On Banach contraction principle in a cone metric space

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

The object of this paper is to establish a generalized form of Banach contraction principle for a cone metric space which is not necessarily normal. This happens to be a generalization of all different forms of Banach contraction Principle, which have been arrived at in L. G. Huang and X. Zhang [L. G. Huang and X. Zhang, J. Math. Anal. Appl 332 (2007), 1468–1476] and Sh. Rezapour, R. Hamlbarani [Sh. Rezapour, R. Hamlbarani, J. Math. Anal. Appl. 345 (2008) 719-724] and D. Ilic, V. Rakocevic [D. Ilic, V. Rakocevic, Applied Mathematics Letters 22 (2009), 728–731]. It also results that the theorem on quasi contraction of Ćirić [L. J. B. Ćirić, Proc. American Mathematical Society 45 (1974), 999–1006]. for a complete metric space also holds good in a complete cone metric space. All the results presented in this paper are new. ©2012. All rights reserved.

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1. Introduction

There has been a number of generalizations of metric space. One such generalization is a cone metric space. In the second half of previous century a lot of work has been done in a K-metric space, which is in the setting of cone in a real normed linear space and variously defined notions of convergence and a Cauchy
sequence [13]. However, another school in U.S.S.R [7] worked in $K$-metric space in the setting of a Banach space $B$ and a closed cone in it in the name of a generalized metric space or a SKS metric space. Recently, in [3] Huang and Zhang defined cone metric space in the same setting of a real Banach space $E$ ordered with a closed cone $P$ in it with $\text{int}P \neq \Phi$ defining convergence and a Cauchy sequence with respect to interior points of $P$. In this space they replaced the set of real numbers of a metric space by an ordered metric space. In this paper we prove a common fixed point theorem for a sequence of self maps satisfying a generalized contractive condition assuming the normality of cone metric space.

Recently, Rezapour and Hamlbarani [11] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In [5], the authors introduced the concept of a compatible pair of self maps in a cone metric space and established a basic result for a non-normal cone metric space with an example, while in [6] weakly compatible maps have been studied. In this paper we are proving a common fixed point theorem for a sequence of self maps satisfying a generalized contractive condition for a non-normal cone metric space. It results in a generalized form of Banach contraction principle in this space.

2. Preliminaries

Definition 2.1. [3] Let $E$ be a real Banach space and $P$ be a subset of $E$. $P$ is called a cone if

(i) $P$ is a closed, nonempty and $P \neq \{0\}$;
(ii) $a, b \in R, a, b \geq 0, x, y \in P$ imply $ax + by \in P$;
(iii) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering “$\leq$” in $E$ by $x \leq y$ if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x << y$ to denote $y - x \in P_0$, where $P_0$ stands for the interior of $P$.

$P$ is called normal if for some $M > 0$ for $x, y \in E, 0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.

Proposition 2.2. Let $P$ be a cone in a real Banach space $E$. If for $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = 0$.

Proof: For $a \in P, k \in [0, 1)$ and $a \leq ka$ gives $(k - 1)a \in P$ implies $-(1 - k)a \in P$. Therefore by (ii) we have $-a \in P$, as $1/(1 - k) > 0$. Hence $a = 0$, by (iii).

Proposition 2.3. [4] Let $P$ be a cone is a real Banach space $E$ with non-empty interior. If for $a \in E$ and $a << c$, for all $c \in P_0$, then $a = 0$.

Remark 2.4. [11] $\lambda P_0 \subseteq P_0$, for $\lambda > 0$ and $P_0 + P_0 \subseteq P_0$.

Definition 2.5. [3] Let $X$ be a nonempty set and $P$ be a cone in a real Banach space $E$. Suppose the mapping $d : X \times X \rightarrow E$ satisfies:

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$, and $(X, d)$ is called a cone metric space. If $P$ is normal, then $(X, d)$ is said to be a normal cone metric space.

Example 2.6. [2] Let $E = R^2, P = \{(x, y) \in E : x \geq 0, y \geq 0\}$ and $X = R$. For $x, y \in R$ define $d(x, y) = |x - y|(1, \alpha)$ where $\alpha \geq 0$ is some fixed constant. Then $(X, d)$ is a cone metric space.

Example 2.7. Let $E = C_R^0[0, 1]$ with the norm $\|f\| = \|f\|_{\infty} + \|f'\|_{\infty}$. Consider the cone $P = \{f \in E : f \geq 0\}$. Then $P$ is not a normal cone as shown in [14]. Taking $X = \{1, 1/2, 1/3 \ldots\}$ we define $d : X \times X \rightarrow P$ by $d(t, 0) = f_m$, where $f_m(t) = \frac{1}{m} - \frac{1}{n} |t|$, for all $t \in [0, 1]$. Then $(X, d)$ is a non-normal cone metric space. $(X, d)$ is not a metric space as it is not normal.
Definition 2.8. Let \((X, d)\) be a cone metric space with respect to a cone in a real Banach space \(E\) with non-empty interior. Let \(\{x_n\}\) be a sequence in \(X\) and \(x \in X\). If for every \(c \in E\) with \(0 << c\) there is a positive integer \(N_c\) such that for all \(n > N_c, d(x_n, x) << c\), then the sequence \(\{x_n\}\) is said to converge to \(x\), and \(x\) is called limit of \(\{x_n\}\). We write \(\lim_{n \to \infty} x_n = x\) or \(x_n \to x\), as \(n \to \infty\).

Definition 2.9. Let \((X, d)\) be a cone metric space with respect to a cone with nonempty interior in a real Banach space \(E\). Let \(\{x_n\}\) be a sequence in \(X\). If for any \(c \in E\) with \(0 << c\) there is a positive integer \(N_c\) such that for all \(n, m > N_c, d(x_n, x_m) << c\), then the sequence \(\{x_n\}\) is said to be a Cauchy sequence in \(X\).

In the following \((X, d)\) will stand for a cone metric space with respect to a cone \(P\) with \(P^0 \neq \emptyset\) in a real Banach space \(E\) and \(\leq\) is partial ordering in \(E\) with respect to \(P\).

Remark 2.10. It follows from above definitions that if \(\{x_{2n}\}\) is a subsequence of a Cauchy sequence \(\{x_n\}\) in a cone metric space \((X, d)\) and \(x_{2n} \to z\) then \(x_n \to z\).

Definition 2.11. Let \((X, d)\) be a cone metric space. If every Cauchy sequence in \(X\) is convergent in \(X\), then \(X\) is called a complete cone metric space.

Proposition 2.12. Let \((X, d)\) be a cone metric space and \(P\) be a cone in a real Banach space \(E\). If \(u \leq v, v << w\) then \(u << w\).

Lemma 2.13. Let \((X, d)\) be a cone metric space and \(P\) be a cone in a real Banach space \(E\) and \(k_1, k_2, k > 0\) are some fixed real numbers. If \(x_n \to x, y_n \to y\) in \(X\) and for some \(a \in P\) \((1.1) ka \leq k_1d(x_n, x) + k_2d(y_n, y)\), for all \(n > N\), for some integer \(N\), then \(a = 0\).

Proof. As \(x_n \to x\), and \(y_n \to y\) for \(c \in P^0\) there exists a positive integer \(N_c\) such that \(\frac{c}{k_1 + k_2} - d(x_n, x), \frac{c}{k_1 + k_2} - d(y_n, y) \in P^0\), for all \(n > N_c\). Therefore by Remark 2.4, we have \(\frac{k_1c}{k_1 + k_2} - k_1d(x_n, x), \frac{k_2c}{k_1 + k_2} - k_2d(y_n, y) \in P^0\), for all \(n > N_c\). Again by adding and Remark 2.4, we have \(c - k_1d(x_n, x) - k_2d(y_n, y) \in P^0\) for all \(n > \max\{N, N_c\}\). From (1.1) and Proposition 2.12 we have \(ka << c\), for each \(c \in P^0\). By Proposition 2.3, we have \(a = 0\), as \(k > 0\).

3. MAIN RESULTS

Theorem 3.1. Let \((X, d)\) be a complete cone metric space with respect to a cone \(P\) contained in a real Banach space \(E\). Let \(\{T_n\}\) be a sequence of self maps on \(X\) satisfying:

\(\text{(3.1.1) For some } \lambda, \mu, \delta, \alpha, \beta \in [0, 1) \text{ with } \lambda + \mu + \delta + 2\alpha < 1, \text{ or else } \lambda + \mu + \delta + 2\beta < 1, \text{ for all } x, y \in X \)

\(d(T_1x, T_1y) \leq \lambda d(x, y) + \mu d(T_1x, y) + \delta d(x, y) + \alpha d(T_1x, T_1y) + \beta d(T_1x, y).\)

For \(x_0 \in X\), let \(x_n = T_n x_{n-1}\), for all \(n\). Then the sequence \(\{x_n\}\) converges in \(X\) and its limit \(u\) is a common fixed point of all the maps of the sequence \(\{T_n\}\). This fixed point is unique if \(\delta + \alpha + \beta < 1\).

Proof. We show that \(\{x_n\}\) is a Cauchy sequence in \(X\).

Step 1: Taking \(x = x_{n-1}, y = x_n\) and \(i = n, j = n + 1\) in (3.1.1) we get,

\(d(T_n x_{n-1}, T_{n+1} x_n) \leq \lambda d(T_n x_{n-1}, x_{n-1}) + \mu d(T_{n+1} x_n, x_n) + \delta d(x_{n-1}, x_n) + \alpha d(T_n x_{n-1}, T_{n+1} x_n) + \beta d(T_n x_{n-1}, x_n).\)

As \(x_n = T_n x_{n-1}\), we have

\(d(x_n, x_{n+1}) \leq \lambda d(x_n, x_{n-1}) + \mu d(x_{n+1}, x_n) + \delta d(x_{n-1}, x_n) + \alpha d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_n),\)

\(\leq \lambda d(x_n, x_{n-1}) + \mu d(x_{n+1}, x_n) + \delta d(x_{n-1}, x_n) + \alpha [d(x_{n-1}, x_n) + d(x_n, x_{n+1})].\)
Writing $d(x_n, x_{n+1}) = d_n$, we have

$$d_n \leq \lambda d_{n-1} + \mu d_n + \delta d_{n-1} + \alpha[d_n + d_{n-1}],$$

i.e.

$$(1 - \mu - \alpha)d_n = (\lambda + \delta + \alpha)d_{n-1},$$

which implies

$$d_n \leq hd_{n-1}, \quad (3.1)$$

if $h = \frac{(\lambda + \delta + \alpha)}{1 - \mu - \alpha}$.  

As $\lambda + \mu + \delta + 2\alpha < 1$ we obtain that $h < 1$.

Now

$$d_n \leq hd_{n-1} \leq h^2d_{n-2} \leq h^3d_{n-3} \leq \ldots \leq h^nd_0, \text{ where } d_0 = d(x_0, x_1).$$

Also

$$d(x_{n+p}, x_n) \leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \ldots + d(x_{n+1}, x_n),$$

i.e.

$$d(x_{n+p}, x_n) \leq d_{n+p-1} + d_{n+p-2} + \ldots + d_n.$$ 

For an arbitrary fixed $m$ we show that

$$T$$

is a common fixed point of all the maps of the sequence $\{T_n\}$. Now

$$d(x_{n+p}, x_n) \leq h^n d_0/(1 - h), \quad (3.2)$$

for all $n > N_c$, for all $p$, by Proposition 2.12. This implies $d(x_{n+p}, x_n) \leq c$, for all $n > N_c$, for all $p$. Hence $\{x_n\}$ is a Cauchy sequence in $X$, which is complete.

Step II: For an arbitrary fixed $m$ we show that $T_mu = u$.

Now,

$$d(T_mu, u) \leq d(T_mu, T_nx_{n-1}) + d(T_nx_{n-1}, u),$$

and

$$d(T_mu, u) \leq d(x_{n+i}, u) + \lambda d(T_nx_{n-1}, x_{n-1}) + \mu d(T_mu, u) + \delta d(u, x_{n-1}) + \alpha d(T_nx_{n-1}, x_{n-1}) + \beta d(u, x_{n-1}).$$

Using (3.1.1) with $x = x_{n-1}$, $y = u$, $i = n$ and $j = m$ we have

$$d(T_mu, u) \leq [\mu + \delta + \alpha]d(x_{n-1}, u) + [1 + \lambda + \beta]d(u, x_{n-1}).$$

As $\{x_n\} \rightarrow u$, $\{x_{n-1}\} \rightarrow u$, and $1 - \mu - \alpha > 0$, using Lemma 2.13, we have $d(T_mu, u) = 0$, and we get $T_mu = u$. Thus $u$ is a common fixed point of all the maps of the sequence $\{T_n\}$.

Step III (Uniqueness): Let $Tnz = z$, for all $n$, be another common fixed point of all the maps of the sequence $\{T_n\}$. Now

$$d(z, u) = d(Tnz, T_nu).$$

Taking $x = z$ and $y = u$ with $i = j = n$ in (3.1.1) we get

$$d(z, u) \leq \lambda d(Tnz, z) + \mu d(T_nu, u) + \delta d(z, u) + \alpha d(z, T_nu) + \beta d(Tnz, u),$$

which gives

$$d(z, u) \leq (\delta + \alpha + \beta)d(z, u).$$
As $\delta + \alpha + \beta < 1$, using Proposition 2.2, we have $d(z,u) = 0$ i.e. $u = z$. Thus $u$ is the unique common fixed point of all the maps of the sequence $\{T_n\}$. To see the sufficiency of the alternate condition $\lambda + \mu + \delta + 2\beta < 1$, in step I we choose $x = u, y = x_{n-1}$ with $i = n + 1$ and $j = n$ in (3.1.1) to obtain $(1 - \lambda - \beta) d_{n} \leq (\mu + \delta + \beta) d_{n-1}$. Thus $d_{n} \leq h'd_{n-1}$, where $h' = \frac{\mu + \delta + \beta}{1 - \lambda - \beta} < 1$.

Again in step II we choose $x = u, y = x_{n-1}$, and in (3.1.1) receiving $(1 - \lambda - \beta) d(T_{m}(u),u) \leq \ldots$ and we get $T_{m}u = u, \forall m$. □

**Theorem 3.2.** Let $(X,d)$ be a complete cone metric space with respect to a cone $P$ contained in a real Banach space $E$. Let $\{A_{n}\}$ be a sequence of self maps in $X$ satisfying:

(3.2.1) For some $\lambda, \mu, \delta, \alpha, \beta \in [0,1)$ with $\lambda + \mu + \delta + 2\alpha < 1$, or else $\lambda + \mu + \delta + 2\beta < 1$ and $\delta + \alpha + \beta < 1$, there exists positive integer $m_{i}$, for each $i$, such that for all $x, y \in X$

$$d(A_{m_{i}}^{n}x, A_{m_{i}}^{n}y) \leq \lambda d(A_{m_{i}}^{n}x, x) + \mu d(A_{m_{i}}^{n}y, y) + \delta(x, y) + \alpha d(x, A_{m_{i}}^{n}x) + \beta d(A_{m_{i}}^{n}x, y).$$

Then all the maps of the sequence $\{A_{n}\}$ have a unique common fixed point in $X$.

**Proof.** In view of (3.2.1) and using Theorem 3.1, all the maps of the sequence $\{A_{m_{i}}^{n}\}$ have a unique common fixed point, say $z$. Hence $A_{m_{i}}^{n}z = z$, for all $i$. Now $A_{m_{i}}^{n}z = z$, implies $A_{m_{i}}^{n}A_{1}z = A_{1}z$. Taking $i = 1, j = 2$ in (3.2.1) we have $A_{2}z = z$. Continuing in similar way it follows that $A_{i}z = z$, for all $i$. Thus $z$ is a common fixed point of all the maps of the sequence $\{A_{i}\}$. Its uniqueness follows from the fact that $A_{i}z = z$, implies $A_{m_{i}}^{n}z = z$, for all $i$. □

**Example 3.3.** (of Theorem 3.2) Let $X = [0,1], E = R^{2}, P = \{(x,y) \in R^{2} : x \geq 0, y \geq 0\} \subseteq R^{2}$, be a cone in $E$. Fix a real number $\gamma > 0$. We define $d : X \times X \to E$ by $d(x,y) = \|x - y\|1, \gamma)$. Then $(X,d)$ is a complete cone metric space. Define $\{A_{n}\}$ on $X$ as follows:

$$A_{n}(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{n+2}], \\ \frac{1}{n+3}, & \text{otherwise.} \end{cases}$$

Taking $m_{i} = 2$, for all $i$. Then the maps $A_{3}^{2}, A_{3}^{2}, A_{3}^{3}, \ldots$ satisfy the condition (3.2.1) for $\lambda = \mu = \delta = \frac{1}{3}$ and $\alpha = \beta = \frac{1}{10}$. Hence by Theorem 3.2, all the maps of the sequence $\{A_{n}\}$ have a unique common fixed point ($u = 0$) in $X$.

Taking $T_{1} = T_{2} = T_{3} = \cdots = T_{n-1} = T_{n} = \cdots = A$ in Theorem 3.1, we get the following general form of Banach contraction principal in a cone metric space which is not necessarily normal

**Theorem 3.4.** Let $(X,d)$ be a complete cone metric space with respect to a cone $P$ contained in a real Banach space $E$ and $A$ be a self map in $X$ satisfying:

(3.4.1) For some $\lambda, \mu, \delta, \alpha, \beta \in [0,1)$ with $\lambda + \mu + \delta + 2\alpha < 1$, or else $\lambda + \mu + \delta + 2\beta < 1$, for all $x, y \in X$

$$d(Ax, Ay) \leq \lambda d(Ax, x) + \mu d(Ay, y) + \delta(x, y) + \alpha d(x, Ay) + \beta d(Ax, y).$$

Then for each $x$ in $X$ the sequence $\{A_{n}x\}$ converges in $X$ and its limit $u$ is a fixed point of $A$. This fixed point is unique if $\delta + \alpha + \beta < 1$.

In [3] L. G. Huang , X. Zhang and in [11] Sh. Rezapour, R. Hambarzumyan proved following various forms of Banach contraction Principle in a normal Cone metric space and in a cone metric space respectively :

**Theorem 1[3] and Theorem 2.3[11]:** Let $(X,d)$ be a complete cone metric space, Suppose the mapping $T : X \times X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq kd(x, y),$$

where $k \in [0,1)$ is a constant. Then $T$ has a unique fixed point in $X$. For each $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

**Theorem 3[3] and Theorem 2.6[11]:** Let $(X,d)$ be a complete cone metric space. Suppose the mapping $T : X \times X \to X$ satisfies the contractive condition

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)]$$

for all $x, y \in X$, where $k \in [0,1/2)$ is a constant. Then $T$ has a unique fixed point in $X$. And for $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.
Theorem 4 [3] and Theorem 2.7 [11]: Let \((X, d)\) be a complete cone metric space. Suppose the mapping \(T: X \times X \rightarrow X\) satisfies the contractive condition
\[
d(Tx, Ty) \leq k[d(Tx, y) + d(Ty, x)]
\]
for all \(x, y \in X\), where \(k \in [0, 1/2)\) is a constant. Then \(T\) has a unique fixed point in \(X\). For each \(x \in X\), the iterative sequence \(\{T^n x\}\) converges to the fixed point.

Theorem 2.8 [11]: Let \((X, d)\) be a complete cone metric space. Suppose the mapping \(T: X \times X \rightarrow X\) satisfies the contractive condition
\[
d(Tx, Ty) \leq kd(x, y) + ld(y, Tx)
\]
for all \(x, y \in X\), where \(k, l \in [0, 1)\) are constants. Then \(T\) has a fixed point in \(X\). Also the fixed point of \(T\) is unique whenever \(k + l < 1\).

Remark 3.5. Above Theorems of [3] and [11] follow from Theorem 3.4 of this paper by taking :
(a) \(\lambda = \mu = \alpha = \beta = 0\) and \(\delta = k\),
(b) \(\lambda = \mu = k\) and \(\delta = \alpha = \beta = 0\),
(c) \(\lambda = \mu = \delta = 0\) and \(\alpha = \beta = k\), and
(d) \(\lambda = \mu = \alpha = 0, \delta = k\), and \(\beta = l\)
respectively in it.

Precisely, Theorem 3.4 synthesizes and generalizes all the results of [3] and [11] for a non-normal cone metric space. Theorem 3.1 is a general form of Banach contraction principle in a complete cone metric space which is not necessarily normal.

Definition 3.6. [4] (Quasi contraction) A self-map \(f\) on a cone metric space \((X, d)\) is said to be a quasi contraction if for a fixed \(\lambda \in (0, 1)\),
\[
d(f(x), f(y)) \leq \lambda u
\]
for every \(x, y \in X\), where
\[
u \in \{d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x))\}.
\]

Theorem 2.1 [4]: Let \((X, d)\) be a complete cone metric space and \(P\) be a normal cone. Then a quasi contraction \(f\) has a unique fixed point in \(X\) and for each \(x \in X\) the iterative sequence \(\{f^n(x)\}\) converges to the fixed point.

Remark 3.7. Keeping one of the constants \(\{\alpha, \beta, \gamma, \delta, \mu\}\) non-zero and all others equal to zero in Theorem 3.4, it follows that the above result of [4] is true even for non-normal complete cone metric space.

Remark 3.8. It has been established in L. J. B. Ćirić [2] that a quasi contraction has a unique fixed point in a complete metric space. It follows from the above Remark that the result of [2] is also true for a complete cone metric space even if it is non-normal.

References


