Weak and strong convergence of an explicit iteration process for an asymptotically quasi-i-nonexpansive mapping in Banach spaces

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Dedicated to George A Anastassiou on the occasion of his sixtieth birthday

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Abstract

In this paper, we prove the weak and strong convergence of an explicit iterative process to a common fixed point of an asymptotically quasi-\textit{I}-nonexpansive mapping \( T \) and an asymptotically quasi-nonexpansive mapping \( I \), defined on a nonempty closed convex subset of a Banach space.

Keywords: Asymptotically quasi-I-nonexpansive self-mappings, explicit iterations, common fixed point, uniformly convex Banach space.

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1. Introduction

Let \( K \) be a nonempty subset of a real normed linear space \( X \) and let \( T: K \to K \) be a mapping. Denote by \( F(T) \) the set of fixed points of \( T \), that is, \( F(T) = \{ x \in K : Tx = x \} \) and we denote by \( D(T) \) the domain of a mapping \( T \). Throughout this paper, we assume that \( X \) is a real Banach space and \( F(T) \neq \emptyset \). Now, we recall some well-known concepts and results.

\textbf{Definition 1.1.} A mapping \( T: K \to K \) is said to be

1. nonexpansive, if \( \| Tx - Ty \| \leq \| x - y \| \) for all \( x, y \in K \).

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2. asymptotically nonexpansive, if there exists a sequence \( \{\lambda_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} \lambda_n = 1 \) such that 
\[ \|T^n x - T^n y\| \leq \lambda_n \|x - y\| \] for all \( x, y \in K \) and \( n \in \mathbb{N} \);
3. quasi-nonexpansive, if \( \|Tx - p\| \leq \|x - p\| \) for all \( x \in K \), \( p \in F(T) \);
4. asymptotically quasi-nonexpansive, if there exists a sequence \( \{\mu_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} \mu_n = 1 \) such that 
\[ \|T^n x - p\| \leq \mu_n \|x - p\| \] for all \( x \in K \), \( p \in F(T) \) and \( n \in \mathbb{N} \).

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space \( H \) : if \( K \) is a closed and convex subset of \( H \) and \( T \) has a fixed point, then every \( x \in K \), \( \{T^n x\} \) is weakly almost convergent, as \( n \to \infty \), to a fixed point of \( T \). It was also shown by Pazy [2] that if \( H \) is a real Hilbert space and \( \frac{1}{n} \sum_{i=0}^{n-1} T^i x \) converges weakly, as \( n \to \infty \), to \( y \in K \), then \( y \in F(T) \).


There are many concepts which generalize a notion of nonexpansive mapping. One of such concepts is \( I \)-nonexpansivity of a mapping \( T \) [20]. Let us recall some notions.

**Definition 1.2.** Let \( T : K \to K \), \( I : K \to K \) be two mappings of nonempty subset \( K \) of a real normed linear space \( X \). Then \( T \) is said to be

1. \( I \)-nonexpansive, if \( \|Tx - Ty\| \leq \|Ix - Iy\| \) for all \( x, y \in K \);
2. asymptotically \( I \)-nonexpansive, if there exists a sequence \( \{\lambda_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} \lambda_n = 1 \) such that 
\[ \|T^n x - T^n y\| \leq \lambda_n \|I^n x - I^n y\| \] for all \( x, y \in K \) and \( n \geq 1 \);
3. quasi-\( I \)-nonexpansive, if \( \|Tx - p\| \leq \|Ix - p\| \) for all \( x \in K \), \( p \in F(T) \cap F(I) \);
4. asymptotically quasi-\( I \)-nonexpansive, if there exists a sequence \( \{\mu_n\} \subset [1, \infty) \) with \( \lim_{n \to \infty} \mu_n = 1 \) such that 
\[ \|T^n x - p\| \leq \mu_n \|I^n x - p\| \] for all \( x \in K \), \( p \in F(T) \cap F(I) \) and \( n \geq 1 \).

**Remark 1.3.** If \( F(T) \cap F(I) \neq \emptyset \) then an asymptotically \( I \)-nonexpansive mapping is asymptotically quasi-\( I \)-nonexpansive.

Best approximation properties of \( I \)-nonexpansive mappings were investigated in [20]. In [21] strong convergence of Mann iterations of \( I \)-nonexpansive mapping has been proved. In [22] the weak and strong convergence of implicit iteration process to a common fixed point of a finite family of \( I \)-asymptotically nonexpansive mappings were proved. In [23] the weak convergence theorems of three-step iterative scheme for an \( I \)-asymptotically nonexpansive mappings in a Banach space has been studied. In [24] a weak convergence theorem for \( I \)-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. Mukhamedov and Saburov [25] studied weak and strong convergence of an implicit iteration process for an asymptotically quasi-\( I \)-nonexpansive mapping in Banach space. In [28] Mukhamedov and Saburov studied strong convergence of an explicit iteration process for a totally asymptotically \( I \)-nonexpansive mapping in Banach spaces. This iteration scheme is defined as follows.

Let \( K \) be a nonempty closed convex subset of a real Banach space \( X \). Consider \( T : K \to K \) an asymptotically quasi-\( I \)-nonexpansive mapping, where \( I : K \to K \) an asymptotically quasi-\( I \)-nonexpansive mapping. Then for two given sequences \( \{\alpha_n\} \), \( \{\beta_n\} \) in \([0, 1]\) we shall consider the following iteration scheme:

\begin{align*}
&x_n \in K, \\
&x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, \ n \geq 0,
\end{align*}

\begin{align*}
&y_n = (1 - \beta_n) x_n + \beta_n I^n x_n. 
\end{align*}

(1.1)
Inspired and motivated by these facts, we study the convergence of an explicit iterative involving an
asymptotically quasi-\(I\)-nonexpansive mapping in nonempty closed convex subset of uniformly convex Banach
spaces.

In this paper, we prove weak and strong convergences of an explicit iterative process \((1.1)\) to a common
fixed point of \(T\) and \(I\).

2. Preliminaries

Recall that a Banach space \(X\) is said to satisfy Opial condition \([25]\) if, for each sequence \(\{x_n\}\) in \(X\) such
that \(\{x_n\}\) converges weakly to \(x\) implies that
\[
\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|
\]
for all \(y \in X\) with \(y \neq x\). It is well known that (see [26]) inequality \((2.1)\) is equivalent to
\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|. \tag{2.2}
\]

Definition 2.1. Let \(K\) be a closed subset of a real Banach space \(X\) and let \(T : K \to K\) be a mapping.

1. A mapping \(T\) is said to be semiclosed (demiclosed) at zero, if for each bounded sequence \(\{x_n\}\) in \(K\),
   the conditions \(x_n\) converges weakly to \(x \in K\) and \(Tx_n\) converges strongly to 0 imply \(Tx = 0\).
2. A mapping \(T\) is said to be semicompact, if for any bounded sequence \(\{x_n\}\) in \(K\) such that \(\|x_n - Tx_n\| \to 0, n \to \infty\),
   then there exists a subsequence \(\{x_{n_k}\} \subset \{x_n\}\) such that \(x_{n_k} \to x^* \in K\) strongly.
3. \(T\) is called a uniformly \(L\)-Lipschitzian mapping, if there exists a constant \(L > 0\) such that \(\|T^n x - T^m y\| \leq L \|x - y\|\) for all \(x, y \in K\) and \(n \geq 1\).

Lemma 2.2. \([17]\) Let \(X\) be a uniformly convex Banach space and let \(b, c\) be two constant with \(0 < b < c < 1\).
Suppose that \(\{t_n\}\) is a sequence in \([b, c]\) and \(\{x_n\}, \{y_n\}\) are two sequence in \(X\) such that
\[
\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = d, \quad \limsup_{n \to \infty} \|x_n\| \leq d, \quad \limsup_{n \to \infty} \|y_n\| \leq d, \tag{2.3}
\]
holds some \(d \geq 0\). Then \(\lim_{n \to \infty} \|x_n - y_n\| = 0\).

Lemma 2.3. \([19]\) Let \(\{a_n\}\) and \(\{b_n\}\) be two sequences of nonnegative real numbers with \(\sum_{n=1}^{\infty} b_n < \infty\). If
one of the following conditions is satisfied:

1. \(a_{n+1} \leq a_n + b_n, \quad n \geq 1,\)
2. \(a_{n+1} \leq (1 + b_n) a_n, \quad n \geq 1,\)

then the limit \(\lim_{n \to \infty} a_n\) exists.

3. Main Results

In this section, we prove convergence theorems of an explicit iterative scheme \((1.1)\) for an asymptotically
quasi-\(I\)-nonexpansive mapping in Banach spaces. In order to prove our main results, the following lemmas
are needed.

Lemma 3.1. Let \(X\) be a real Banach space and let \(K\) be a nonempty closed convex subset of \(X\). Let
\(T : K \to K\) be an asymptotically quasi-\(I\)-nonexpansive mapping with a sequence \(\{\lambda_n\} \subset \{1, \infty\}\) and \(I : K \to K\)
asymptotically quasi-nondecreasing mapping with a sequence \(\{\mu_n\} \subset \{1, \infty\}\) such that \(F = F(T) \cap F(I) \neq \emptyset\).
Suppose \(N = \limsup \lambda_n \geq 1, M = \limsup \mu_n \geq 1\) and \(\{\alpha_n\}, \{\beta_n\}\) are two sequences in \([0, 1]\)
such that \(\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) a_n < \infty\). If \(\{x_n\}\) is an explicit iterative sequence defined by \((1.1)\), then for each
\(p \in F = F(T) \cap F(I)\) the limit \(\lim_{n \to \infty} \|x_n - p\|\) exists.
Proof. Since $p \in F = F(T) \cap F(I)$, for any given $p \in F$, it follows (1.1) that
\[
\|x_{n+1} - p\| \leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \|T^ny_n - p\|
\]
\[
\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \lambda_n \|I^ny_n - p\|
\]
\[
\leq (1 - \alpha_n) \|x_n - p\| + \alpha_n \lambda_n \mu_n \|y_n - p\|.
\] (3.1)

Again from (1.1) we derive that
\[
\|y_n - p\| \leq (1 - \beta_n) \|x_n - p\| + \beta_n \|I^nx_n - p\|
\]
\[
\leq (1 - \beta_n) \|x_n - p\| + \beta_n \mu_n \|x_n - p\|
\]
\[
\leq (1 - \beta_n) \mu_n \|x_n - p\| + \beta_n \mu_n \|x_n - p\|
\]
\[
\leq \mu_n \|x_n - p\|,
\] (3.2)

which means
\[
\|y_n - p\| \leq \mu_n \|x_n - p\| \leq \lambda_n \mu_n \|x_n - p\|.
\] (3.3)

Then from (3.3) we have
\[
\|x_{n+1} - p\| \leq \|1 + \alpha_n (\lambda_n^2 \mu_n^2 - 1)\| \|x_n - p\|.
\] (3.4)

By putting $b_n = \alpha_n (\lambda_n^2 \mu_n^2 - 1)$ the last inequality can be rewritten as follows:
\[
\|x_{n+1} - p\| \leq (1 + b_n) \|x_n - p\|.
\] (3.5)

By hypothesis we find
\[
\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \alpha_n (\lambda_n^2 \mu_n^2 - 1)
\]
\[
= \sum_{n=1}^{\infty} (\lambda_n \mu_n + 1) (\lambda_n \mu_n - 1) \alpha_n
\]
\[
\leq (NM + 1) \sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty.
\]

Defining $a_n = \|x_n - p\|$ in (3.5) we have
\[
a_{n+1} \leq (1 + b_n) a_n,
\] (3.6)

and Lemma [2,3] implies the existence of the limit $\lim_{n \to \infty} a_n$. The means the limit
\[
\lim_{n \to \infty} \|x_n - p\| = d
\] (3.7)

exists, where $d \geq 0$ constant. This completes the proof. \hfill \Box

**Theorem 3.2.** Let $X$ be a real Banach space and let $K$ be a nonempty closed convex subset of $X$. Let $T : K \to K$ be a uniformly $L_1$-Lipschitzian asymptotically quasi-$I$-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$ and $I : K \to K$ be a uniformly $L_2$-Lipschitzian asymptotically quasi-$I$-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}, \{\beta_n\}$ are two sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. Then an explicit iterative sequence $\{x_n\}$ defined by (1.1) converges strongly to a common fixed point in $F = F(T) \cap F(I)$ if and only if
\[
\lim_{n \to \infty} \inf d(x_n, F) = 0.
\] (3.8)
Proof. The necessity of condition (3.7) is obvious. Let us prove the sufficiency part of theorem. Since $T, I : K \to K$ are uniformly $L$-Lipschitzian mappings, so $T$ and $I$ are continuous mappings. Therefore the sets $F(T)$ and $F(I)$ are closed. Hence $F = F(T) \cap F(I)$ is a nonempty closed set.

For any given $p \in F$, we have
\[
\|x_{n+1} - p\| \leq (1 + b_n) \|x_n - p\|,
\]
which means that the strong convergence limit of the sequence $\{x_n\}$ is $p$.

From (3.10) due to Lemma 2.3, we obtain the existence of the limit $\lim_{n \to \infty} d(x_n, F)$. By condition (3.7), we get
\[
\lim_{n \to \infty} d(x_n, F) = \liminf_{n \to \infty} d(x_n, F) = 0.
\]
Let us prove that the sequence $\{x_n\}$ converges to a common fixed point of $T$ and $I$. In fact, due to $1 + t \leq \exp(t)$ for all $t > 0$, and from (3.8), we obtain
\[
\|x_{n+1} - p\| \leq \exp(b_n) \|x_n - p\|.
\]
Hence, for any positive integers $m, n$ from (3.11) with $\sum_{n=1}^{\infty} b_n < \infty$ we find
\[
\|x_{n+m} - p\| \leq \exp(b_{n+m-1}) \|x_{n+m-1} - p\| \leq \exp\left(\sum_{i=n}^{n+m-1} b_i\right) \|x_n - p\| \leq \exp\left(\sum_{i=1}^{\infty} b_i\right) \|x_n - p\|,
\]
which means that
\[
\|x_{n+m} - p\| \leq W \|x_{n} - p\|
\]
for all $p \in F$, where $W = \exp\left(\sum_{i=1}^{\infty} b_i\right) < \infty$.

Since $\lim_{n \to \infty} d(x_n F) = 0$, then for any given $\varepsilon > 0$, there exists a positive integer number $n_0$ such that
\[
d(x_{n_0}, F) < \frac{\varepsilon}{W}.
\]
Therefore there exists $p_1 \in F$ such that
\[
\|x_{n_0} - p_1\| < \frac{\varepsilon}{W}.
\]
Consequently, for all $n \geq n_0$ from (3.14) we derive
\[
\|x_n - p_1\| \leq W \|x_{n_0} - p_1\| < W \cdot \frac{\varepsilon}{W} = \varepsilon,
\]
which means that the strong convergence limit of the sequence $\{x_n\}$ is a common fixed point $p_1$ of $T$ and $I$. This completes the proof.

Lemma 3.3. Let $X$ be a real uniformly Banach space and let $K$ be a nonempty closed convex subset of $X$. Let $T : K \to K$ be a uniformly $L_1$-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$ and $I : K \to K$ be a uniformly $L_2$-Lipschitzian asymptotically quasi-
nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[t, 1-t]$ for some $t \in (0, 1)$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. Then an explicit iterative sequence $\{x_n\}$ defined by (3.1) satisfies the following:
\[
\lim_{n \to \infty} \|x_n - Tx_n\| = 0, \quad \lim_{n \to \infty} \|x_n - Ix_n\| = 0.
\]
Proof. First, we will prove that
\[
\lim_{n \to \infty} \| x_n - T^n x_n \| = 0, \quad \lim_{n \to \infty} \| x_n - I^n x_n \| = 0. \tag{3.19}
\]
According to Lemma 3.1 for any \( p \in F = F(T) \cap F(I) \) we have \( \lim_{n \to \infty} \| x_n - p \| = d \). It follows from (1.1) that
\[
\| x_{n+1} - p \| = \| (1 - \alpha_n) (x_n - p) + \alpha_n (T^n y_n - p) \| \to d, \quad n \to \infty. \tag{3.20}
\]
By means of asymptotically quasi-\( I \)-nonexpansivity of \( T \) and asymptotically quasi-nonexpansivity of \( I \) from (3.3) we get
\[
\lim_{n \to \infty} \sup_{x \in \mathcal{W}} \| T^n y_n - p \| \leq \lim_{n \to \infty} \sup_{x \in \mathcal{W}} \lambda_n \mu_n \| y_n - p \| \leq \lim_{n \to \infty} \lambda_n^2 \mu_n^2 \| x_n - p \| = d. \tag{3.31}
\]
Now using
\[
\lim_{n \to \infty} \sup_{x \in \mathcal{W}} \| x_n - p \| = d, \tag{3.22}
\]
with (3.21) and applying Lemma 2.2 to (3.20) we obtain
\[
\lim_{n \to \infty} \| x_n - T^n y_n \| = 0. \tag{3.23}
\]
Now from (1.1) and (3.22) we infer that
\[
\lim_{n \to \infty} \| x_{n+1} - x_n \| = \lim_{n \to \infty} \| \alpha_n (T^n y_n - x_n) \| = 0. \tag{3.24}
\]
From (3.23) and (3.24) we get
\[
\lim_{n \to \infty} \| x_{n+1} - T^n y_n \| \leq \lim_{n \to \infty} \| x_{n+1} - x_n \| + \lim_{n \to \infty} \| x_n - T^n y_n \| = 0. \tag{3.25}
\]
On the other hand, we have
\[
\| x_n - p \| \leq \| x_n - T^n y_n \| + \| T^n y_n - p \|
\leq \| x_n - T^n y_n \| + \lambda_n \mu_n \| y_n - p \|, \tag{3.26}
\]
which implies
\[
\| x_n - p \| - \| x_n - T^n y_n \| \leq \lambda_n \mu_n \| y_n - p \|. \tag{3.27}
\]
The last inequality with (3.3) yields that
\[
\| x_n - p \| - \| x_n - T^n y_n \| \leq \lambda_n \mu_n \| y_n - p \| \leq \lambda_n^2 \mu_n^2 \| x_n - p \|. \tag{3.28}
\]
Then (3.22) and (3.23) with the Squeeze Theorem imply that
\[
\lim_{n \to \infty} \| y_n - p \| = d. \tag{3.29}
\]
Again from (1.1) we can see that
\[
\| y_n - p \| = \| (1 - \beta_n) (x_n - p) + \beta_n (I^n x_n - p) \| \to \infty, \quad n \to \infty. \tag{3.30}
\]
From (3.7) one finds
\[
\lim_{n \to \infty} \sup_{x \in \mathcal{W}} \| I^n x_n - p \| \leq \lim_{n \to \infty} \sup_{x \in \mathcal{W}} \mu_n \| x_n - p \| = d. \tag{3.31}
\]
Now applying Lemma 2.2 to (3.29) we obtain
\[
\lim_{n \to \infty} \| x_n - I^n x_n \| = 0. \tag{3.32}
\]
From (3.24) and (3.32) we have
\[
\lim_{n \to \infty} \|x_{n+1} - I^a x_n\| \leq \lim_{n \to \infty} \|x_{n+1} - x_n\| + \lim_{n \to \infty} \|x_n - I^a x_n\| = 0. \tag{3.33}
\]

It follows from (1.1) that
\[
\|y_n - x_n\| = \beta_n \|x_n - I^a x_n\|. \tag{3.34}
\]

Hence, from (3.32) and (3.34) we obtain
\[
\lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.35}
\]

Consider
\[
\|x_n - T^a x_n\| \leq \|x_n - T^a y_n\| + L_1 \|y_n - x_n\|. \tag{3.36}
\]

Then from (3.23) and (3.35) we obtain
\[
\lim_{n \to \infty} \|x_n - T^a x_n\| = 0. \tag{3.37}
\]

From (3.24) and (3.35) we have
\[
\lim_{n \to \infty} \|x_{n+1} - y_n\| \leq \lim_{n \to \infty} \|x_{n+1} - x_n\| + \lim_{n \to \infty} \|y_n - x_n\| = 0. \tag{3.38}
\]

Finally, from
\[
\|x_n - T x_n\| \leq \|x_n - T^a x_n\| + L_1 \|x_n - y_{n-1}\| + L_1 \|T^{n-1} y_{n-1} - x_n\|, \tag{3.39}
\]

which with (3.25), (3.37) and (3.38) we get
\[
\lim_{n \to \infty} \|x_n - T x_n\| = 0. \tag{3.40}
\]

Similarly, one has
\[
\|x_n - I x_n\| \leq \|x_n - I^a x_n\| + L_2 \|x_n - x_{n-1}\| + L_2 \|I^{n-1} x_{n-1} - x_n\|, \tag{3.41}
\]

which with (3.24), (3.32) and (3.33) implies
\[
\lim_{n \to \infty} \|x_n - I x_n\| = 0. \tag{3.42}
\]

This completes the proof. \hfill \Box

**Theorem 3.4.** Let $X$ be a real uniformly convex Banach space satisfying Opial condition and let $K$ be a nonempty closed convex subset of $X$. Let $E : X \to X$ be an identity mapping, let $T : K \to K$ be a uniformly $L_1$-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$, and $I : K \to K$ be a uniformly $L_2$-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $F = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_n \lambda_n \geq 1$, $M = \lim_n \mu_n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[t, 1-t]$ for some $t \in (0, 1)$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. If the mappings $E - T$ and $E - I$ are semiclosed at zero, then an explicit iterative sequence $\{x_n\}$ defined by (1.1) converges weakly to a common fixed point of $T$ and $I$.

**Proof.** Let $p \in F$, then according to Lemma 3.1 the sequence $\{\|x_n - p\|\}$ converges. This provides that $\{x_n\}$ is a bounded sequence. Since $X$ is uniformly convex, then every bounded subset of $X$ is weakly compact. Since $\{x_n\}$ is a bounded sequence in $K$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q \in K$. Hence, from (3.40) and (3.42) it follows that
\[
\lim_{n_k \to \infty} \|x_{n_k} - T x_{n_k}\| = 0, \quad \lim_{n_k \to \infty} \|x_{n_k} - I x_{n_k}\| = 0. \tag{3.43}
\]
Since the mappings $E - T$ and $E - I$ are semiclosed at zero, therefore, we find $Tq = q$ and $Iq = q$, which means $q \in \mathcal{F} = F(T) \cap F(I)$.

Finally, let us prove that $\{x_n\}$ converges weakly to $q$. In fact, suppose the contrary, that is, there exists some subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q_1 \in K$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in \mathcal{F} = F(T) \cap F(I)$.

Taking $p = q$ and $p = q_1$ and using the same argument given in the proof of (3.7), we can prove that the limits $\lim_{n \to \infty} \|x_n - q\|$ and $\lim_{n \to \infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n \to \infty} \|x_n - q\| = d, \quad \lim_{n \to \infty} \|x_n - q_1\| = d_1,$$

(3.44)

where $d$ and $d_1$ are two nonnegative numbers. By virtue of the Opial condition of $X$, we obtain

$$d = \lim_{n_k \to \infty} \sup \|x_{n_k} - q\| \leq \lim_{n_k \to \infty} \sup \|x_{n_k} - q_1\| = d_1$$

$$= \lim_{n_j \to \infty} \sup \|x_{n_j} - q_1\| \leq \lim_{n_j \to \infty} \sup \|x_{n_j} - q\|.$$

(3.45)

This is a contradiction. Hence $q_1 = q$. This implies that $\{x_n\}$ converges weakly to $q$. This completes the proof.

**Theorem 3.5.** Let $X$ be a real uniformly convex Banach space and let $K$ be a nonempty closed convex subset of $X$. Let $T : K \to K$ be a uniformly $L_1$-Lipschitzian asymptotically quasi-I-nonexpansive mapping with a sequence $\{\lambda_n\} \subset [1, \infty)$, and $I : K \to K$ be a uniformly $L_2$-Lipschitzian asymptotically quasi-nonexpansive mapping with a sequence $\{\mu_n\} \subset [1, \infty)$ such that $\mathcal{F} = F(T) \cap F(I) \neq \emptyset$. Suppose $N = \lim_{n \to \infty} \lambda_n \geq 1$, $M = \lim_{n \to \infty} \mu_n \geq 1$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[t, 1 - t]$ for some $t \in (0, 1)$ such that $\sum_{n=1}^{\infty} (\lambda_n \mu_n - 1) \alpha_n < \infty$. If at least one mapping of the mappings $T$ and $I$ is semicompact, then an explicit iterative sequence $\{x_n\}$ defined by (3.1) converges strongly to a common fixed point of $T$ and $I$.

**Proof.** Without any loss of generality, we may assume that $T$ is semicompact. This with (3.40) means that there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^*$ strongly and $x^* \in K$. Since $T$, $I$ are continuous, then from (3.40) and (3.42) we find

$$\|x^* - Tx^*\| = \lim_{n_k \to \infty} \|x_{n_k} - Tx_{n_k}\| = 0, \quad \|x^* - Ix^*\| = \lim_{n_k \to \infty} \|x_{n_k} - Ix_{n_k}\| = 0.$$ 

(3.46)

This shows that $x^* \in \mathcal{F} = F(T) \cap F(I)$. According to Lemma 3.1 the limit $\lim_{n \to \infty} \|x_n - x^*\|$ exists. Then

$$\lim_{n \to \infty} \|x_n - x^*\| = \lim_{n_k \to \infty} \|x_{n_k} - x^*\| = 0,$$

which means that $\{x_n\}$ converges to $x^* \in \mathcal{F}$. This completes the proof. \qed

**References**


