



A method of differential and sensitivity properties for weak vector variational inequalities

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Abstract

In this paper, by virtue of a contingent derivative and a Φ -contingent cone, we investigate differential properties of a class of set-valued maps in a more general setting utilizing Hadamard directional differentials. Then, by means of a gap function, sensitivity properties are discussed for a weak vector variational inequality. We also show that our results extend some existing results in the literature. ©2015 All rights reserved.

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1. Introduction

The concept of vector variational inequalities was firstly introduced by Giannessi [8] in a finite-dimensional space. It has numerous applications in different fields, including optimization, convex analysis and functional analysis. In last years, vector variational inequalities are discussed extensively. Many important results on existences and stabilities for various kinds of vector variational inequalities have been established, see e.g. [3, 4, 5, 7, 9, 10, 15, 17] and the references therein.

Recently, some authors have investigated the sensitivity of vector variational inequalities. By using a coerciveness condition, Li, et al. [11] investigated differential and sensitivity properties of set-valued gap functions for vector variational inequalities and weak vector variational inequalities. By using the similar

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method of [11], Li and Li [13] also investigated second-order differential and sensitivity properties of set-valued gap functions for weak vector variational inequalities. By using a Φ -contingent cone instead of the coerciveness condition, Meng and Li [14] investigated the differential and sensitivity properties of set-valued gap functions for Minty vector variational inequalities and Minty weak vector variational inequalities. By virtue of an asymptotic second-order contingent derivative and an asymptotic second-order Φ -contingent cone, Li and Zhai [12] investigated differential properties of a class of set-valued maps and obtained an explicit expression of their asymptotic second-order contingent derivatives. By virtue of a contingent derivative and a Φ -contingent cone, Zhu, et al. [16] investigated differential properties of set-valued gap functions for a generalized weak vector variational inequality, which is a generalization of a weak vector variational inequality and a Minty weak vector variational inequality.

Motivated by the work reported in [11, 12, 16], in this paper, we will study differential and sensitivity properties of the gap function of weak vector variational inequalities in a more general setting utilizing Hadamard directional differentials. To this end, by using Hadamard directional differentials and Φ -contingent cone, we first obtain the differential properties of a class of set-valued maps and get an explicit expression of the contingent derivatives. Then, by using a gap function of a weak vector variational inequality, we establish sensitivity properties for the weak vector variational inequality. We also show that our results generalize the corresponding results obtained in [11, 12].

The organization of this paper is as follows. In Section 2, we recall some basic concepts and properties. In Section 3, we obtain differential properties of a class of set-valued maps and generalize the corresponding results in [11, 12, 16] without any coerciveness condition. In Section 4, we investigate sensitivity properties for a weak vector variational inequality.

2. Preliminaries

Throughout this paper, let the set of nonnegative real numbers be denoted by \mathbb{R}_+ , the origins of all real normed spaces be denoted by 0, and the norms of all real normed spaces be denoted by $\|\cdot\|$. Let X and Y be two real normed spaces, where the space Y is partially ordered by a nontrivial closed convex cone $C \subseteq Y$ with nonempty interior $\text{int } C$. Moreover, let M be a set-valued map from X to Y . The domain and graph of M are defined by $\text{dom } M := \{x \in X \mid M(x) \neq \emptyset\}$ and $\text{gph } M := \{(x, y) \in X \times Y \mid y \in M(x)\}$, respectively.

First, let us recall some important definitions and give some auxiliary results.

Definition 2.1. [1] Let S be a nonempty subset of Y . An element $a \in S$ is called a weak maximal element of the set S , if

$$(\{a\} + \text{int } C) \cap S = \emptyset.$$

We denote by $\text{WMax}(S, \text{int } C)$ the set of all the weak maximal elements of S .

Definition 2.2. [2] Let $f : X \rightarrow Y$ be a vector-valued function. f is said to be Hadamard directionally differentiable at $x \in X$ in a direction $u \in X$, iff there exists $d_H f(x, u) \in Y$ such that for any sequences $\{u_n\} \subseteq X$ and $\{h_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $u_n \rightarrow u$ and $t_n \rightarrow 0$, we have

$$d_H f(x; u) = \lim_{n \rightarrow +\infty} \frac{f(x + t_n u_n) - f(x)}{t_n}.$$

Remark 2.3. It is shown in [2] that the directionally differentiability in the Hadamard sense is weaker than the Fréchet differentiability.

Definition 2.4. [1] Let Q be a nonempty subset of X and $\hat{x} \in \text{cl } Q$, where $\text{cl } Q$ denotes the closure of Q .

- (i) The set $T(Q, \hat{x})$ is called the contingent cone of Q at \hat{x} , iff, for any $x \in T(Q, \hat{x})$, there exist sequences $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $t_n \rightarrow 0$ and $\{x_n\} \subseteq X$ with $x_n \rightarrow x$, such that $\hat{x} + t_n x_n \in Q$, for all n .
- (ii) The set $T^b(Q, \hat{x})$ is called the adjacent cone of Q at \hat{x} , iff, for any $x \in T^b(Q, \hat{x})$ and any sequence $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $t_n \rightarrow 0$, there exists a sequence $\{x_n\} \subseteq X$ with $x_n \rightarrow x$, such that $\hat{x} + t_n x_n \in Q$, for all n .

(iii) The set Q is derivable at \hat{x} , iff $T(Q, \hat{x}) = T^b(Q, \hat{x})$.

Definition 2.5. [1] Let $M : X \rightrightarrows Y$ be a set-valued map and $(\hat{x}, \hat{y}) \in \text{gph } M$.

- (i) A set-valued map $DM(\hat{x}, \hat{y}) : X \rightrightarrows Y$ is called the contingent derivative of M at (\hat{x}, \hat{y}) , iff $\text{gph } DM(\hat{x}, \hat{y}) := T(\text{gph } M, (\hat{x}, \hat{y}))$.
- (ii) A set-valued map $D^bM(\hat{x}, \hat{y}) : X \rightrightarrows Y$ is called the adjacent derivative of M at (\hat{x}, \hat{y}) , iff $\text{gph } D^bM(\hat{x}, \hat{y}) := T^b(\text{gph } M, (\hat{x}, \hat{y}))$.
- (iii) M is said to be proto-differentiable at (\hat{x}, \hat{y}) , iff $T(\text{gph } M, \hat{x}) = T^b(\text{gph } M, \hat{x})$, i.e., $\text{gph } M$ is derivable at (\hat{x}, \hat{y}) .

Next, we recall a generalized contingent cone called Φ -contingent cone introduced by Meng and Li [14].

Definition 2.6. [14] Let Q be a nonempty subset of X and $\hat{x} \in \text{cl } Q$. Consider a vector-valued map $\Phi : X \rightarrow Y$.

- (i) The set $T_\Phi(Q, \hat{x})$ is called the Φ -contingent cone of Q at \hat{x} , iff, for any $x \in T_\Phi(Q, \hat{x})$, there exist sequences $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $t_n \rightarrow 0$ and $\{x_n\} \subseteq X$ with $x_n \rightarrow \hat{x}$, such that $\frac{\Phi(x_n) - \Phi(\hat{x})}{t_n} \rightarrow x$, for all n .
- (ii) The set $T_\Phi^b(Q, \hat{x})$ is called the adjacent cone of Q at \hat{x} , iff, for any $x \in T_\Phi^b(Q, \hat{x})$ and any sequence $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $t_n \rightarrow 0$, there exists a sequence $\{x_n\} \subseteq X$ with $x_n \rightarrow \hat{x}$, such that $\frac{\Phi(x_n) - \Phi(\hat{x})}{t_n} \rightarrow x$, for all n .
- (iii) The set Q is Φ -derivable at \hat{x} , iff $T_\Phi(Q, \hat{x}) = T_\Phi^b(Q, \hat{x})$.

3. Differential Properties of a Class of Set-Valued Maps

In remainder sections, let K be a nonempty compact subset of a real normed vector space X , $F : X \rightarrow Y$ be a continuous vector-valued function and

$$G(x) := \langle F(x), x - K \rangle = \bigcup_{z \in K} \langle F(x), x - z \rangle.$$

In this section, we will discuss the differential properties of G . To this end, let $\hat{x} \in X$ and $\hat{F}(x) := \langle F(\hat{x}), \hat{x} - x \rangle$. If $(\hat{x}, \hat{y}) \in \text{gph } G$, we can define a nonempty compact subset $\Omega(\hat{x}, \hat{y})$ of K by

$$\Omega(\hat{x}, \hat{y}) := \{x \in K \mid \hat{F}(x) = \hat{y}\}.$$

At first, we give the following lemma used later.

Lemma 3.1. [12] Let X be a finite dimensional space, Q be a nonempty subset of X and $\hat{x}, \bar{x} \in K$. Suppose that the following coerciveness condition (see [11]) holds:

$$\lim_{\|z\| \rightarrow \infty, z \in X} \|\langle F(\hat{x}), z \rangle\| = \infty.$$

Then

$$T_\Phi(Q, \bar{x}) = - \bigcup_{x^* \in T(Q, \bar{x})} \langle F(\hat{x}), x^* \rangle.$$

The following theorem is the main result of this section.

Theorem 3.2. Let $(\hat{x}, \hat{y}) \in \text{gph } G$. Assume that F is Hadamard directionally differentiable at \hat{x} in any direction $x \in \text{dom } DG(\hat{x}, \hat{y})$. Then,

$$DG(\hat{x}, \hat{y})(x) = \langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + T_{\hat{F}}(K, \bar{x}) \right). \tag{3.1}$$

Proof. Suppose that $x \in \text{dom } DG(\hat{x}, \hat{y})$ and $y \in DG(\hat{x}, \hat{y})(x)$. Then, there exist sequences $\{(x_n, y_n)\} \subseteq X \times Y$ and $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $(x_n, y_n) \rightarrow (x, y)$ and $t_n \rightarrow 0$, such that

$$\hat{y} + t_n y_n \in G(\hat{x} + t_n x_n) = \bigcup_{z \in K} \langle F(\hat{x} + t_n x_n), \hat{x} + t_n x_n - z \rangle.$$

So, for every $n \in \mathbb{N}$, there exists a $\bar{x}_n \in K$, such that

$$\hat{y} + t_n y_n = \langle F(\hat{x} + t_n x_n), \hat{x} + t_n x_n - \bar{x}_n \rangle. \tag{3.2}$$

Since K is compact, without loss of generality, we can assume that $\bar{x}_n \rightarrow \bar{x} \in K$. From the continuity of F and (3.2), we have $\hat{y} = \langle F(\hat{x}), \hat{x} - \bar{x} \rangle = \hat{F}(\bar{x})$, that is $\bar{x} \in \Omega(\hat{x}, \hat{y})$ since $\bar{x} \in K$. Then, it follows from (3.2) that

$$y_n - \langle F(\hat{x} + t_n x_n), x_n \rangle = \frac{\langle F(\hat{x} + t_n x_n), \hat{x} - \bar{x}_n \rangle - \langle F(\hat{x}), \hat{x} - \bar{x} \rangle}{t_n}, \tag{3.3}$$

which follows that

$$y_n - \langle F(\hat{x} + t_n x_n), x_n \rangle = \frac{\langle F(\hat{x}), \hat{x} - \bar{x}_n \rangle - \langle F(\hat{x}), \hat{x} - \bar{x} \rangle}{t_n} + \left\langle \frac{F(\hat{x} + t_n x_n) - F(\hat{x})}{t_n}, \hat{x} - \bar{x}_n \right\rangle,$$

and hence

$$y_n - \langle F(\hat{x} + t_n x_n), x_n \rangle = \frac{\hat{F}(\bar{x}_n) - \hat{F}(\bar{x})}{t_n} + \left\langle \frac{F(\hat{x} + t_n x_n) - F(\hat{x})}{t_n}, \hat{x} - \bar{x}_n \right\rangle. \tag{3.4}$$

Obviously,

$$\lim_{n \rightarrow \infty} (y_n - \langle F(\hat{x} + t_n x_n), x_n \rangle) = y - \langle F(\hat{x}), x \rangle. \tag{3.5}$$

Moreover, by the Hadamard directional differentiability of F at \hat{x} in the direction x , we get

$$\lim_{n \rightarrow \infty} \left\langle \frac{F(\hat{x} + t_n x_n) - F(\hat{x})}{t_n}, \hat{x} - \bar{x}_n \right\rangle = \langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle. \tag{3.6}$$

Then, by (3.4), (3.5), (3.6) and the definition of \hat{F} -contingent cone, we have

$$y - \langle F(\hat{x}), x \rangle - \langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle \in T_{\hat{F}}(K, \bar{x}).$$

Therefore, we obtain that

$$y \in \langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + T_{\hat{F}}(K, \bar{x}) \right).$$

Conversely, suppose that

$$y \in F(\hat{x}, x) + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + T_{\hat{F}}(K, \bar{x}) \right).$$

Then, there exist $\bar{x} \in \Omega(\hat{x}, \hat{y})$ and $y^* \in T_{\hat{F}}(K, \bar{x})$ such that

$$y = \langle F(\hat{x}), x \rangle + \langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + y^*.$$

By $y^* \in T_{\hat{F}}(K, \bar{x})$, we obtain that there exist sequences $\{\bar{x}_n\} \subseteq K$ and $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $\bar{x}_n \rightarrow \bar{x}$ and $t_n \rightarrow 0$ such that

$$\frac{\hat{F}(\bar{x}_n) - \hat{F}(\bar{x})}{t_n} = \frac{\langle F(\hat{x}), \hat{x} - \bar{x}_n \rangle - \langle F(\hat{x}), \hat{x} - \bar{x} \rangle}{t_n} \rightarrow y^*.$$

On the other hand, considering an arbitrary sequence $\{x_n\} \subseteq X$ satisfying $x_n \rightarrow x$, we get

$$\left\langle \frac{F(\hat{x} + t_n x_n) - F(\hat{x})}{t_n}, \hat{x} - \bar{x}_n \right\rangle \rightarrow \langle d_H F(\hat{x}, \bar{x}; x), \hat{x} - \bar{x} \rangle.$$

Setting

$$y_n = \langle F(\hat{x} + t_n x_n), x_n \rangle + \left\langle \frac{F(\hat{x} + t_n x_n) - F(\hat{x})}{t_n}, \hat{x} - \bar{x}_n \right\rangle + \frac{\langle F(\hat{x}), \hat{x} - \bar{x}_n \rangle - \langle F(\hat{x}), \hat{x} - \bar{x} \rangle}{t_n}.$$

So, $y_n \rightarrow y$ and

$$\hat{y} + t_n y_n = \langle F(\hat{x} + t_n x_n), \hat{x} + t_n x_n - \bar{x}_n \rangle \in G(\hat{x} + t_n x_n).$$

These imply that

$$y \in DG(\hat{x}, \hat{y})(x).$$

This completes the proof. □

Remark 3.3. When X and Y are finite dimensional spaces, and F is Fréchet differentiable, Li and Zhai [12, Corollary 4.1] obtain a similar formula for contingent derivative of the set-valued map G . However, in Theorem 3.2, we obtain the formula for contingent derivative of G in the most general setting, namely, when the spaces X and Y are real normed spaces, and the map F is Hadamard directionally differentiable. Obviously, our results improve and generalize the corresponding results in Li and Zhai [12].

The following corollary is a generalization of Li, et al. [11, Theorem 3.1].

Corollary 3.4. *Let X be a finite dimensional space and $(\hat{x}, \hat{y}) \in \text{gph } G$. Assume that F is Hadamard directionally differentiable at \hat{x} in any direction $x \in \text{dom } DG(\hat{x}, \hat{y})$. If the following coerciveness condition (see [11]) holds:*

$$\lim_{\|z\| \rightarrow \infty, z \in X} \|\langle F(\hat{x}), z \rangle\| = \infty.$$

Then,

$$DG(\hat{x}, \hat{y})(x) = \langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle - \bigcup_{x^* \in T(Q, \bar{x})} \langle F(\hat{x}), x^* \rangle \right). \tag{3.7}$$

Proof. It follows directly from Lemma 3.1 and Theorem 3.2. □

Remark 3.5. If we replace directionally differentiability in the Hadamard sense with Fréchet differentiability in Corollary 3.4, then, the equality (3.7) becomes

$$DG(\hat{x}, \hat{y})(x) = \langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle \nabla F(\hat{x})x, \hat{x} - \bar{x} \rangle - \bigcup_{x^* \in T(Q, \bar{x})} \langle F(\hat{x}), x^* \rangle \right),$$

which is the result obtained by Li, et al. in [11, Theorem 3.1]. As we know, the directionally differentiability in the Hadamard sense is weaker than the Fréchet differentiability even in finite dimensional spaces. So, Corollary 3.4 is more general than [11, Theorem 3.1].

Similarly to the proof of Theorem 3.2, we obtain the following theorem which provides a formula for adjacent derivative of the set-valued map G .

Theorem 3.6. *Let $(\hat{x}, \hat{y}) \in \text{gph } G$. Assume that F is Hadamard directionally differentiable at \hat{x} in a direction $x \in \text{dom } DG(\hat{x}, \hat{y})$. If K is \hat{F} -derivable on $\Omega(\hat{x}, \hat{y})$, then,*

$$D^b G(\hat{x}, \hat{y})(x) = \langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + T_F^b(K, \bar{x}) \right).$$

From Theorems 3.2 and 3.6, we can easily obtain the following result. This result will play an important role in the following section.

Proposition 3.7. *Let $(\hat{x}, \hat{y}) \in \text{gph } G$. Assume that F is Hadamard directionally differentiable at \hat{x} in any direction $x \in \text{dom } DG(\hat{x}, \hat{y})$. If K is \hat{F} -derivable on $\Omega(\hat{x}, \hat{y})$, then,*

$$DG(\hat{x}, \hat{y})(x) = D^bG(\hat{x}, \hat{y})(x).$$

4. Sensitivity Properties of Weak Vector Variational Inequalities

In this section, we investigate the sensitivity properties of the gap function for a weak vector variational inequalities (WVVI), which is to find $x^* \in K$ such that

$$\langle F(x^*), x^* - x \rangle \notin \text{int } C, \forall x \in K.$$

At first, we recall the gap function for the weak vector variational inequalities.

Definition 4.1. [6] A set-valued map W defined from X to Y is said to be a gap function for the (WVVI), iff

- (a) $0 \in W(\hat{x})$ if and only if \hat{x} solves the (WVVI);
- (b) $W(x) \cap (-\text{int } C) = \emptyset, \forall x \in K$.

Consider a set-valued map $W : X \rightrightarrows Y$ defined by $W(x) := \text{WMax}(G(x), C)$. By Theorem 2 in [6], W is a gap function for the (WVVI). Now, in the remainder of this section, we investigate the relationship between contingent derivative DW of the set-valued map W and the contingent derivative DG of the set-valued map G .

Theorem 4.2. *Let $(\hat{x}, \hat{y}) \in \text{gph } W$. Assume that K is \hat{F} -derivable on $\Omega(\hat{x}, \hat{y})$, and F is Hadamard directionally differentiable at \hat{x} in any direction $x \in \text{dom } DW(\hat{x}, \hat{y})$. Then,*

$$DW(\hat{x}, \hat{y})(x) \subseteq \text{WMax}(DG(\hat{x}, \hat{y})(x), \text{int } C). \tag{4.1}$$

Proof. Let $y \in DW(\hat{x}, \hat{y})(x)$. Then, there exist sequences $\{t_n\} \subseteq \mathbb{R}_+ \setminus \{0\}$ with $t_n \rightarrow 0$ and $\{(x_n, y_n)\} \subseteq X \times Y$ with $(x_n, y_n) \rightarrow (x, y)$ such that

$$\hat{y} + t_n y_n \in W(\hat{x} + t_n x_n) \subseteq G(\hat{x} + t_n x_n). \tag{4.2}$$

So, $y \in DG(\hat{x}, \hat{y})(x)$. Suppose that $y \notin \text{WMax}(DG(\hat{x}, \hat{y})(x), \text{int } C)$. Then, there exists a $\bar{y} \in DG(\hat{x}, \hat{y})(x)$ such that

$$\bar{y} - y \in \text{int } C. \tag{4.3}$$

On the other hand, by Proposition 3.7, we have $\bar{y} \in D^bG(\hat{x}, \hat{y})(x)$. Then, for the above-mentioned sequence $\{t_n\}$, there exists a sequence $\{(\bar{x}_n, \bar{y}_n)\} \subseteq X \times Y$ with $(\bar{x}_n, \bar{y}_n) \rightarrow (\bar{x}, \bar{y})$ such that

$$\hat{y} + t_n \bar{y}_n \in G(\hat{x} + t_n \bar{x}_n) = \bigcup_{z \in K} \langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - z \rangle.$$

So, for every $n \in \mathbb{N}$, there exists a $x'_n \in K$, such that

$$\hat{y} + t_n \bar{y}_n = \langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - x'_n \rangle. \tag{4.4}$$

Since K is compact, without loss of generality, we can assume that $x'_n \rightarrow x' \in K$. From the continuity of F and (4.4), we have $\hat{y} = \langle F(\hat{x}), \hat{x} - x' \rangle = \hat{F}(x')$, that is $x' \in \Omega(\hat{x}, \hat{y})$. Then, it follows from (4.4) that

$$\begin{aligned} \hat{y} + t_n \left(\bar{y}_n - \frac{\langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - x'_n \rangle - \langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - x'_n \rangle}{t_n} \right) \\ = \langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - x'_n \rangle \in G(\hat{x} + t_n \bar{x}_n), \end{aligned} \tag{4.5}$$

Setting

$$\alpha(n) = \frac{\langle F(\hat{x} + t_n \bar{x}_n), \hat{x} + t_n \bar{x}_n - x'_n \rangle - \langle F(\hat{x}), \hat{x} - x'_n \rangle}{t_n}$$

and

$$\beta(n) = \frac{\langle F(\hat{x} + t_n x_n), \hat{x} + t_n x_n - x'_n \rangle - \langle F(\hat{x}), \hat{x} - x'_n \rangle}{t_n}.$$

Then, by (4.5), we get

$$\hat{y} + t_n (\bar{y}_n - \alpha(n) + \beta(n)) \in G(\hat{x} + t_n x_n), \quad (4.6)$$

By the definition of W , (4.2) and (4.6), we get

$$(\hat{y} + t_n (\bar{y}_n - \alpha(n) + \beta(n))) - (\hat{y} + t_n y_n) \notin \text{int } C,$$

and hence,

$$\bar{y}_n - \alpha(n) + \beta(n) - y_n \notin \text{int } C.$$

Moreover, by the Hadamard directional differentiability of F at \hat{x} in any direction x , we get $\alpha(n) - \beta(n) \rightarrow 0$. Thus, $\bar{y} - y \notin \text{int } C$, which contradicts (4.3). Therefore, $y \in \text{WMax}(DG(\hat{x}, \hat{y})(x), \text{int } C)$ and (4.1) holds. This completes the proof. \square

Remark 4.3. It is important to note that the result obtained in Theorem 4.2 has been investigated by Li, et al. [11, Theorem 4.1] under a coerciveness condition. However, the coerciveness condition is not necessary for Theorem 3.2, but essential for [11, Theorem 4.1]. So, our result is more general than [11, Theorem 4.1].

By Theorems 3.2 and 4.2, the following corollary is easily obtained.

Corollary 4.4. *Let $(\hat{x}, \hat{y}) \in \text{gph } W$. Assume that K is \hat{F} -derivable on $\Omega(\hat{x}, \hat{y})$, and F is Hadamard directionally differentiable at \hat{x} in any direction $x \in \text{dom } DW(\hat{x}, \hat{y})$. Then,*

$$DW(\hat{x}, \hat{y})(x) \subseteq \text{WMax} \left(\langle F(\hat{x}), x \rangle + \bigcup_{\bar{x} \in \Omega(\hat{x}, \hat{y})} \left(\langle d_H F(\hat{x}, x), \hat{x} - \bar{x} \rangle + T_{\hat{F}}(K, \bar{x}) \right), \text{int } C \right).$$

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