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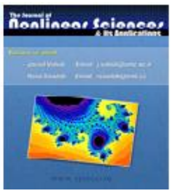

Coupled coincidence points for compatible mappings satisfying mixed monotone property

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

We establish coupled coincidence and coupled fixed point results for a pair of mappings satisfying a compatibility hypothesis in partially ordered metric spaces. An example is given to illustrate our obtained results.

Keywords: Compatible mappings, Coupled fixed point, mixed monotone property, partially ordered set
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1. Introduction and Preliminaries

In the last years, fixed points of mappings in partially ordered metric spaces have been investigated by many researchers [1, 2, 3, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22]. The first result in this direction was given by Ran and Reurings [17, Theorem 2.1] who presented its applications to linear and nonlinear metric spaces. Subsequently, Nieto and Rodŕiguez-López [15] extended the result of Ran and Reurings [17] for non-decreasing mappings and applied it to obtain a unique solution for a first order ordinary differential equation with periodic boundary conditions. Similar applications based on a version of Theorems 2.1-2.5 [15] for a mixed monotone mapping $F : X \times X \to X$ were given by Bhaskar and Lakshmikantham [4]. In

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[4], Bhaskar and Lakshmikantham introduced the notion of a coupled fixed point and proved some coupled fixed point theorems for mappings satisfying a mixed monotone property. They discussed the problem of uniqueness of coupled fixed point and applied their theorems to problems of existence and uniqueness of solution for a periodic boundary value problem. Recently, Lakshmikantham and Ćirić [11] introduced the concept of mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems for commuting mappings, extending the theorems due to Bhaskar and Lakshmikantham [4]. Successively, Choudhury and Kundu [5], introduced the notion of compatibility of mappings in a partially ordered metric space and used this notion to establish a coupled coincidence point result which extends the works of Bhaskar and Lakshmikantham [4] and Lakshmikantham and Ćirić [11].

Now, we recall some definitions introduced in [4, 5, 11].

Let $(X, \preceq)$ be a partially ordered set and $F : X \to X$ be a mapping. The mapping F is said to be non-decreasing if for all $x, y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$. Similarly, F is said to be non-increasing, if for all $x, y \in X$, $x \preceq y$ implies $F(x) \succeq F(y)$.

Bhaskar and Lakshmikantham [4] introduced the following notions of mixed monotone mapping and coupled fixed point.

Definition 1.1. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for all $x_1, x_2 \in X$, $x_1 \preceq x_2$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $y_1 \preceq y_2$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

The concept of the mixed monotone property is generalized by Lakshmikantham and Ćirić [11] as follows.

Definition 1.2. [11]. Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for all $x_1, x_2 \in X$, $g(x_1) \preceq g(x_2)$ implies $F(x_1, y) \preceq F(x_2, y)$, for any $y \in X$ and for all $y_1, y_2 \in X$, $g(y_1) \preceq g(y_2)$ implies $F(x, y_1) \succeq F(x, y_2)$, for any $x \in X$.

Clearly, if g is the identity mapping, then Definition 1.2 reduces to Definition 1.1.

Definition 1.3. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.4. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 1.5. Let (X, d) be a metric space, $F : X \times X \to X$ and $g : X \to X$. Then, F and g are compatible if

$$\lim_{n\to+\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to+\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X, such that

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = x \text{ and } \lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y$$

for all $x, y \in X$.

In this paper, we generalize the results of Bhaskar and Lakshmikantham [4] by considering generalized contractive conditions for a pair of mappings and prove results concerning coupled coincidence point and coupled fixed point. We give also an example to illustrate our results.

2. The Main Result

Our first result is the following coupled coincidence point theorem.

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β, L with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d(F(x,y), F(u,v)) \quad &\leq \quad \alpha \min\{d(F(x,y), g(x)), d(F(u,v), g(x))\} + \beta \min\{d(F(x,y), g(u)), d(F(u,v), g(u))\} \\ &\quad + L \min\{d(F(x,y), g(u)), d(F(u,v), g(x))\}, \end{aligned} \tag{2.1}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose that $F(X \times X) \subseteq g(X)$, g is continuous non-decreasing, g and F are compatible and also either

(a) F is continuous or

(b) X has the following property:

(i) if a non-decreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,

(ii) if a non-increasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,

holds. Then, there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Let $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$.

Analogously, there exist $x_2, y_2 \in X$ such that $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$.

Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$g(x_{n+1}) = F(x_n, y_n) \text{ and } g(y_{n+1}) = F(y_n, x_n) \ \forall \ n \geq 0. \tag{2.2}$$

Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \text{ and } g(y_n) \succeq g(y_{n+1}). \tag{2.3}$$

We shall use the mathematical induction. Let $n = 0$, since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (2.3) holds for $n = 0$. We assume that (2.3) hold for some $n > 0$. As F has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (2.2), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, y_n), \ \ F(y_{n+1}, x_n) \preceq F(y_n, x_n) = g(y_{n+1}). \tag{2.4}$$

Also for the same reason we have

$$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n), \ F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2}).$$

Merging the above results, we obtain $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$.

Thus by the mathematical induction, we conclude that (2.3) holds for all $n \geq 0$.

We check easily that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \preceq g(x_{n+1}) \preceq \cdots$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_{n+1}) \succeq \cdots .$$

Since $g(x_n) \succeq g(x_{n-1})$ and $g(y_n) \preceq g(y_{n-1})$, from (2.1) and (2.2), we have

$$\begin{aligned} d(g(x_{n+1}), g(x_n)) &= d(F(x_n, y_n), F(x_{n-1}, y_{n-1}) \\ &\leq \alpha \min\{d(F(x_n, y_n), g(x_n)), d(F(x_{n-1}, y_{n-1}), g(x_n))\} \\ &\quad +\beta \min\{d(F(x_n, y_n), g(x_{n-1})), d(F(x_{n-1}, y_{n-1}), g(x_{n-1}))\} \\ &\quad +L \min\{d(F(x_n, y_n), g(x_{n-1})), d(F(x_{n-1}, y_{n-1}), g(x_n))\}, \end{aligned}$$

that is

$$d(g(x_{n+1}), g(x_n)) \leq \beta d(g(x_n), g(x_{n-1})). \tag{2.5}$$

Similarly, since $g(y_{n-1}) \succeq g(y_n)$ and $g(x_{n-1}) \preceq g(x_n)$, from (2.1) and (2.2), we have

$$d(g(y_n), g(y_{n+1})) \leq \alpha d(g(y_n), g(y_{n-1})). \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned} d(g(x_{n+1}), g(x_n)) + d(g(y_n), g(y_{n+1})) &\leq \beta d(g(x_n), g(x_{n-1})) + \alpha d(g(y_n), g(y_{n-1})) \\ &\leq (\alpha+\beta) d(g(x_n), g(x_{n-1})) + (\alpha+\beta) d(g(y_n), g(y_{n-1})) \\ &= (\alpha+\beta)[d(g(x_n), g(x_{n-1})) + d(g(y_n), g(y_{n-1}))]. \end{aligned}$$

Set $\rho_n = d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))$ and $\delta = \alpha + \beta$, then the sequence $\{\rho_n\}$ is decreasing as

$$0 \leq \rho_n \leq \delta\rho_{n-1} \leq \delta^2\rho_{n-2} \leq \cdots \leq \delta^n \rho_0$$

which implies

$$\lim_{n\to+\infty} \rho_n = \lim_{n\to+\infty} [d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))] = 0. \tag{2.7}$$

Thus,

$$\lim_{n\to+\infty} d(g(x_{n+1}), g(x_n)) = 0 \text{ and } \lim_{n\to+\infty} d(g(y_{n+1}), g(y_n))] = 0.$$

In what follows, we shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.

For each $m \geq n$, we have

$$d(g(x_m), g(x_n)) \leq d(g(x_m), g(x_{m-1})) + d(g(x_{m-1}), g(x_{m-2})) + \cdots + d(g(x_{n+1}), g(x_n))$$

and

$$d(g(y_m), g(y_n)) \leq d(g(y_m), g(y_{m-1})) + d(g(y_{m-1}), g(y_{m-2})) + \cdots + d(g(y_{n+1}), g(y_n)).$$

Therefore

$$\begin{aligned} d(g(x_m), g(x_n)) + d(g(y_m), g(y_n)) &\leq \rho_{m-1} + \rho_{m-2} + \cdots + \rho_n \\ &\leq (\delta^{m-1} + \delta^{m-2} + \cdots + \delta^n)\rho_0 \\ &\leq \frac{\delta^n}{1-\delta}\rho_0 \end{aligned}$$

which implies that

$$\lim_{n,m\to+\infty} [d(g(x_m), g(x_n)) + d(g(y_m), g(y_n))] = 0.$$

This imply that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X. Now, since (X, d) is a complete metric space, there exists $(x, y) \in X \times X$ such that

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = x \quad \text{and} \quad \lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y. \tag{2.8}$$

From (2.8) and the continuity of g, we get

$$\lim_{n\to+\infty} g(g(x_n)) = g(x) \text{ and } \lim_{n\to+\infty} g(g(y_n)) = g(y). \tag{2.9}$$

From (2.2) and the compatibility of F and g, we have

$$\begin{cases} \lim_{n\to+\infty} d(g(F(x_n,y_n)),F(g(x_n),g(y_n))) = 0, \\ \lim_{n\to+\infty} d(g(F(y_n,x_n)),F(g(y_n),g(x_n))) = 0. \end{cases} \tag{2.10}$$

Finally, we claim that (x,y) is a coupled coincidence point of F and g.

Taking the limit as $n \to +\infty$ in (2.10), by (2.2), (2.8), (2.9) and the continuity of F, we get

$$\begin{aligned} g(x) = \lim_{n\to+\infty} g(g(x_{n+1})) &= \lim_{n\to+\infty} F(g(x_n),g(y_n)) \\ &= F(\lim_{n\to+\infty} g(x_n), \lim_{n\to+\infty} g(y_n)) = F(x,y), \end{aligned}$$

$$\begin{aligned} g(y) = \lim_{n\to+\infty} g(g(y_{n+1})) &= \lim_{n\to+\infty} F(g(y_n),g(x_n)) \\ &= F(\lim_{n\to+\infty} g(y_n), \lim_{n\to+\infty} g(x_n)) = F(y,x). \end{aligned}$$

Thus, we proved that $F(x,y) = g(x)$ and $F(y,x) = g(y)$.
Now, suppose that (b) holds. Since $\{g(x_n)\}$ is non-decreasing and $g(x_n) \to x$, and $\{g(y_n)\}$ is non-increasing and $g(y_n) \to y$, by assumption (b), we have $g(gx_n) \preceq g(x)$ and $g(gy_n) \succeq g(y)$ for all n. Then, we get

$$\begin{aligned} d(g(x),F(x,y)) &\leq d(g(x),g(g(x_{n+1}))) + d(g(g(x_{n+1})),F(x,y)) \\ &= d(g(x),g(g(x_{n+1}))) + d(g(F(x_n,y_n)),F(x,y)) \\ &= d(g(x),g(g(x_{n+1}))) + d(g(F(x_n,y_n)),F(g(x_n),g(y_n))) \\ &\quad + d(F(g(x_n),g(y_n)),F(x,y)) \\ &\leq d(g(x),g(g(x_{n+1}))) + d(g(F(x_n,y_n)),F(g(x_n),g(y_n))) \\ &\quad + \alpha \min\{d(F(g(x_n),g(y_n)),g(gx_n)), d(F(x,y),g(g(x_n)))\} \\ &\quad + \beta \min\{d(F(g(x_n),g(y_n)),g(x)), d(F(x,y),g(x))\} + \\ &\quad + L \min\{F(g(x_n),g(y_n)),g(x)), d(F(x,y),g(g(x_n)))\}. \end{aligned}$$

Taking the limit as $n \to +\infty$ in the above inequality and using (2.8) and (2.10), we get $d(g(x),F(x,y)) = 0$. Hence $g(x) = F(x,y)$. Similarly, one can show that $g(y) = F(y,x)$. Thus F and g have a coupled coincidence point. This makes end to the proof. □

If $g = I$, the identity mapping in Theorem 2.1, then we deduce the following result of coupled fixed point.

Corollary 2.2. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a mapping such that F has the mixed monotone property on X and there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Also suppose there exist non-negative real numbers α, β and L with $\alpha + \beta < 1$ such that*

$$\begin{aligned} d(F(x,y),F(u,v)) \leq\ & \alpha \min\{d(F(x,y),x), d(F(u,v),x)\} + \beta \min\{d(F(x,y),u), d(F(u,v),u)\} \\ & + L \min\{d(F(x,y),u), d(F(u,v),x)\}, \end{aligned}$$

for all $(x,y),(u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$ and either (a) or (b) of Theorem 2.1 holds. Then, there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$, that is, F has a coupled fixed point $(x,y) \in X \times X$.

Remark 2.3. By choosing α, β and L suitably, one can deduce some corollaries from Theorem 2.1.

For example, if $\alpha = \beta = 0$ in Theorem 2.1, then we can state the following corollary.

Corollary 2.4. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X and there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose that there exists a non-negative real number L such that*

$$d(F(x,y), F(u,v)) \leq L \min\{d(F(x,y), g(u)), d(F(u,v), g(x))\},$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous non-decreasing, g and F are compatible, and also suppose either (a) or (b) of Theorem 2.1 holds. Then, there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Now we give sufficient conditions for uniqueness of the coupled coincidence point. If $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x,y), (u,v) \in X \times X, \quad (u,v) \preceq (x,y) \Leftrightarrow x \succeq u,\ y \preceq v.$$

Theorem 2.5. *In addition to the hypotheses of Theorem 2.1, suppose that $L = 0$ and for every $(x, y), (x^*, y^*) \in X \times X$ there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F and g have a unique coupled coincidence point, that is, there exists a unique $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.*

Proof. From Theorem 2.1, the set of coupled coincidence points of F and g is non-empty. Suppose that (x, y) and (x^*, y^*) are coupled coincidence points of F and g, that is, $g(x) = F(x, y)$, $g(y) = F(y, x)$, $g(x^*) = F(x^*, y^*)$ and $g(y^*) = F(y^*, x^*)$, then we show that

$$g(x) = g(x^*) \text{ and } g(y) = g(y^*). \tag{2.11}$$

By assumption, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Put $u_0 = u$, $v_0 = v$, and choose $u_1, v_1 \in X$ so that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, proceeding as in the proof of Theorem 2.1, we can inductively define sequences $\{g(u_n)\}$, $\{g(v_n)\}$ such that

$$g(u_{n+1}) = F(u_n, v_n) \text{ and } g(v_{n+1}) = F(v_n, u_n)\ \forall\, n \geq 0.$$

Further, set $x_0 = x, y_0 = y$, $x_0^* = x^*, y_0^* = y^*$ and, on the same way, define the sequences $\{g(x_n)\}$, $\{g(y_n)\}$, $\{g(x_n^*)\}$ and $\{g(y_n^*)\}$. Then it is easy to show that

$$g(x_n) \to F(x,y),\ \ g(y_n) \to F(y,x),\ \ g(x_n^*) \to F(x^*,y^*) \text{ and } g(y_n^*) \to F(y^*,x^*)$$

as $n \to +\infty$.

Since $(F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y))$ and $(F(u, v), F(v, u)) = (g(u_1), g(v_1))$ are comparable, then $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$. It is easy to show that $(g(x), g(y))$ and $(g(u_n), g(v_n))$ are comparable, that is, $g(x) \preceq g(u_n)$ and $g(y) \succeq g(v_n)$ for all $n \geq 1$. Thus from (2.1), we have

$$\begin{aligned} d(g(x), g(u_{n+1})) &= d(F(x,y), F(u_n,v_n)) \\ &\leq \alpha \min\{d(F(x,y), g(x)), d(F(u_n,v_n), g(x))\} \\ &\quad + \beta \min\{d(F(x,y), g(u_n)), d(F(u_n,v_n), g(u_n))\}. \end{aligned}$$

Now, since $F(x, y) = g(x)$, we get

$$d(g(x), g(u_{n+1})) \leq \beta \min\{d(g(x), g(u_n)), d(F(u_n,v_n), g(u_n))\}$$

and hence

$$d(g(x), g(u_{n+1})) \leq \beta d(g(x), g(u_n)). \tag{2.12}$$

Again from (2.1), we have

$$\begin{aligned} d(g(v_{n+1}),g(y)) &= d(F(v_n,u_n),F(y,x)) \\ &\leq \alpha\min\{d(F(v_n,u_n),g(v_n)),d(F(y,x),g(v_n))\} \\ &\quad +\beta\min\{d(F(v_n,u_n),g(y)),d(F(y,x),g(y))\}. \end{aligned}$$

Since $F(y,x)=g(y)$, we get

$$d(g(v_{n+1}),g(y)) \leq \alpha\min\{d(F(v_n,u_n),g(v_n)),d(g(y),g(v_n))\},$$

and so

$$d(g(v_{n+1}),g(y)) \leq \alpha d(g(v_n),g(y)). \tag{2.13}$$

From (2.12) and (2.13), we have

$$\begin{aligned} d(g(x),g(u_{n+1}))+d(g(y),g(v_{n+1})) &\leq \beta d(g(x),g(u_n))+\alpha d(g(v_n),g(y)) \\ &\leq (\alpha+\beta)[d(g(x),g(u_n))+d(g(y),g(v_n))] \\ &\leq (\alpha+\beta)^2[d(g(x),g(u_{n-1}))+d(g(y),g(v_{n-1}))] \\ &\quad \cdots \\ &\leq (\alpha+\beta)^{n+1}[d(g(x),g(u_0))+d(g(y),g(v_0))]. \end{aligned}$$

Taking the limit as $n\to+\infty$, we get $\lim_{n\to+\infty}[d(g(x),g(u_n))+d(g(y),g(v_n))]=0$. It implies that

$$\lim_{n\to+\infty} d(g(x),g(u_n)) = \lim_{n\to+\infty} d(g(y),g(v_n)) = 0. \tag{2.14}$$

Following the same lines as above, one can show that

$$\lim_{n\to+\infty} d(g(x^*),g(u_n)) = \lim_{n\to+\infty} d(g(y^*),g(v_n)) = 0. \tag{2.15}$$

By the triangle inequality, (2.14) and (2.15), we get

$$d(g(x),g(x^*) \leq d(g(x),g(u_{n+1}))+d(g(x^*),g(u_{n+1})) \to 0 \ \ as \ \ n\to+\infty,$$

$$d(g(y),g(y^*) \leq d(g(y),g(v_{n+1}))+d(g(y^*),g(v_{n+1})) \to 0 \ \ as \ \ n\to+\infty.$$

Therefore, we have $g(x)=g(x^*)$ and $g(y)=g(y^*)$ and so (2.11) holds. □

If $g=I$, the identity mapping in Theorem 2.5, then we deduce the following corollary.

Corollary 2.6. *In addition to the hypotheses of Corollary 2.2, suppose that $L=0$ and for every $(x,y),(x^*,y^*)\in X\times X$ there exists a $(u,v)\in X\times X$ such that $(F(u,v),F(v,u))$ is comparable to $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F has a unique coupled fixed point, that is, there exists a unique $(x,y)\in X\times X$ such that $x=F(x,y)$ and $y=F(y,x)$.*

Now, we state and prove the last theorem of this paper.

Theorem 2.7. *In addition to the hypotheses of Theorem 2.1, if $g(x_0)$ and $g(y_0)$ are comparable and $L=0$, then F and g have a coupled coincidence point (x,y) such that $g(x)=F(x,y)=F(y,x)=g(y)$.*

Proof. By Theorem 2.1, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_n)\to g(x)$ and $g(y_n)\to g(y)$, where (x,y) is a coincidence point of F and g. Suppose $g(x_0)\preceq g(y_0)$, then it is an easy matter to show that

$$g(x_n)\preceq g(y_n)\ \forall\, n\geq 0.$$

Thus, by (2.1) we have

$$\begin{aligned} d(g(x_n), g(y_n)) &= d(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\le \alpha \min\{d(F(x_{n-1}, y_{n-1}), g(x_{n-1})), d(F(y_{n-1}, x_{n-1}), g(x_{n-1}))\} \\ &\quad + \beta \min\{d(F(x_{n-1}, y_{n-1}), g(y_{n-1})), d(F(y_{n-1}, x_{n-1}), g(y_{n-1}))\} \\ &= \alpha \min\{d(g(x_n), g(x_{n-1})), d(g(y_n), g(x_{n-1}))\} \\ &\quad + \beta \min\{d(g(x_n), g(y_{n-1})), d(g(y_n), g(y_{n-1}))\}. \end{aligned}$$

By taking the limit as $n \to +\infty$, we get $d(g(x), g(y)) = 0$. Hence $F(x, y) = g(x) = g(y) = F(y, x)$. Similar arguments can be used if $g(y_0) \preceq g(x_0)$. To avoid repetitions details are omitted. This makes end to the proof. □

If we assume $g = I$ in Theorem 2.7, then we deduce the following corollary.

Corollary 2.8. *In addition to the hypotheses of Corollary 2.1, if x_0 and y_0 are comparable and $L = 0$, then F has a coupled fixed point, that is, there exists x such that $F(x, x) = x$.*

Example 2.9. Let $X = [0, +\infty)$. Then $(X, \preceq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for $x, y \in X$. Define $g : X \to X$ by $g(x) = \frac{4x^2}{\min\{\alpha,\beta\}}$ with $0 < \alpha + \beta < 1$, and $F : X \times X \to X$ by

$$F(x, y) = \begin{cases} \frac{x^2-y^2}{4} & \text{if} \quad x > y, \\ 0 & \text{if} \quad x \le y. \end{cases}$$

Denote $\delta = \min\{\alpha, \beta\}$. By routine calculations, the reader can easily verify that the following assumptions hold:

(I) (X, d) is a complete metric space;
(II) F has the mixed g-monotone property;
(III) $(x_0, y_0) = (0, \sqrt{\delta}) \Rightarrow g(x_0) = 0 = F(x_0, y_0)$ and $g(y_0) = 4 > \frac{\delta}{4} = F(y_0, x_0)$ (as $x_0 < y_0$);
(IV) $F(X \times X) \subseteq g(X)$;
(V) F and g are continuous; g is non-decreasing.

Here, we show only that F and g are compatible and condition (2.1) in Theorem 2.1 is satisfied for all real numbers α, β, with $0 < \alpha + \beta < 1$, and $L \ge 0$.
• F and g are compatible.
Consider two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = x \in X \tag{2.16}$$

and

$$\lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y \in X. \tag{2.17}$$

We have to prove that

$$\begin{cases} \lim_{n\to+\infty} d(g(F(x_n, y_n)), F(g(x_n), g(y_n))) = 0, \\ \lim_{n\to+\infty} d(g(F(y_n, x_n)), F(g(y_n), g(x_n))) = 0. \end{cases} \tag{2.18}$$

We claim that $(x, y) = (0, 0)$. In fact, suppose that $x > 0$. From (2.16) and the definition of F, there exists $n_0 \in \mathbb{N}$ such that $x_n > y_n$ for all $n \ge n_0$. Then, from (2.17) and the definition of F, we get

$$0 = \lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = y.$$

From the definition of g, this implies that

$$\lim_{n\to+\infty} g(y_n) = \lim_{n\to+\infty} \frac{4y_n^2}{\delta} = y = 0,$$

that is,

$$\lim_{n\to+\infty} y_n = y = 0.$$

Now, using (2.16) and the definition of F, we obtain

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} \frac{x_n^2 - y_n^2}{4} = \lim_{n\to+\infty} \frac{x_n^2}{4} = x.$$

Using (2.16) and the definition of g, we have

$$\lim_{n\to+\infty} \frac{4x_n^2}{\delta} = x.$$

From the uniqueness of the limit, we get

$$4x = \frac{\delta x}{4},$$

that is,

$$(16-\delta)x = 0.$$

Since $0 < \delta < 1$, we obtain that $x = 0$, which is a contradiction. Then, $x = 0$. Similarly, one can also show that $y = 0$. Then, we have

$$\lim_{n\to+\infty} F(x_n, y_n) = \lim_{n\to+\infty} g(x_n) = \lim_{n\to+\infty} F(y_n, x_n) = \lim_{n\to+\infty} g(y_n) = 0. \tag{2.19}$$

Now, (2.18) follows immediately from (2.19), the continuity of F, the continuity of g and the continuity of d. Thus we proved that F and g are compatible.

• Condition (2.1) holds, for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$.

We distinguish the following four cases:

Case 1. If $x \le y$ and $u \le v$, then we have

$$\begin{aligned} d(F(x,y), F(u,v)) &= d(0,0) = 0 \le \alpha d\left(0, \frac{4x^2}{\delta}\right) + \beta d\left(0, \frac{4u^2}{\delta}\right) \\ &\quad + L \min\left\{ d\left(0, \frac{4u^2}{\delta}\right), d\left(0, \frac{4x^2}{\delta}\right)\right\}. \end{aligned}$$

Case 2. if $x \le y$ and $u > v$, then we have

$$\begin{aligned} d(F(x,y), F(u,v)) &= d\left(0, \frac{u^2 - v^2}{4}\right) = \frac{u^2 - v^2}{4} \le \frac{u^2}{4} < \beta \frac{15u^2}{4\delta} \\ &< \alpha \min\left\{ d\left(0, \frac{4x^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4x^2}{\delta}\right)\right\} \\ &\quad + \beta \min\left\{ d\left(0, \frac{4u^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4u^2}{\delta}\right)\right\} \\ &\quad + L \min\left\{ d\left(0, \frac{4u^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4x^2}{\delta}\right)\right\}. \end{aligned}$$

Case 3. If $x > y$ and $u > v$, without restriction of generality we suppose $x < u$, and then we have

$$\begin{aligned} d(F(x,y), F(u,v)) &= d\left(\frac{x^2 - y^2}{4}, \frac{u^2 - v^2}{4}\right) = \frac{1}{4}|u^2 - v^2 - x^2 + y^2| \le \frac{1}{2}u^2 \\ &< \beta \frac{15u^2}{4\delta} < \alpha \min\left\{ d\left(\frac{x^2 - y^2}{4}, \frac{4x^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4x^2}{\delta}\right)\right\} \\ &\quad + \beta \min\left\{ d\left(\frac{x^2 - y^2}{4}, \frac{4u^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4u^2}{\delta}\right)\right\} \\ &\quad + L \min\left\{ d\left(\frac{x^2 - y^2}{4}, \frac{4u^2}{\delta}\right), d\left(\frac{u^2 - v^2}{4}, \frac{4x^2}{\delta}\right)\right\}. \end{aligned}$$

Case 4. If $x > y$ and $u \leq v$, then, from $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, it follows $v \geq u \geq x > y \geq v$. This is a contradiction, and so this case must not be considered.

Thus condition (2.1) holds in all the cases. Hence by Theorem 2.1, F and g have a coupled coincidence point $(0,0) \in X \times X$. (Moreover, $(0,0)$ is a coupled fixed point of F).

On the other hand, we have

$$d(F(2,1), F(3,1/2)) = \frac{23}{16} \quad \text{and} \quad \frac{d(2,3) + d(1,1/2)}{2} = \frac{3}{4}.$$

Then, there is no $k \in [0,1)$ such that

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}\,[d(x,u) + d(y,v)] \text{ for all } x \leq u,\ y \geq v.$$

Then, Theorem 2.1 of Bhaskar and Lakshmikantham [4] cannot be applied in this case. Moreover, there is no function $\phi : [0,+\infty) \to [0,+\infty)$, with $\phi(t) < t$ and $\lim_{r\to t^+} \phi(r) < t$ for each $t > 0$, such that

$$d(F(x,y), F(u,v)) \leq \phi\left(\frac{d(x,u) + d(y,v)}{2}\right) \text{ for all } x \leq u,\ y \geq v.$$

Then, also Theorem 2.1 of Lakshmikantham and Ćirić [11] cannot be applied in this case.

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