Some new results on complete $U^*_n$-metric space

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Abstract

In this paper, we give some new definitions of $U^*_n$-metric spaces and we prove a common fixed point theorem for two mappings under the condition of weakly compatible and establish common fixed point for sequence of generalized contraction mappings in complete $U^*_n$-metric space. ©2013 All rights reserved.

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1. Introduction and Preliminaries

Recently Sedghi et. al. \cite{11} introduced the concept of $D^*$-metric spaces and proved some common fixed point theorems (see also \cite{3, 12}). In the present work, we introduce a new notion of generalized $D^*$-metric space called $U^*$-metric space of dimension $n$ and study some fixed point results for two self-mappings $f$ and $g$ on $U^*_n$-metric spaces. Some fundamental properties of the proposed metric are studied.

**Definition 1.1.** \cite{2} Let $G$ be an ordered group. An ordered group metric (or OG-metric ) on a nonempty set $X$ is a symmetric nonnegative function $d_G$ from $X \times X$ into $G$ such that $d_G(x, y) = 0$ if and only if $x = y$ and such that the triangle inequality is satisfied; the pair $(X, d_G)$ is an ordered group metric space (or OG-metric space).

For $n \geq 2$, let $X^n$ denotes the cartesian product $X \times \ldots \times X$ and $\mathbb{R}^+ = [0, +\infty)$. We begin with the following definition.

**Definition 1.2.** Let $X$ be a non-empty set. Let $U^*_n : X^n \to G^+$ be a function that satisfies the following conditions:
(U1) $U_n^*(x_1, \ldots, x_n) = 0$ if $x_1 = \ldots = x_n$,

(U2) $U_n^*(x_1, \ldots, x_n) > 0$ for all $x_1, \ldots, x_n$ with $x_i \neq x_j$, for some $i, j \in \{1, \ldots, n\}$,

(U3) $U_n^*(x_1, \ldots, x_n) = U_n^*(x_{\pi_1}, \ldots, x_{\pi_n})$, for every permutation $(\pi(1), \ldots, \pi(n))$ of $(1, 2, \ldots, n)$,

(U4) $U_n^*(x_1, x_2, \ldots, x_n) \leq U_n^*(x_1, \ldots, x_{n-1}, a) + U_n^*(a, x_n, \ldots, x_n)$, for all $x_1, \ldots, x_n, a \in X$.

The function $U_n^*$ is called a universal ordered group metric of dimension $n$, or more specifically an $OU_n^*$-metric on $X$, and the pair $(X, U_n^*)$ is called an $OU_n^*$-metric space.

For example we can place $G^+ = \mathbb{Z}^+$ or $\mathbb{R}^+$. In the sequel, for simplicity we assume that $G^+ = \mathbb{R}^+$.

**Example 1.3.** (a) Let $(X, d)$ be a usual metric space, then $(X, S_n)$ and $(X, M_n)$ are $U_n^*$-metric spaces, where

$$S_n(x_1, \ldots, x_n) = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} d(x_i, x_j),$$

$$M_n(x_1, \ldots, x_n) = \max\{d(x_i, x_j) : 1 \leq i < j \leq n\}.$$

(b) Let $\phi$ be a non-decreasing and concave function with $\phi(0) = 0$. If $(X, d)$ is a usual metric space, then $(X, \phi_n)$ defined by

$$\phi_n(x_1, \ldots, x_n) = \phi^{-1}\left(\sum_{1 \leq i < j \leq n} \phi(d(x_i, x_j))\right)$$

is a $U_n^*$-metric.

(c) Let $X = C([0, T])$ be the set of all continuous functions defined on $[0, T]$. Defined $I_n : X^n \rightarrow \mathbb{R}^+$ by

$$I_n(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} \sup_{t \in [0, T]} |x_i(t) - x_j(t)|.$$

Then $(X, I_n)$ is a $U_n^*$-metric space.

(d) Let $X = \mathbb{R}^n$ defined $L_n : \mathbb{R}^n \rightarrow \mathbb{R}^+$ by

$$L_n(x_1, \ldots, x_n) = \sum_{1 \leq i < j \leq n} \|x_i - x_j\|^{\frac{1}{r}}$$

For every $r \in \mathbb{R}^+$. Then $(X, L_n)$ is a $U_n^*$-metric space.

(e) Let $X = \mathbb{R}$ defined $K_n : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$K_n(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_1 = \cdots = x_n \\ \max\{x_1, \cdots, x_n\} & \text{otherwise} \end{cases}$$

Then $(X, K_n)$ is a $U_n^*$-metric space.

**Remark 1.4.** In a $U_n^*$-metric space, we prove that $U^*(x, \ldots, x, y) = U^*(x, y, \ldots, y)$. For

(i) $U^*(x, \ldots, x, y) \leq U^*(x, \ldots, x) + U^*(x, y, \ldots, y) = U^*(x, y, \ldots, y)$ and similary

(ii) $U^*(y, \ldots, y, x) \leq U^*(y, \ldots, y) + U^*(y, x, \ldots, x) = U^*(y, x, \ldots, x)$.

Hence by (i),(ii) we get $U^*(x, \ldots, x, y) = U^*(x, y, \ldots, y)$.

**Proposition 1.5.** Let $(X, U)$ and $(Y, V)$ be two $U_n^*$-metric spaces. Then $(Z, W)$ is also a $U_n^*$-metric space, where $Z = X \times Y$ and $W(z_1, \ldots, z_n) = \max\{U(x_1, \ldots, x_n), V(y_1, \ldots, y_n)\}$ for $z_i = (x_i, y_i) \in Z$ with $x_i \in X, y_i \in Y, i = 1, \ldots, n$. 
Definition 1.6. A \( U^*_n \)-metric space \( X \) is said to be bounded if there exists a constant \( M > 0 \) such that \( U_n^*(x_1, ..., x_n) \leq M \) for all \( x_1, ..., x_n \in X \). A \( U_n^* \)-metric space \( X \) is said to be unbounded if it is not bounded.

Proposition 1.7. Let \((X, U^*_n)\) be a \( U^*_n \)-metric space and let \( M > 0 \) be a fixed positive real number. Then \((X, V)\) is a bounded \( U^*_n \)-metric space with bound \( M \), where the function \( V \) is given by

\[
V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))}
\]

for all \( x_1, ..., x_n \in X \) and with \( k > 0 \).

Proof. Obviously (U1-U3) conditions are satisfied. We only prove the (U4) inequality. Let \( x_1, ..., x_n \in X \),

\[
V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))} = M - \frac{Mk}{(k + U^*(x_1, ..., x_n))} \leq M - \frac{Mk}{(k + U^*(x_1, ..., x_n-1, a) + U^*(a, x_n, ..., x_n))}
\]

\[
= M - \frac{Mk}{(k + U^*(x_1, ..., x_n-1, a) + U^*(a, x_n, ..., x_n))}
\]

\[
= \frac{M(U^*(x_1, ..., x_n-1, a))}{(k + U^*(x_1, ..., x_n-1, a) + U^*(a, x_n, ..., x_n))}
\]

\[
+ \frac{M(U^*(x_1, ..., x_n-1, a))}{(k + U^*(x_1, ..., x_n-1, a) + U^*(a, x_n, ..., x_n))}
\]

\[
\leq M(U^*(x_1, ..., x_{n-1}, a)) + \frac{M(U^*(a, x_n, ..., x_n))}{(k + U^*(a, x_n, ..., x_n))}
\]

\[
= V(x_1, ..., x_{n-1}, a) + V(a, x_n, ..., x_n).
\]

Hence \((X, V)\) is a \( U^*_n \)-metric space.

Let \( x_1, ..., x_n \in X \). Then we have,

\[
V(x_1, ..., x_n) = \frac{MU^*(x_1, ..., x_n)}{(k + U^*(x_1, ..., x_n))} \leq \frac{MU^*(x_1, ..., x_n)}{(U^*(x_1, ..., x_n))} = M
\]

This show that \((X, V)\) is bounded with \( U^*_n \)-bound \( M \).

Definition 1.8. Let \((X, U^*_n)\) be a \( U^*_n \)-metric space, then for \( x_0 \in X \), \( r > 0 \), the \( U^*_n \)-ball with center \( x_0 \) and radius \( r \) is

\[
B_{U^*_n}(x_0, r) = \{ y \in X : U^*_n(x_0, y, ..., y) < r \}.
\]
Definition 1.9. Let \((X, U^*_n)\) be a \(U^*_n\)-metric space and \(Y \subset X\).

(1) If for every \(y \in Y\) there exist \(r > 0\) such that \(B_U^*(y, r) \subset Y\), then subset \(Y\) is called open subset of \(X\).

(2) Subset \(Y\) of \(X\) is said to be \(U^*_n\)-bounded if there exists \(r > 0\) such that \(U^*(x, y, ..., y) < r\) for all \(x, y \in Y\).

(3) A sequence \(\{x_k\}\) in \(X\) converges to \(x\) if and only if

\[
U^*(x_k, ..., x_k, x) = U^*(x, ..., x, x_k) \to 0 \quad \text{as} \quad k \to \infty.
\]

That is for each \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that

\[
\forall k \geq N \implies U^*(x, ..., x, x_k) < \varepsilon \quad (\ast).
\]

This is equivalent with, for each \(\varepsilon > 0\) there exists \(N \in \mathbb{N}\) such that

\[
\forall l_1, ..., l_{n-1} \geq N \implies U^*(x, x_{l_1}, ..., x_{l_{n-1}}) < \varepsilon \quad (\ast\ast).
\]

(4) Let \((X, U^*_n)\) be a \(U^*_n\)-metric space, then a sequence \(\{x_k\} \subseteq X\) is said to be \(U^*_n\)-Cauchy if for every \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(U^*_n(x_k, x_m, ..., x_l) < \varepsilon\) for all \(k, m, ..., l \geq N\). The \(U^*_n\)-metric space \((X, U^*_n)\) is said to be complete if every \(U^*_n\) Cauchy sequence is convergent.

Remark 1.10. (i) Let \(\tau\) be the set of all \(Y \subset X\) with \(y \in Y\) if and only if there exists \(r > 0\) such that \(B_U^*(y, r) \subset Y\). Then \(\tau\) is a topology on \(X\) induced by the \(U^*_n\)-metric.

(ii) If have (*) of Definition 1.9 then for each \(\varepsilon > 0\) there exists, \(N_1 \in \mathbb{N}\) such that for every \(l_1 \geq N_1 \implies U^*(x, ..., x, x_{l_1}) < \frac{\varepsilon}{n-1}\).

\(N_2 \in \mathbb{N}\) such that for every \(l_2 \geq N_2 \implies U^*(x, ..., x, x_{l_2}) < \frac{\varepsilon}{n-1}\),

and similary there exist \(N_{n-1} \in \mathbb{N}\) such that for every \(l_{n-1} \geq N_{n-1} \implies U^*(x, ..., x, x_{l_{n-1}}) < \frac{\varepsilon}{n-1}\).

Let

\[
N_0 = \max\{N_1, ..., N_{n-1}\} \quad \text{and} \quad K_0 = \min\{l_1, ..., l_{n-1}\}.
\]

For \(K_0 > N_0\) we have

\[
U^*(x, x_{l_1}, ..., x_{l_{n-1}}) \leq U^*(x, x_{l_1}, ..., x_{l_{n-2}}, x) + U^*(x, x_{l_{n-1}}, ..., x_{l_{n-1}})
\]

\[
\leq U^*(x, x, x_{l_1}, ..., x_{l_{n-2}}, x) + U^*(x, x, x_{l_{n-1}}, ..., x_{l_{n-1}})
\]

\[
\leq \quad \vdots
\]

\[
\leq \sum_{i=1}^{n-1} U^*(x, x_i, ..., x_i)
\]

\[
< \frac{(n-1)\varepsilon}{n-1} = \varepsilon.
\]

Conversely, set \(l_1 = \cdots = l_{n-1} = k\) in (\(\ast\ast\)) we have \(U^*(x, ..., x, x_k) < \varepsilon\).

Proposition 1.11. In a \(U^*_n\)-metric space, \((X, U^*_n)\), the following are equivalent. 

(i) The sequence \(\{x_k\}\) is \(U^*_n\)-Cauchy.

(ii) For each \(\varepsilon > 0\), there exists \(N \in \mathbb{N}\) such that \(U^*_n(x_k, x_l, x_l) < \varepsilon\), for all \(k, l \geq N\).

Lemma 1.12. Let \((X, U^*_n)\) be a \(U^*_n\)-metric space.

(1) If \(r > 0\), then the ball \(B_U^*(x, r)\) with center \(x \in X\) and radius \(r\) is the open ball.

(2) If sequence \(\{x_k\}\) in \(X\) converges to \(x\), then \(x\) is unique.

(3) If sequence \(\{x_k\}\) in \(X\) converges to \(x\), then sequence \(\{x_k\}\) is a Cauchy sequence.

(4) The function of \(U^*_n\) is continuous on \(X^n\).
Proof. proof 1)
Let \( w \in B_{U^*}(x, r) \) so that \( U^*(x, w, ..., w) < r \). If set \( U^*(x, w, ..., w) = \delta \) and \( r' = r - \delta \) then we prove that \( B_{U^*}(w, r') \subseteq B_{U^*}(x, r) \). Let \( y \in B_{U^*}(w, r') \), by \((U_4)\) we have \( U^*(x, y, ..., y) = U^*(y, y, y, ..., y) \leq U^*(y, ..., y, w) + U^*(w, x, ..., x) < r' + \delta = r \).

proof 2)
Let \( x_k \rightarrow y \) and \( y \neq x \). Since \( \{x_k\} \) converges to \( x \) and \( y \), for each \( \varepsilon > 0 \) there exists, \( N_1 \in \mathbb{N} \) such that for every \( k \geq N_1 \implies U^*(x, ..., x, x_k) < \frac{\varepsilon}{2} \) and

\[ U^*(y, ..., y, x_k) < \frac{\varepsilon}{2} \]

If set \( N_0 = \text{max}\{N_1, N_2\} \), then for every \( k \geq N_0 \) by \((U_4)\) we have

\[ U^*(x, ..., y, x) \leq U^*(x, ..., x, x_k) + U^*(x_k, y, ..., y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

then \( U^*(x, ..., x, y) = 0 \) is a contradiction. So \( x = y \).

proof 3)
Since \( x_k \rightarrow x \) for each \( \varepsilon > 0 \) there exists, \( N_1 \in \mathbb{N} \) such that for every \( k \geq N_1 \implies U^*(x, ..., x, x_k) < \frac{\varepsilon}{2} \) and

\[ U^*(y, ..., y, x_k) < \frac{\varepsilon}{2}. \]

If set \( N_0 = \text{max}\{N_1, N_2\} \), then for every \( k, l \geq N_0 \) by \((U_4)\) we have

\[ U^*(x, ..., x, x_l) \leq U^*(x, ..., x, x_k) + U^*(x_k, x_l, ..., x_l) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

Hence sequence \( \{x_k\} \) is a Cauchy sequence.

proof 4)
Let the sequence \( \{(x_1)_k, ..., (x_n)_k\} \) in \( X^n \) converges to a point \((z_1, ..., z_n)\) i.e.

\[ \lim_{k \to \infty} (x_i)_k = z_i \quad i = 1, ..., n \]

for each \( \varepsilon > 0 \) there exists, \( N_1 \in \mathbb{N} \) such that for every \( k > N_1 \implies U^*((z_1, ..., z_1, (x_1)_k) < \frac{\varepsilon}{n} \)

\[ N_2 \in \mathbb{N} \] such that for every \( k > N_2 \implies U^*((z_2, ..., z_2, (x_2)_k) < \frac{\varepsilon}{n} \)

\[ \vdots \]

\[ N_n \in \mathbb{N} \] such that for every \( k > N_n \implies U^*((z_n, ..., z_n, (x_n)_k) < \frac{\varepsilon}{n} \).

If set \( N_0 = \text{max}\{N_1, ..., N_n\} \), then for every \( k \geq N_0 \) we have

\[ U^*((x_1)_k, ..., (x_n)_k) \leq U^*((x_1)_k, ..., (x_{n-1})_k, (z_n)) + U^*((z_n, (x_n)_k, ..., (x_n)_k)

\[ \leq U^*((x_1)_k, ..., (x_{n-2})_k, (z_n, (z_{n-1})_k, ..., (x_{n-1})_k)

\[ + U^*(z_n, (x_n)_k, ..., (x_n)_k)

\[ \leq \vdots \]

\[ \leq U^*(z_1, ..., z_n) + \sum_{i=1}^{n} U^*(z_i, (x_i)_k, ..., (x_i)_k)

\[ \leq U^*(z_1, ..., z_n) + \frac{n\varepsilon}{n} = U^*(z_1, ..., z_n) + \varepsilon. \]

Hence we have

\[ U^*((x_1)_k, ..., (x_n)_k) - U^*(z_1, ..., z_n) < \varepsilon \]
Remark 1.15. Let $f$ and $g$ be mappings from a $U_n^*$-metric space $(X, U_n^*)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is $fx = gx$ implies that $f(x) = g(x)$.

Definition 1.14. Let $(X, U_n^*)$ be a $U_n^*$-metric space, for $A_1, ..., A_n \subseteq X$, define

$$\Delta_{U^*}(A_1, ..., A_n) = \sup\{U^*(a_1, ..., a_n) | a_i \in A_i, i = 1, ..., n\}.$$

Remark 1.15. It follows immediately from the definition that

(i) If $A_i$ consists of a single point $a_i$ we write

$$\Delta_{U^*}^*(A_1, ..., A_{i-1}, a_i, A_{i+1}, ..., A_n) = \Delta_{U^*}^*(A_1, ..., A_{i-1}, a_i, A_{i+1}, ..., A_n).$$

If $A_1, ..., A_n$ also consists of a single point $a_1, ..., a_n$ respectively, we write

$$\Delta_{U^*}^*(A_1, ..., A_n) = \Delta_{U^*}^*(a_1, ..., a_n).$$

Also we have

$$\Delta_{U^*}^*(A_1, ..., A_n) = 0 \iff A_1 = \cdots = A_n = \{a\},$$

$$\Delta_{U^*}^*(A_1, ..., A_n) = \Delta_{U^*}^*(A_{\pi_1}, ..., A_{\pi_n}),$$

for every permutation $(\pi_1, ..., \pi_n)$ of $(1, 2, ..., n)$.

In particular for $\emptyset \neq A_1 = \cdots = A_n \subseteq X$,

$$\Delta_{U^*}^*(A_1) = \sup\{U^*(b_1, ..., b_n) | b_1, ..., b_n \in A_1\}.$$

(ii) If $A \subseteq B$, then $\Delta_{U^*}^*(A) \leq \Delta_{U^*}^*(B)$.

(iii) For a sequence $A_k = \{x_k, x_{k+1}, x_{k+2}, \cdots\}$ in $U_n^*$-metric space $(X, U_n^*)$, let $a_k = \Delta_{U^*}^*(A_k)$ for $k \in \mathbb{N}$. Then

(a) Since $A_{k+1} \subseteq A_k$ hence $\Delta_{U^*}^*(A_{k+1}) \leq \Delta_{U^*}^*(A_k)$, for every $k \geq 1$.

(b) $U^*(x_l, ..., x_n) \leq \Delta_{U^*}^*(A_k) = a_k$ for every $l_1, ..., l_n \geq k$.

(c) $0 \leq \Delta_{U^*}^*(A_k) = a_k$.

Therefore, $\{a_k\}$ is decreasing and bounded for all $k \in \mathbb{N}$, and so there exists an $0 \leq a$ such that $\lim_{k \to \infty} a_k = a$. 

\[ U^*(z_1, ..., z_n) \leq U^*(z_1, ..., z_{n-1}, (x_n)_{k}) + U^*((x_n)_{k}, z_n, ..., z_n) \]

\[ \leq U^*(z_1, ..., z_{n-2}, (x_n)_{k}, (x_{n-1})_{k}) + U^*((x_{n-1})_{k}, z_{n-1}, ..., z_{n-1}) \]

\[ + U^*((x_n)_{k}, z_n, ..., z_n) \]

\[ \leq \]

\[ \leq U^*((x_1)_{k}, ..., (x_n)_{k}) + \sum_{i=1}^{n} U^*((x_i)_{k}, z_i, ..., z_i) \]

\[ \leq U^*((x_1)_{k}, ..., (x_n)_{k}) + \frac{n \varepsilon}{n} = U^*((x_1)_{k}, ..., (x_n)_{k}) + \varepsilon. \]

That is,

$$U^*(z_1, ..., z_n) - U^*((x_1)_{k}, ..., (x_n)_{k}) < \varepsilon.$$
Lemma 1.16. Let \((X, U_n^*)\) be an \(U_n^*\)-metric space. If \(\lim_{k \to \infty} a_k = 0\), then sequence \(\{x_k\}\) is a Cauchy sequence.

Proof. Since \(\lim_{k \to \infty} a_k = 0\), we have that for every \(\varepsilon > 0\), there exists a \(N_0 \in \mathbb{N}\) such that for every \(k > N_0\), \(|a_k - 0| < \varepsilon\). That is \(a_k = \Delta_{U_n^*}(A_k) < \varepsilon\). Then for \(l_1, \ldots, l_n \geq k > N_0\) by (b) of Remark 1.15 we have

\[
U^*(x_{i1}, \ldots, x_{in}) \leq \sup\{U^*(x_i, \ldots, x_j) \mid x_i, \ldots, x_j \in A_k\} = a_k < \varepsilon.
\]

Therefore, \(\{x_k\}\) is a Cauchy sequence in \(X\).

2. Main results

Theorem 2.1. Let \(X\) be a \(U_n^*\)-complete metric space

I) If \(f\) and \(g\) be self-mappings of a complete \(U_n^*\)-metric space \((X, U_n^*)\) satisfying:

i) \(g(X) \subseteq f(X)\), and \(f(X)\) is closed subset of \(X\),

ii) the pair \((f, g)\) is weakly compatible,

iii) \(U^*(g_{z_1}, \ldots, g_{z_n}) \leq \psi(U^*(f_{z_1}, \ldots, f_{z_n}))\), for every \(z_1, \ldots, z_n \in X\), where \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) is a nondecreasing continuous function with \(\psi(t) < t\) for every \(t > 0\).

Then \(f\) and \(g\) have a unique common fixed point in \(X\).

II) If \(f_k : X \rightarrow X\) be a sequence maps such that

\[
U^*(f_{l_1 z_1}, f_{l_2 z_2}, \ldots, f_{l_n z_{n-1}}, z_n) \leq \beta U^*(z_1, \ldots, z_n)
\]

for all \(i \neq j\) and \(z_1, \ldots, z_n \in X\) with \(0 \leq \beta < \frac{1}{2}\). Then \(\{f_k\}\) have a unique common fixed point.

Proof. proof I)

Let \(x_0\) be an arbitrary point in \(X\). By (i), we can choose a point \(x_1\) in \(X\) such that \(y_0 = gx_0 = f x_1\) and \(y_1 = gx_1 = f x_2\). In general, there exists a sequence \(\{y_k\}\) such that, \(y_k = gx_k = f x_{k+1}\), for \(k = 0, 1, 2, \ldots\). We prove that sequence \(\{y_k\}\) is a Cauchy sequence. Let \(A_k = \{y_k, y_{k+1}, y_{k+2}, \ldots\}\) and \(a_k = \Delta_{U_n^*}(A_k)\), \(k \in \mathbb{N}\). Then we know \(\lim_{k \to \infty} a_k = a\) for some \(a \geq 0\).

Taking \(z_i = x_{l_1+t}\) in (iii) for \(l \geq 1\) and \(l_1, \ldots, l_n \geq 0\)

\[
U^*(y_{l_1+t}, \ldots, y_{l_n+t}) = U^*(gx_{l_1+t}, \ldots, gx_{l_n+t}) \leq \psi(U^*(f_{x_{l_1+t}}, \ldots, f_{x_{l_n+t}})) = \psi(U^*(y_{l_1+t-1}, \ldots, y_{l_n+t-1}))
\]

Since \(U^*(y_{l_1+t-1}, \ldots, y_{l_n+t-1}) \leq a_{l-1}\), for every \(l_1, \ldots, l_n \geq 0\) and \(\psi\) is increasing in \(t\), we get

\[
U^*(y_{l_1+t}, \ldots, y_{l_n+t}) \leq \psi(U^*(y_{l_1+t-1}, \ldots, y_{l_n+t-1})).
\]

Therefore

\[
\sup_{l_1, \ldots, l_n \geq 0} \{U^*(y_{l_1+t}, \ldots, y_{l_n+t}) \leq \psi(a_{l-1}).
\]

Hence, we have \(a_l \leq \psi(a_{l-1})\). Letting \(l \to \infty\), we get \(a \leq \psi(a)\). If \(a \neq 0\), then \(a \leq \psi(a) < a\), which is a contradiction. Thus \(a = 0\) and hence \(\lim_{k \to \infty} a_k = 0\). Thus Lemma 1.16 \(\{y_k\}\) is a Cauchy sequence in \(X\). By the completeness of \(X\), there exists a \(v \in X\) such that

\[
\lim_{k \to \infty} y_k = \lim_{k \to \infty} g x_k = \lim_{k \to \infty} f x_{k+1} = v.
\]

Let \(f(X)\) is closed, there exist \(w \in X\) such that \(f w = v\). Now we show that \(g w = v\). For this it is enough set \(x_k, \ldots, x_k, w\) replacing \(z_1, \ldots, z_n\) respectively, in inequality (iii) we get

\[
U^*(g x_k, \ldots, g x_k, g w) \leq \psi(U^*(f x_k, \ldots, f x_k, f w))
\]
Taking $k \to \infty$, we get
\[ U^*(v, ..., v, gw) \leq \psi(U^*(0)) = 0, \]
it implies $gw = v$.

Since the pair $(f, g)$ are weakly compatible, hence we get, $gfw = fgw$. Thus $fv = gv$. Now we prove that $gv = v$. If we substitute $z_1, ..., z_n$ in $(iii)$ by $x_k$, $x_k$ and $v$ respectively, we get
\[ U^*(gx_k, ..., gx_k, gu) \leq \psi(U^*(fx_k, ..., fx_k, fv)) \]
Taking $k \to \infty$, we get
\[ U^*(v, ..., v, gv) \leq \psi(U^*(v, ..., v, gv)). \]
If $gv \neq v$, then $U^*(v, ..., v, gv) < U^*(v, ..., v, gv)$, is contradiction. Therefore,
\[ fv = gv = v. \]

For the uniqueness, let $v$ and $v'$ be fixed points of $f, g$. Taking $z_1 = ... = z_{n-1} = v$ and $z_n = v'$ in $(iii)$, we have
\[ U^*(v, ..., v, v') = U^*(gv, ..., gv, gv') \]
\[ \leq \psi(U^*(fv, ..., fv, fv')) \]
\[ = \psi(U^*(v, ..., v, v')) \]
\[ < U^*(v, ..., v, v'), \]
which is a contradiction. Thus we have $v = v'$.

Proof II)

Let $x_0 \in X$ be any fixed arbitrary element define a sequence $\{x_k\}$ in $X$ as. $x_{k+1} = f_{k+1}x_k$ for all $k = 0, 1, 2, \cdots$.

Let $d_k = U^*(x_k, x_{k+1}, ..., x_{k+1})$ for all $k = 0, 1, 2, \cdots$.

Now
\[ d_{k+1} = U^*(x_{k+1}, x_{k+2}, ..., x_{k+2}) \]
\[ = U^*(f_{k+1}x_k, f_{k+2}x_{k+1}, ..., f_{k+2}x_{k+1}, x_{k+2}) \]
\[ \leq \beta U^*(x_k, x_{k+1}, ..., x_{k+1}, x_{k+2}) \]
\[ \leq \beta U^*(x_k, x_{k+1}, ..., x_{k+1}, x_{k+1}) + \beta U^*(x_{k+1}, x_{k+2}, ..., x_{k+2}) \]
\[ = \beta d_k + \beta d_{k+1}. \]

Hence
\[ d_{k+1} \leq \frac{\beta}{1 - \beta} d_k, \]
\[ d_k \leq \frac{\beta}{1 - \beta} d_{k-1} \text{ for all } n = 1, 2, \cdots. \] Let $\alpha = \frac{\beta}{1 - \beta}$, we have
\[ d_k \leq \alpha d_{k-1} \leq \alpha^n d_0 \to 0 \text{ as } k \to \infty. \] Therefore
\[ \lim_{k \to \beta} d_k = 0. \] Thus
\[ \lim_{k \to \beta} U^*(x_k, x_{k+1}, ..., x_{k+1}) = 0. \]

Now we shall prove that $\{x_k\}$ is a $U^*_n$-Cauchy sequence in $X$.

Let $l > k > N_0$ for some $N_0 \in \mathbb{N}$. Now
\[ U^*(x_k, ..., x_k, x_l) \leq U^*(x_k, ..., x_k, x_{k+1}) + U^*(x_{k+1}, ..., x_{k+1}, x_l) \]
\[ \leq \sum_{l=\infty} U^*(x_l, ..., x_{l+1}) \to 0 \text{ as } k, l \to \infty. \]
Hence $\lim_{k,l \to \infty} U^*(x_k, \ldots, x_{k+l}) = 0$. 

Thus $\{x_k\}$ is $U_n^*$-Cauchy sequence in $X$. 

Since $X$ is $U_n^*$-complete $x_k \to x$ in $X$. We prove that $x$ is a fixed point of $f_k$ for all $k$ suppose there exist a $k'$ such that $f_{k'} x \neq x$. Then

$$U^*(f_{k'}, x, \ldots, x) = \lim_{k \to \infty} U^*(f_{k'} x, x_{k+1}, \ldots, x_{k+1}, x)$$

$$= \lim_{k \to \infty} U^*(f_{k'} x, f_{k+1} x_{k}, \ldots, f_{k+1} x_{k}, x)$$

$$\leq \beta \lim_{k \to \infty} U^*(x, x_{k+1}, \ldots, x_{k+1}, x) = 0.$$ 

Therefore $U^*(f_{k'}, x, \ldots, x) = 0$. Therefore $f_k x = x$ for all $k$. Thus $x$ is common fixed point of $\{f_k\}$ for all $k$. For the uniqueness, suppose $x \neq y$ such that $f_k y = y$ for all $k$. Then

$$U^*(x, y, \ldots, y) = U^*(f_k x, f_k y, \ldots, f_k y, y)$$

$$\leq \beta U^*(x, y, \ldots, y)$$

This implies $(1 - \beta)U^*(x, y, \ldots, y) \leq 0$.

Since $x \neq y$ we have $U^*(x, y, \ldots, y) > 0$ her $(1 - \beta) < 0$.

This implies $\beta > 1$ which contraction to $\beta < \frac{1}{2}$.

Thus $\{f_k\}$ have a unique common fixed point.

**Corollary 2.2.** Let $f$ be self-mapping of a complete $U_n^*$-metric space $(X, U_n^*)$ satisfying:

$$U^*(z_1, \ldots, z_n) \leq \psi(U^*(f^m z_1, \ldots, f^m z_n)),$$

for every $z_1, \ldots, z_n \in X$, $f$ is surjective and $m \in \mathbb{N}$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.

Then $f$ have a unique fixed point in $X$.

**Proof.** If we define $g = I$ identity map in Theorem 2.1. There exists a unique $v \in X$ such that $f^m v = v$.

Thus

$$f^m(f v) = f(f^m v) = f v.$$ 

Since $v$ is unique, we have $f v = v$. 

**Corollary 2.3.** Let $g$ be self-mapping of a complete $U_n^*$-metric space $(X, U_n^*)$ satisfying:

$$U^*(g^m z_1, \ldots, g^m z_n) \leq \psi(U^*(z_1, \ldots, z_n)),$$

for every $z_1, \ldots, z_n \in X$ and $m \in \mathbb{N}$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.

Then $g$ have a unique fixed point in $X$.

**Proof.** If we define $f = I$ identity map in Theorem 2.1. There exists a unique $v \in X$ such that $g^m v = v$.

Thus

$$g^m(g v) = g(g^m v) = g v.$$ 

Since $v$ is unique, we have $g v = v$. 

**Corollary 2.4.** Let $f$ and $g$ be self-mappings of a complete $U_n^*$-metric space $(X, U_n^*)$ satisfying:

(i) $g^r(X) \subseteq f^s(X)$, and $f^s(X)$ is closed subset of $X$, 

(ii) the pair $(f^*, g^*)$ is weakly compatible and $f^* g = g^* f = f g^*$, 

(iii) $U^*(g^r z_1, \ldots, g^r z_n) \leq \psi(U^*(f^s z_1, \ldots, f^s z_n))$, for every $z_1, \ldots, z_n \in X$ and $r, s \in \mathbb{N}$ where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is
a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.
Then $f$ and $g$ have a unique common fixed point in $X$.

**Proof.** By Theorem 2.1 there exists a fixed point $v \in X$ such that $f^*v = g^*v = v$. On the other hand, we have

$$gv = g(g^*v) = g^*(gv)$$
$$gv = g(f^*v) = f^*(gv).$$

Since $v$ is unique, we have $gv = v$. Similarly, we have $f^*v = v$.

**Corollary 2.5.** Let $f$, $g$ and $h$ be self-mappings of a complete $U^*_n$-metric space $(X, U^*_n)$ satisfying:
(i) $g(X) \subseteq fh(X)$, and $fh(X)$ is closed subset of $X$,
(ii) the pair $(fh, g)$ is weakly compatible and $fh = hf, gh = hg$,
(iii) $U^*(gz_1, ..., gz_n) \leq \psi(U^*(fhz_1, ..., fhz_n))$, for every $z_1, ..., z_n \in X$, where $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a nondecreasing continuous function with $\psi(t) < t$ for every $t > 0$.
Then $f$, $g$ and $h$ have a unique common fixed point in $X$.

**Proof.** By Theorem 2.1 there exists a fixed point $v \in X$ such that $fhv = gv = v$. Now, we prove that $hv = v$. If $hv \neq v$ in (iii), then we have

$$U^*(hv, v, ..., v) = U^*(hgv, gv, ..., gv)$$
$$= U^*(ghv, gv, ..., gv)$$
$$\leq \psi(U^*(fhhv, fhv, ..., fhv))$$
$$= \psi(U^*(hv, v, ..., v))$$
$$< U^*(hv, v, ..., v),$$

which is a contradiction. Thus we have $hv = v$. Therefore,

$$fv = fhv = v = hv = gv.$$

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**References**


