n+1} & = P_{C_{n+1}}x_0.
\end{align*}
\]
Assume that the control sequences \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \((0, 1)\) satisfy the restrictions:

(a) \( \beta_n \leq \alpha_n, \forall n \geq 1; \)

(b) \( \lim inf_{n \to \infty} \alpha_n > 1; \)

(c) \( \lim sup_{n \to \infty} \alpha_n \leq \alpha < \frac{1}{\sqrt{1+L^2+1}}, \forall n \geq 0. \)

Then the sequence \( \{x_n\} \) converges strongly to \( P_{F(T)}x_0. \)

Remark 2.4. Comparing Corollary 2.3 with Theorem 3.6 of Zhou [20], we do not require that the mapping \( I - T \) is demi-closed at zero. From the computation point of view, we remove the iterative step \( Q_n, \) see [20] for more details.

Remark 2.5. Corollary 2.3 also gives an affirmative answer to the problem proposed by Marino and Xu [8].

References


