Convergence theorems of a general approximation method for common fixed points of a finite family of asymptotically-quasi nonexpansive mappings in Banach spaces

Qiao-Li Dong\textsuperscript{a,*}, Songnian He\textsuperscript{a}, Bin-Chao Deng\textsuperscript{b}

\textsuperscript{a}College of Science, Civil Aviation University of China, Tianjin 300300, China.
\textsuperscript{b}Technical economy and management specialty, School of Management, Tianjin University, Tianjin 300072, China.

This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this paper, we consider a general iterative scheme to approximate a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings. Several strong and weak convergence results are presented in Banach spaces and an finite family of asymptotically quasi-nonexpansive mappings is constructed. Our results generalize and extend many known results in the current literature. ©2012 NGA. All rights reserved.

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1. Introduction and Preliminaries

Let $C$ be a nonempty subset of a real Banach space $X$ and $T$ a selfmapping of $C$. Denote by $F(T)$ the set of fixed points of $T$. Throughout this paper, we assume that $F(T) \neq \emptyset$. The mapping $T$ is said to be
(i) nonexpansive if \(|Tx - Ty| \leq |x - y|\), for all \(x, y \in C\);
(ii) quasi-nonexpansive if \(|Tx - p| \leq |x - p|\), for all \(x \in C\) and \(p \in F(T)\);
(iii) asymptotically nonexpansive if there exists a sequence \(\{r_n\}\) in \([0, \infty)\) with \(\lim_{n \to \infty} r_n = 0\) and \(|T^n x - T^n y| \leq (1 + r_n)|x - y|\), for all \(x, y \in C\) and \(n = 1, 2, 3, \ldots\);
(iv) asymptotically quasi-nonexpansive if there exists a sequence \(\{r_n\}\) in \([0, \infty)\) with \(\lim_{n \to \infty} r_n = 0\) and \(|T^n x - p| \leq (1 + r_n)|x - p|\), for all \(x \in C\), \(p \in F(T)\) and \(n = 1, 2, 3, \ldots\);
(v) uniformly \(L\)-Lipschitzian if there exists constant \(L > 0\) such that \(|T^n x - T^n y| \leq L|x - y|\), for all \(x, y \in C\) and \(n = 1, 2, 3, \ldots\);
(vi) \((L - \gamma)\) uniform Lipschitz if there are constants \(L > 0\) and \(\gamma > 0\) such that \(|T^n x - T^n y| \leq L|x - y|^{\gamma}\), for all \(x, y \in C\) and \(n = 1, 2, 3, \ldots\);
(vii) semi-compact if for a sequence \(\{x_n\}\) in \(C\) with \(\lim_{n \to \infty} ||x_n - Tx_n|| = 0\), there exists a subsequence \(\{x_{n_i}\}\) of \(\{x_n\}\) such that \(x_{n_i} \to p \in C\).

From the definitions, it is easy to see that,
(i) a nonexpansive mapping must be quasi-nonexpansive and asymptotically nonexpansive;
(ii) an asymptotically nonexpansive mapping is an asymptotically quasi-nonexpansive and uniformly \(L\)-Lipschitzian;
(iii) a uniformly \(L\)-Lipschitzian mapping is \((L - 1)\) uniform Lipschitz.

**Condition (A)**. Let \(C\) be a subset of a normed space \(X\). A family of self-mappings \(\{T_i : i = 1, 2, \ldots, k\}\) of \(C\) is said to have Condition (A) if there exists a nondecreasing function \(f : [0, \infty) \to [0, \infty)\) with \(f(0) = 0\) and \(f(r) > 0\) for all \(r \in (0, \infty)\) such that \(|x - T_i x| \geq f(d(x, F))\) for some \(1 \leq i \leq k\) and for all \(x \in C\), where

\[
d(x, F) = \inf \left\{ ||x - p|| : p \in F = \bigcap_{i=1}^{k} F(T_i) \right\}.
\]

The map \(T : C \to X\) is said to be demiclosed at 0 if for each sequence \(\{x_n\}\) in \(C\) converging weakly to \(x \in C\) and \(Tx_n\) converging strongly to 0, we get \(Tx = 0\).

A Banach space \(X\) is said to have Opial’s property if for each sequence \(\{x_n\}\) converging weakly to \(x \in C\) and \(x \neq y\), we have the condition

\[
\lim_{n \to \infty} ||x_n - x|| < \liminf_{n \to \infty} ||x_n - y||.
\]


The problem of finding common fixed points of nonlinear mapping is of practical importance. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. Many researchers [4-7] are interested in studying approximation method for finding common fixed points of nonlinear mapping.


Fukhar-ud-din and Khan [10] studied a iterative process with errors to approximate the common fixed points of two asymptotically quasi-nonexpansive mappings.
Recently, Khan et al. [11] introduced an iterative process for a finite family of mappings as follows: Let $C$ be a convex subset of a Banach space $X$ and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of self-mappings of $C$. Suppose that $a_{in} \in [0, 1]$ for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$. For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

\[
\begin{align*}
  x_{n+1} &= (1 - a_{kn}) x_n + a_{kn} T^k_n y_{(k-1)n}, \\
  y_{(k-1)n} &= (1 - a_{(k-1)n}) x_n + a_{(k-1)n} T_{k-1}^n y_{(k-2)n}, \\
  y_{(k-2)n} &= (1 - a_{(k-2)n}) x_n + a_{(k-2)n} T_{k-2}^n y_{(k-3)n}, \\
  &\vdots \\
  y_{2n} &= (1 - a_{2n}) y_{1n} + a_{2n} T_2^n y_{1n}, \\
  y_{1n} &= (1 - a_{1n}) y_{0n} + a_{1n} T_1^n y_{0n},
\end{align*}
\]

(1.1)

where $y_{0n} = x_n$ for all $n$. The iterative process (1.1) is the generalized form of the modified Mann (one-step) iterative process by Schu [12], the modified Ishikawa (two-step) iterative process by Tan and Xu [13], and the three-step iterative process by Xu and Noor [14].

Very recently, Kettapun et al. [15] introduced an iterative process for a finite family of mappings as follows: Let $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

\[
\begin{align*}
  x_{n+1} &= (1 - a_{kn}) y_{(k-1)n} + a_{kn} T^k_n y_{(k-1)n}, \\
  y_{(k-1)n} &= (1 - a_{(k-1)n}) y_{(k-2)n} + a_{(k-1)n} T_{k-1}^n y_{(k-2)n}, \\
  y_{(k-2)n} &= (1 - a_{(k-2)n}) y_{(k-3)n} + a_{(k-2)n} T_{k-2}^n y_{(k-3)n}, \\
  &\vdots \\
  y_{2n} &= (1 - a_{2n}) y_{1n} + a_{2n} T_2^n y_{1n}, \\
  y_{1n} &= (1 - a_{1n}) y_{0n} + a_{1n} T_1^n y_{0n},
\end{align*}
\]

(1.2)

where $y_{0n} = x_n$ for all $n$.

Motivated by Khan et al. [11] and Kettapun et al. [15], we introduce a new iterative scheme for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings as follows: for $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

\[
\begin{align*}
  x_{n+1} &= a_{kn} x_n + b_{kn} y_{(k-1)n} + c_{kn} T^k_n y_{(k-1)n}, \\
  y_{(k-1)n} &= a_{(k-1)n} x_n + b_{(k-1)n} y_{(k-2)n} + c_{(k-1)n} T_{k-1}^n y_{(k-2)n}, \\
  y_{(k-2)n} &= a_{(k-2)n} x_n + b_{(k-2)n} y_{(k-3)n} + c_{(k-2)n} T_{k-2}^n y_{(k-3)n}, \\
  &\vdots \\
  y_{2n} &= a_{2n} x_n + b_{2n} y_{1n} + c_{2n} T_2^n y_{1n}, \\
  y_{1n} &= a_{1n} x_n + b_{1n} y_{0n} + c_{1n} T_1^n y_{0n},
\end{align*}
\]

(1.3)

where $y_{0n} = x_n$, $(a_{(i)n}, b_{(i)n}, c_{(i)n}) \in I \times I \times I$, $I = [0, 1]$ and $a_{(i)n} + b_{(i)n} + c_{(i)n} = 1$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$.

The aim of this paper is to obtain some strong and weak convergence results for the iterative process (1.3) of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

We need the following useful known lemmas for the development of our convergence results.

**Lemma 1.1.** (Cf. [7, Lemma 2.2]). Let the sequences $\{a_n\}$ and $\{\delta_n\}$ of real numbers satisfy:

\[ a_{n+1} \leq (1 + \delta_n) a_n, \quad \text{where } a_n \geq 0, \delta_n \geq 0 \text{ for all } n = 1, 2, 3, \ldots, \]

and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then

(i) $\lim_{n \to \infty} a_n$ exists;

(ii) if $\liminf_{n \to \infty} a_n = 0$, then $\lim_{n \to \infty} a_n = 0$.

**Lemma 1.2.** (See [13 Lemma 1.3]). Let $X$ be a uniformly convex Banach space. Assume that $0 < b \leq t_n \leq c < 1$, $n = 1, 2, 3, \ldots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in $X$ be such that $\limsup_{n \to \infty} \|x_n\| \leq a$, $\limsup_{n \to \infty} \|y_n\| \leq a$ and $\lim_{n \to \infty} \|t_n x_n + (1-t_n) y_n\| = a$, where $a \geq 0$. Then $\lim_{n \to \infty} \|x_n - y_n\| = 0$. 
2. Convergence in Banach spaces

The aim of this section is to establish the strong convergence of the iterative scheme \( (1.3) \) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space under some appropriate conditions.

**Lemma 2.1.** Let \( C \) be a nonempty closed convex subset of a real Banach space \( X \), and \( \{T_i : i = 1, 2, \ldots, k\} \) be a family of asymptotically quasi-nonexpansive self-mappings of \( C \), i.e., \( \|T_i^n x - p_i\| \leq (1 + r_n)\|x - p_i\| \), for all \( x \in C \) and \( p_i \in F(T_i) \), \( i = 1, 2, \ldots, k \). Let \( (a(i)_n, b(i)_n, c(i)_n) \in I \times I \times I \), where \( I = [0, 1] \) and \( a(i)_n + b(i)_n + c(i)_n = 1 \), for all \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, \ldots, k \). Suppose that \( F = \bigcap_{i=1}^k F(T_i) \neq \emptyset \), \( x_1 \in C \), and the iterative sequence \( \{x_n\} \) is defined by \( (1.3) \). Then for \( p \in F \), we get

(i) \( \|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\| \), for all \( i = 1, 2, \ldots, k \);
(ii) \( \|y_n - p\| \leq (1 + r_n)\|x_n - p\| \), for \( i = 1, 2, \ldots, k - 1 \);
(iii) \( \|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| \);
(iv) if \( \sum_{n=1}^{\infty} r_n < \infty \), \( \lim_{n \to \infty} \|x_n - p\| \) exists,

where \( r_n = \max_{1 \leq i \leq k} \{r_{in}\} \) and \( \delta_n = \sum_{i=1}^{k} (ki) r_n^i \).

**Proof.** Let \( p \in F \).

(i) For \( i = 1, 2, \ldots, k \), we have

\[
\|T_i^n y_{(i-1)n} - p\| \leq (1 + r_n)\|y_{(i-1)n} - p\|.
\]

(ii) Set \( y_{kn} = x_{n+1} \). Using part (i), we obtain

\[
\|y_{1n} - p\| = \|a_{1n}x_n + b_{1n}x_n + c_{1n}T_1^n x_n - p\|
\leq a_{1n}\|x_n - p\| + b_{1n}\|x_n - p\| + c_{1n}\|T_1^n x_n - p\|
\leq (1 - c_{1n})\|x_n - p\| + c_{1n}(1 + \gamma_n)\|x_n - p\|
\leq (1 + \gamma_n)\|x_n - p\|.
\]

Assume that \( \|y_{jn} - p\| \leq (1 + \gamma_n)^j\|x_n - p\| \) holds for some \( 1 \leq j \leq k - 1 \). We have

\[
\|y_{(j+1)n} - p\| = \|a_{(j+1)n}x_n + b_{(j+1)n}y_n + c_{(j+1)n}T_1^n y_{jn} - p\|
\leq a_{(j+1)n}\|x_n - p\| + b_{(j+1)n}\|y_n - p\| + c_{(j+1)n}\|T_1^n y_{jn} - p\|
\leq a_{(j+1)n}\|x_n - p\| + b_{(j+1)n}\|y_n - p\| + c_{(j+1)n}(1 + r_n)\|y_n - p\|
\leq a_{(j+1)n}\|x_n - p\| + (1 - a_{(j+1)n})(1 + r_n)\|y_n - p\|
\leq (1 + r_n)^{j+1}\|x_n - p\| + (1 - a_{(j+1)n})(1 + r_n)^{j+1}\|x_n - p\|
\leq (1 + r_n)^{j+1}\|x_n - p\|.
\]

Thus, by induction, we have

\[
\|y_n - p\| \leq (1 + r_n)^j\|x_n - p\| \quad \text{for } i = 1, 2, \ldots, k.
\]

(iii) By part part (ii), we get

\[
\|x_{n+1} - p\| = \|y_{kn} - p\| \leq (1 + r_n)^k\|x_n - p\|
\leq (1 + \delta_n)\|x_n - p\|.
\]

(iv) By (iii), we have \( \|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\| \). From \( \sum_{n=1}^{\infty} r_n < \infty \), we also have \( \sum_{n=1}^{\infty} r_n^i < \infty \), for \( i = 1, 2, \ldots, k \). It follows that \( \sum_{n=1}^{\infty} \delta_n = \sum_{n=1}^{\infty} [\sum_{i=1}^{k} (ki) r_n^i] = \sum_{i=1}^{k} [(ki) \sum_{n=1}^{\infty} r_n^i] < \infty \). By Lemma 1, we get \( \lim_{n \to \infty} \|x_n - p\| \) exists.
Theorem 2.2. Let $C$ be a nonempty closed convex subset of a real Banach space $X$, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of asymptotically quasi-nonexpansive self-mappings of $C$, i.e., $\|T_i^n x - p_i\| \leq (1 + r_{in}) \|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$ is defined by (1.3). Let $(a_{(i)n}, b_{(i)n}, c_{(i)n}) \in I \times I \times I$, where $I = [0, 1]$ and $a_{(i)n} + b_{(i)n} + c_{(i)n} = 1$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

Proof. Since the necessity is obvious, we will only prove the sufficiency. Take the infimum over $F$ in (2.2), we have

$$d(x_{n+1}, F) \leq (1 + \delta_n) d(x_n, F).$$

By $\sum_{n=1}^{\infty} \delta_n < \infty$, $\liminf_{n \to \infty} d(x_n, F) = 0$, and Lemma 1, we get that $\lim_{n \to \infty} d(x_n, F) = 0$.

Next, we show that $\{x_n\}$ is a Cauchy sequence in $C$. Let $p \in F$. By Lemma 3(iv), $\lim_{n \to \infty} \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. Set $M = \sup_{n \geq 1} \{\|x_n - p\|\}$. From Lemma 3(iii), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M \delta_n. \quad (2.3)$$

Thus, for positive integers $m$ and $n$, we have

$$\|x_{n+m} - p\| \leq \|x_{n+m-1} - p\| + M \delta_{n+m-1} \leq \|x_{n+m-2} - p\| + M (\delta_{n+m-1} + \delta_{n+m-2}) \ldots \leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \delta_i.$$ 

Therefore, for any $p \in F$, we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\| + M \sum_{i=n}^{n+m-1} \delta_i.$$ 

Hence,

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) + M \sum_{i=n}^{n+m-1} \delta_i.$$ 

By $\lim_{n \to \infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, we get that $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, $x_n \to q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and $C$ is a closed subset of $X$. Next we show that $q \in F$. Since $F(T_i)$ is a closed subset in $C$ for all $i = 1, 2, \ldots, k$, so is $F = \bigcap_{i=1}^{k} F(T_i)$. From the continuity of $d(x, F)$ with $d(x_n, F) \to 0$ and $x_n \to q$ as $n \to \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete.

Since any asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive, the next corollary is obtained immediately from Theorem 4.

Corollary 2.3. Let $C$ be a nonempty closed convex subset of a real Banach space $X$, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of asymptotically nonexpansive self-mappings of $C$, i.e., $\|T_i^n x - T_i^n y\| \leq (1 + r_{in}) \|x - y\|$, for all $x, y \in C$, $i = 1, 2, \ldots, k$. Let $(a_{(i)n}, b_{(i)n}, c_{(i)n}) \in I \times I \times I$, where $I = [0, 1]$ and $a_{(i)n} + b_{(i)n} + c_{(i)n} = 1$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$ is defined by (1.3). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.
3. Results in uniformly convex Banach spaces

In this section, we establish some weak and strong convergence results for the iterative scheme by removing the condition \( \lim \inf_{n \to \infty} d(x_n, F) = 0 \) from the results obtained in Section 3. Instead, we consider the class of \((L - \gamma)\) uniform Lipschitz and asymptotically quasi-nonexpansive mappings on a uniformly convex Banach space.

**Lemma 3.1.** Let \( C \) be a nonempty closed convex subset of an uniformly convex real Banach space \( X \), and \( \{T_i : i = 1, 2, \ldots, k\} \) be a family of \((L - \gamma_i)\) uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of \( C \), i.e., \( \|T^n_i x - T^n_i y\| \leq L \|x - y\|^{\gamma_i} \), and \( \|T^n_i x - p_i\| \leq (1 + r_i) \|x - p_i\| \), for all \( x, y \in C \) and \( p_i \in F(T_i), i = 1, 2, \ldots, k \). Suppose that \( F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset \), \( x_1 \in C \), and the iterative sequence \( \{x_n\} \) is defined by \( (3.3) \). For \( n = 1, 2, 3, \ldots \) and \( i = 1, 2, \ldots, k \), let \( (a(i)_n, b(i)_n, c(i)_n) \in I \times I \times I \), where \( I = [0, 1] \) and \( a(i)_n + b(i)_n + c(i)_n = 1 \), \( a_n \in [\delta, 1 - \delta] \) for some \( \delta \in (0, \frac{1}{2}) \), and \( b_n \in [a, b] \) for some \( 0 < a < b < 1 \). Assume that \( \sum_{n=1}^{\infty} r_n < \infty \), where \( r_n = \max_{1 \leq i \leq k} \{r(i)_n\} \). Then,

(i) \( \lim_{n \to \infty} \|x_n - T^n_i y(i-1)_n\| = 0 \), for all \( i = 1, 2, \ldots, k \);

(ii) \( \lim_{n \to \infty} \|x_n - T^n_i x_n\| = 0 \), for all \( i = 1, 2, \ldots, k \).

**Proof.** (i) Let \( p \in F \). By Lemma 3(iv), we obtain that \( \lim_{n \to \infty} \|x_n - p\| \) exists and we then suppose that

\[
\lim_{n \to \infty} \|x_n - p\| = c.
\]  

By (3.1) and Lemma 3(ii), we have

\[
\lim \sup_{n \to \infty} \|y(i)_n - p\| \leq c, \quad \text{for } i = 1, 2, \ldots, k - 1.
\]  

We also notice that

\[
\|x_{n+1} - p\| = \|a_kx_n + b_ky(k-1)_n + c_kT^n_k y(k-1)_n - p\|
\leq a_k\|x_n - p\| + b_k\|y(k-1)_n - p\| + c_k(1 + r_n)\|y(k-1)_n - p\|
\leq a_k\|x_n - p\| + (1 - a_k)(1 + r_n)\|y(k-1)_n - p\|
\leq \left(1 - (1 - a_k)(1 - a(k-1)_n) \cdots (1 - a_{(i+1)}_n)\right)(1 + r_n)^{k-i}\|x_n - p\|
+ (1 - a_k)(1 - a(k-1)_n) \cdots (1 - a_{(i+1)}_n)(1 + r_n)^{k-i}\|y(i)_n - p\|
\]

which implies

\[
\|x_n - p\| \leq \frac{\|x_n - p\|}{(1 - a_k)(1 - a(k-1)_n) \cdots (1 - a_{(i+1)}_n)}
- \frac{\|x_{n+1} - p\|}{(1 - a_k)(1 - a(k-1)_n) \cdots (1 - a_{(i+1)}_n)(1 + r_n)^{k-i} + \|y(i)_n - p\|}
\leq \frac{\|x_n - p\|}{(1 - a_k)(1 - a(k-1)_n) \cdots (1 - a_{(i+1)}_n)(1 + r_n)^{k-i}} + \|y(i)_n - p\|.
\]

Therefore,

\[
\|x_n - p\| \leq \frac{\|x_n - p\|(1 + r_n)^k - \|x_{n+1} - p\|}{\delta^{k-i}(1 + r_n)^{k-i}} + \|y(i)_n - p\|
\]

and hence

\[
c \leq \lim \inf_{n \to \infty} \|y(i)_n - p\|, \quad \text{for } i = 1, 2, \ldots, k - 1.
\]
Combining (3.2) and (3.3), we have
\[
\lim_{n \to \infty} \|y_{kn} - p\| = c, \quad \text{for} \ i = 1, 2, \ldots, k - 1.
\]
Let \( y_{kn} = x_{n+1} \), then we have \( \lim_{n \to \infty} \|y_{kn} - p\| = \lim_{n \to \infty} \|x_{n+1} - p\| = c \). So, we get, for \( i = 1, 2, \ldots, k \),
\[
\lim_{n \to \infty} \|a_{in}(x_n - p) + b_{in}(y_{(i-1)n} - p) + c_{in}T^n_i y_{(i-1)n} - p\|
\]
\[
= \lim_{n \to \infty} \|a_{in}(x_n - p) + (1 - a_{in}) \left( \frac{b_{in}}{1 - a_{in}} (y_{(i-1)n} - p) + \frac{c_{in}}{1 - a_{in}} (T^n_i y_{(i-1)n} - p) \right) \|
\]
\[
= \lim_{n \to \infty} \|c_{in}(T^n_i y_{(i-1)n} - p) + (1 - c_{in}) \left( \frac{a_{in}}{1 - c_{in}} (x_n - p) + \frac{b_{in}}{1 - c_{in}} (y_{(i-1)n} - p) \right) \|
\]
\[
= c.
\]
Also, from Lemma 3(i) and (3.2), we obtain
\[
\limsup_{n \to \infty} \|T^n_i y_{(i-1)n} - p\| \leq c, \quad \text{for} \ i = 1, 2, \ldots, k.
\]
Thus, we can get, for \( i = 1, 2, \ldots, k \),
\[
\limsup_{n \to \infty}\left[ \frac{b_{in}}{1 - a_{in}} (y_{(i-1)n} - p) + \frac{c_{in}}{1 - a_{in}} (T^n_i y_{(i-1)n} - p) \right]
\]
\[
\leq \limsup_{n \to \infty}\left[ \frac{b_{in}}{1 - c_{in}} (x_n - p) + \frac{c_{in}}{1 - c_{in}} (y_{(i-1)n} - p) \right]
\]
\[
\leq c,
\]
and
\[
\limsup_{n \to \infty}\left[ \frac{a_{in}}{1 - c_{in}} (x_n - p) + \frac{b_{in}}{1 - c_{in}} (y_{(i-1)n} - p) \right]
\]
\[
\leq \limsup_{n \to \infty}\left[ \frac{a_{in}}{1 - a_{in}} (y_{(i-1)n} - p) + \frac{b_{in}}{1 - a_{in}} (x_n - p) \right]
\]
\[
\leq c.
\]
By Lemma 2, we obtain, for \( i = 1, 2, \ldots, k \),
\[
\lim_{n \to \infty} \frac{1}{1 - a_{in}} \left[ (1 - a_{in})x_n - (b_{in}y_{(i-1)n} + c_{in}T^n_i y_{(i-1)n}) \right] = 0,
\]
and
\[
\lim_{n \to \infty} \frac{1}{1 - c_{in}} \left[ (1 - c_{in})T^n_i y_{(i-1)n} - (a_{in}x_n + b_{in}y_{(i-1)n}) \right] = 0.
\]
So, we have, for \( i = 1, 2, \ldots, k \),
\[
\lim_{n \to \infty} \|x_n - y_{in}\|
\]
\[
= \lim_{n \to \infty} \|x_n - (b_{in}y_{(i-1)n} + a_{in}x_n + c_{in}T^n_i y_{(i-1)n})\|
\]
\[
= (1 - a_{in}) \frac{1}{1 - a_{in}} \| (1 - a_{in})x_n - (b_{in}y_{(i-1)n} + c_{in}T^n_i y_{(i-1)n}) \|
\]
\[
= 0,
\]
and
\[
\lim_{n \to \infty} \|T^n_i y_{(i-1)n} - y_{in}\|
\]
\[
= \lim_{n \to \infty} \|T^n_i y_{(i-1)n} - (a_{in}x_n + b_{in}y_{(i-1)n} + c_{in}T^n_i y_{(i-1)n})\|
\]
\[
= (1 - c_{in}) \frac{1}{1 - c_{in}} \| (1 - c_{in})T^n_i y_{(i-1)n} - (a_{in}x_n + b_{in}y_{(i-1)n}) \|
\]
\[
= 0.
\]
Combining (3.6)-(3.7), we can get
\[
\lim_{n \to \infty} \|x_n - T^n_i y_{(i-1)n}\| = 0, \quad \text{for} \ i = 1, 2, \ldots, k.
\]
(ii) From part (i), for $i = 1$, we have
\[
\lim_{n \to \infty} \|T_i^n x_n - x_n\| = 0. \tag{3.9}
\]
For $i = 2, 3, \ldots, k$, we get
\[
\lim_{n \to \infty} \|T_i^n x_n - x_n\| \leq \|T_i^n x_n - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - x_n\| \\
\leq \|x_n - y_{(i-1)n}\|^{\gamma_i} + \|T_i^n y_{(i-1)n} - x_n\|.
\]
By part (i) and (3.6), we conclude that
\[
\lim_{n \to \infty} \|T_i^n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k.
\tag{3.10}
\]
From (1.3), we have
\[
\|x_{n+1} - x_n\| \leq b_{kn}\|y_{(k-1)n} - x_n\| + c_{kn}\|T_k^n y_{(k-1)n} - x_n\|.
\]
From (3.6) and (3.8), we have
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}
\]
For $i = 1, 2, \ldots, k$, we have
\[
\|x_n - T_i x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^{n+1} x_{n+1} - T_i^n x_n\| \\
\quad + \|T_i^n x_n - T_i x_n\| \\
\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i^n x_n - x_n\|^{\gamma_i} \\
\quad + L\|T_i^n x_n - x_n\|^{\gamma_i}.
\]
Using (3.10) and (3.11), we obtain
\[
\lim_{n \to \infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k.
\]

**Theorem 3.2.** Under the hypotheses of Lemma 6, suppose that $\{T_i : i = 1, 2, \ldots, k\}$ satisfies Condition (A). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

**Proof.** By using Condition (A), there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that
\[
\|x_n - T_j x_n\| \geq f(d(x_n, F))
\]
for some $1 \leq j \leq k$. From Lemma 6(ii), we have
\[
\lim_{n \to \infty} \|x_n - T_j x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k.
\]
So, we get $\lim_{n \to \infty} f(d(x_n, F)) = 0$ which implies
\[
\lim_{n \to \infty} d(x_n, F) = 0.
\]
By Theorem 4, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$.

**Theorem 3.3.** Under the hypotheses of Lemma 6, assume that $T_j^m$ is semi-compact for some positive integers $m$ and $1 \leq j \leq k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \ldots, k\}$. 
Proof. Suppose that $T_j^m$ is semi-compact for some positive integers $m$ and $1 \leq j \leq k$. We have

$$\|T_j^m x_n - x_n\| \leq \|T_j^m x_n - T_j^{m-1} x_n\| + \|T_j^{m-1} x_n - T_j^{m-2} x_n\| + \cdots + \|T_j^2 x_n - T_j x_n\| + \|T_j x_n - x_n\|$$

$$\leq (m-1)L\|T_j x_n - x_n\|^\gamma + \|T_j x_n - x_n\|.$$ 

Then, by Lemma 6(ii), we get $\|T_j^m x_n - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and $T_j^m$ is semi-compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \to q \in C$ as $l \to \infty$.

By continuity of $T_i$ and Lemma 6(ii), we obtain

$$\|q - T_i q\| = \lim_{l \to \infty} \|x_{n_l} - T_i x_{n_l}\| = 0,$$ for all $j = 1, 2, \ldots, k$.

Therefore, $q \in F$, and $\lim \inf_{n \to \infty} d(x_n, F) = 0$ and then Theorem 4 implies that $\{x_n\}$ converges strongly to a common fixed point $q$ of the family $\{T_i : i = 1, 2, \ldots, k\}$.

Theorem 8 is very useful in the case that one of $T_i : i = 1, 2, \ldots, k$, is semi-compact.

**Theorem 3.4.** Let $C$ be a nonempty closed convex subset of an uniformly convex real Banach space $X$ satisfying the Opial property, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymptotically quasi-nonexpansive self-mappings of $C$, i.e., $\|T_i^m x - T_i^n y\| \leq L\|x - y\|^\gamma_i$, and $\|T_i^m x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i)$, $i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$ is defined by (1.3). For $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$, let $(a(i)n, b(i)n, c(i)n) \in I \times I \times I$, where $I = [0, 1]$ and $a(i)n + b(i)n + c(i)n = 1$, $a_n \in [\delta, 1 - \delta]$ for some $\delta \in (0, \frac{1}{2})$, and $\delta_n \in [a, b]$ for some $0 < a < b < 1$. Assume that $\sum_{n=1}^\infty r_n < \infty$, where $r_n = \max_{1 \leq i \leq k} \{r_{in}\}$. If $I - T_i$, $i = 1, 2, \ldots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

**Proof.** Let $p \in F$. By Lemma 3(iv), we get $\lim_{n \to \infty} \|x_n - p\|$ exists. Then we follow the proof of Theorem 3.2 by Khan et al. [11] until we can conclude that $\{x_n\}$ converges weakly to a common fixed point $p \in F$.

**Remark 3.5.** It is clear that Theorems 4, 7, 8 and 9 can be used for any asymptotically nonexpansive mapping.

4. An example

In this section, we will construct a finite family of quasi-nonexpansive mappings satisfying the conditions of Theorem 7.

Let $X = l^2$ with the norm $\| \cdot \|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^\infty x_i^2}, \quad \forall x = (x_1, x_2, \ldots, x_n, \ldots) \in X,$$

and $C = \{x = (x_1, x_2, \ldots, x_n, \ldots) | x_1 \leq 0, \; x_i \in R^1, \; i = 2, 3, \ldots\}$. Then it is obvious that $C$ is a nonempty closed convex subset of $X$.

Now, for any $x = (x_1, x_2, \ldots, x_n, \ldots) \in C$, define an finite family of mappings $T_i : C \to C$, $i = 1, 2, \ldots, k$ as follows:

$T_1(x) = (0, 4x_1, 0, \ldots, 0),$

$T_2(x) = (0, 0, 4x_1, 0, \ldots, 0),$

$\vdots$
Therefore we get the desired result.

It is easy to see that if we get \( F.Q.-L. \) Dong, S. He, B.-C. Deng, J. Nonlinear Sci. Appl. 5 (2012), 232–242 241

It is easy to see that

where

So

Combining the result (i), we obtain (4.1).

For

\( T_i(x) = (0, \ldots, 0, 4x_1, \ldots, 0) \)

we get \( F(T_i) = \{0\} \) and

\[
T^n_i(x) = (0, 0, 0, \ldots, 0, \ldots), \quad \forall n = 2, 3, \ldots, \quad i = 1, 2, \ldots, k.
\]

For \( i = 1, 2, \ldots, k, \forall n \geq 1 \), take \( r_{in} \in [0, \infty) \) with \( r_{i1} = 3 \) and \( \lim_{n \to \infty} r_{in} = 0 \), and \( \forall p \in F(T_i) \), we have

\[
\begin{align*}
\|T_i(x) - p\| - (1 + r_{i1})\|x - p\| &= \|(0, 0, 0, 4x_1, \ldots, 0)\| - (1 + 3)\|(x_1, x_2, \ldots, x_n, \ldots)\|
\end{align*}
\]

\[
= -4x_1 - 4 \sum_{i=1}^{\infty} x_i^2 \leq 0,
\]

and, for all \( n = 2, 3, \ldots, \)

\[
\|T^n_i(x) - p\| - (1 + r_{in})\|x - p\| = 0 - (1 + r_{in})\|x\| \leq 0.
\]

So \( T_i, i = 1, 2, \ldots, k, \) is an asymptotically quasi-nonexpansive mapping.

Next we prove the following results:

(i) \( T_i, i = 1, \ldots, k, \) is a Lipschitzian mapping.

For \( \forall x, y \in C \), we have

\[
\begin{align*}
\|T_i x - T_i y\| &\leq \|(0, 0, 0, 4x_1, \ldots, 0) - (0, 0, 0, 4y_1, \ldots, 0)\|
\end{align*}
\]

\[
= 4|x_1 - y_1|
\]

\[
\leq 4\|x - y\|,
\]

where \( i = 1, 2, \ldots, k. \) This indicates that \( T_i, i = 1, 2, \ldots, k, \) is a Lipschitzian mapping.

(ii) \( T_i, i = 1, \ldots, k, \) satisfies

\[
\|T^n_i x - T^n_i y\| \leq L\|x - y\|^\gamma, \quad \forall n \geq 1,
\]

where \( L = 4 \) and \( \gamma = 1. \)

For \( n \geq 2, \forall x, y \in C, \) we have

\[
\begin{align*}
\|T^n_i x - T^n_i y\| &= \|0 - 0\|
\end{align*}
\]

\[
\leq 4\|x - y\|.
\]

Combining the result (i), we obtain (4.1).

(iii) \( T_i, i = 1, 2, \ldots, k, \) is a semi-compact mapping.

Set a sequence \( \{x_n = (x_n^1, x_n^2, \ldots, x_n^m, \ldots)\}_{n=1}^\infty \), then we have, for \( i = 1, 2, \ldots, k, \)

\[
\|x_n - T_i x_n\| = \sqrt{(x_n^1)^2 + \ldots + (x_n^{i+1} - 4x_n^1)^2 + \ldots + (x_n^m)^2 + \ldots}
\]

It is easy to see that if \( \|x_n - T_i x_n\| \to 0 \), then we have \( x_n^m \to 0, m = 1, 2, \ldots, \) which yields \( \|x_n\| \to 0. \)

Therefore we get the desired result.
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