Solvability of multi-point boundary value problems on the half-line

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This paper is dedicated to Professor Ljubomir Ćirić

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Abstract

In this work, using the Leray-Schauder continuation principle, we study the existence of at least one solution to the quasilinear second-order multi-point boundary value problems on the half-line.

Keywords: Solvability, m-point boundary value problem, p-Laplacian, half-line

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1. Introduction

Boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and in various applications such as an unsteady flow of gas through a semi-infinite porous media, theory of drain flows and plasma physics. There have been many works concerning the existence of solutions for the boundary value problems on the half-line. We refer the reader to [1] [2] [3] [4] [6] [8] [9] [10] [11] [12] [13] [14] [15] [16] [19] [20] [22] [23] and the references therein.

Recently, Lian and Ge [19] studied the second-order three-point boundary value problem

\[ x''(t) + g(t, x(t), x'(t)) = 0, \quad a.e. \ t \in \mathbb{R}_+, \]

\[ x(0) = \alpha x(\eta), \quad \lim_{t \to \infty} x'(t) = 0, \]

where \( \mathbb{R}_+ = [0, \infty) \), \( \alpha \neq 1 \) and \( \eta > 0 \). The authors investigated the existence of at least one solution under the assumption that \( g(t, \cdot, \cdot) \) and \( tg(t, \cdot, \cdot) \) are Carathéodory with respect to \( L^1(\mathbb{R}_+) \).

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More recently, Kosmatov ([10]) studied the second-order nonlinear differential equation

\[(q(t)y'(t))' = k(t, y(t), y'(t)), \text{ a.e. } t \in \mathbb{R}_+ ,\]

satisfying two sets of boundary conditions:

\[y'(0) = 0, \quad \lim_{t \to \infty} y(t) = 0\]

and

\[y(0) = 0, \quad \lim_{t \to \infty} y(t) = 0,\]

where \(k : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is Carathéodory with respect to \(L^1(\mathbb{R}_+), \mathbb{R} = (-\infty, \infty), q \in C(\mathbb{R}_+ \cap C^1(0, \infty), 1/q \in L^1(\mathbb{R}_+)\) and \(q(t) > 0\) for all \(t \in \mathbb{R}_+\). The author obtained the existence of at least one solution to the above problems using the Leray-Schauder continuation principle. In the end of the paper, the author pointed out that the assumption \(q(0) > 0\) could be omitted, in which case one would have to work in a Banach space equipped with a weighted norm after the boundary conditions are adjusted accordingly.

Motivated by the above works ([10, 11]), we study the quasilinear second-order nonlinear differential equation

\[(w(t)\varphi_p(u'(t)))' + f(t, u(t), u'(t)) = 0, \text{ a.e. } t \in \mathbb{R}_+ ,\]  

(P)

satisfying the following four sets of boundary conditions:

\[u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad \lim_{t \to \infty} (\varphi_p^{-1}(w)u')(t) = 0, \quad (BC_1)\]

\[u(0) = \sum_{i=1}^{m-2} a_i (\varphi_p^{-1}(w)u')(\xi_i), \quad \lim_{t \to \infty} (\varphi_p^{-1}(w)u')(t) = 0, \quad (BC_2)\]

\[\lim_{t \to 0^+} (\varphi_p^{-1}(w)u')(t) = 0, \quad \lim_{t \to \infty} u(t) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad (BC_3)\]

\[\lim_{t \to 0^+} (\varphi_p^{-1}(w)u')(t) = 0, \quad \lim_{t \to \infty} u(t) = \sum_{i=1}^{m-2} a_i (\varphi_p^{-1}(w)u')(\xi_i), \quad (BC_4)\]

where \(\varphi_p(s) = |s|^{p-2}s, p > 1, \xi_i \in \mathbb{R}_+\) with \(0 \leq \xi_1 < \xi_2 < \cdots < \xi_{m-2}\), \(a_i \in \mathbb{R}\) with \(\sum_{i=1}^{m-2} a_i \neq 1, w \in C(\mathbb{R}_+, \mathbb{R})\) and \(f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function such that \(f = f(t, u, v)\) is Lebesgue measurable in \(t\) for all \((u, v) \in \mathbb{R} \times \mathbb{R}\) and continuous in \((u, v)\) for almost all \(t \in \mathbb{R}_+\). We further assume the following conditions hold.

(F) There exist measurable functions \(\alpha, \beta\) and \(\gamma\) such that

\[\alpha, \beta/w, \gamma \in L^1(\mathbb{R}_+)\]

and

\[|f(t, u, v)| \leq \alpha(t) |u|^{p-1} + \beta(t) |v|^{p-1} + \gamma(t), \text{ a.e. } t \in \mathbb{R}_+.\]

(W) \(\varphi_p^{-1}(1/w) \in L^1(\mathbb{R}_+)\) and \(Z_w = \{t \in \mathbb{R}_+ \mid w(t) = 0\}\) is a finite set.

By a solution to problem \((P), (BC_i)\), we understand a function \(u \in C(\mathbb{R}_+) \cap C^1(\mathbb{R}_+ \setminus Z_w)\) with \(w\varphi_p(u') \in AC(\mathbb{R}_+)\) satisfying \((P), (BC_i)\) \((i = 1, 2, 3, 4)\).

To the author’s knowledge, the multi-point boundary value problems with sign-changing weight \(w\) have not been investigated until now. The purpose of this paper is to establish the existence of at least one solution to \(p\)-Laplacian boundary value problems \((P), (BC_i)\) \((i = 1, 2, 3, 4)\) with sign-changing weight \(w\).
Since $\mathbb{R}_+$ is not compact, the related compactness principle on a bounded interval $[0,1]$ does not hold. In addition, solutions $u$ of $(P), (BC_i)$ ($i = 1, 2, 3, 4$) may not be in $C^{1}(\mathbb{R}_+)$ since $w$ may have zeros in $\mathbb{R}_+$. In order to overcome these difficulties, a new Banach space equipped with a weighted norm is introduced, and then we can proceed with the Leray-Schauder continuation principle which was used in many works (see, e.g., [5, 7, 9, 10, 11, 17]) in order to prove the existence of a solution for the problems $(P), (BC_i)$ ($i = 1, 2, 3, 4$).

The rest of this paper is organized as follows. In Section 2, a weighted Banach space and corresponding operators to problems $(P), (BC_i)$ ($i = 1, 2, 3, 4$) are introduced, and lemmas are presented. In Section 3, our main results are given, and also an example to illustrate our results is presented.

2. Preliminaries

Let $X$ be the Banach space $X = \{ u \in C^{1}(\mathbb{R}_+ \setminus Z_w) \mid u$ and $\varphi_p^{-1}(w)u'$ are continuous and bounded functions on $\mathbb{R}_+ \}$ equipped with norm $\|u\| = \|u\|_{\infty} + \|\varphi_p^{-1}(w)u'\|_{\infty},$ where $\|v\|_{\infty} = \sup_{t \in \mathbb{R}_+} |v(t)|$ and let $Y$ be the Banach space $L^1(\mathbb{R}_+)$ equipped with norm $\|h\|_1 = \int_0^{\infty} |h(s)| ds.$

For convenience, we will use the following constants

$A = 1 - \sum_{i=1}^{m-2} a_i,$

$B = |A|^{-1} \sum_{i=1}^{m-2} |a_i| \int_0^{\xi_i} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds + \int_0^{\infty} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds,$

$C = \sum_{i=1}^{m-2} |a_i| + \int_0^{\infty} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds,$

$D = |A|^{-1} \sum_{i=1}^{m-2} |a_i| \int_{\xi_i}^{\infty} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds + \int_0^{\infty} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds.$

For each $h \in Y,$ we define, for $t \in \mathbb{R}_+,$

$(T_1 h)(t) = A^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \left( \frac{1}{w(s)} \right) \int_s^{\infty} h(\tau)d\tau \right) ds$

$+ \int_0^{t} \varphi_p^{-1} \left( \frac{1}{w(s)} \right) \int_s^{\infty} h(\tau)d\tau \right) ds,$

$(T_2 h)(t) = \sum_{i=1}^{m-2} a_i \varphi_p^{-1} \left( \int_{\xi_i}^{\infty} h(s) ds \right)$

$+ \int_0^{t} \varphi_p^{-1} \left( \frac{1}{w(s)} \right) \int_s^{\infty} h(\tau)d\tau \right) ds,$

$(T_3 h)(t) = A^{-1} \sum_{i=1}^{m-2} a_i \int_{\xi_i}^{\infty} \varphi_p^{-1} \left( \frac{1}{w(s)} \right) \int_0^{s} h(\tau)d\tau \right) ds$

$+ \int_t^{\infty} \varphi_p^{-1} \left( \frac{1}{w(s)} \right) \int_0^{s} h(\tau)d\tau \right) ds.$
and
\[
(T_i h)(t) = -\sum_{i=1}^{m-2} a_i \varphi_p^{-1} \left( \int_0^{\xi_i} h(s) ds \right) + \int_t^\infty \varphi_p^{-1} \left( \frac{1}{w(s)} \int_0^s h(\tau) d\tau \right) ds.
\]

Then \( T_i : Y \to X \) is well defined and for each \( h \in Y \), \( T_i h \) is the unique solution of the differential equation
\[
(w(t)\varphi_p(u'(t)))' + h(t) = 0, \text{ a.e. } t \in \mathbb{R}_+,
\]
subject to the boundary conditions \((BC_i) (i = 1, 2, 3, 4)\).

**Lemma 2.1.** Let \( h \in Y \). Then \( T_1 h \) satisfies
\[
\|T_1 h\|_\infty \leq B\|h\|_1^{1/(p-1)} \tag{2.1}
\]
and
\[
\|\varphi_p^{-1}(w)(T_1 h)'\|_\infty \leq \|h\|_1^{1/(p-1)} \tag{2.2}
\]

**Proof.** Let \( h \in Y \). Then, for all \( t \in \mathbb{R}_+ \), one has
\[
|T_1 h(t)| \leq \left| |A|^{-1} \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds \right| \left( \int_0^\infty |h(s)| ds \right)^{1/(p-1)}
+ \left( \int_0^t \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds \right) \left( \int_0^\infty |h(s)| ds \right)^{1/(p-1)}
\leq B\|h\|_1^{1/(p-1)}.
\]

Similarly, for all \( t \in \mathbb{R}_+ \), one has
\[
|\varphi_p^{-1}(w)(T_1 h)'(t)| = \left| \varphi_p^{-1} \left( \int_t^\infty h(s) ds \right) \right|
\leq \|h\|_1^{1/(p-1)}.
\]

Thus the proof is complete. \( \square \)

The following lemmas can be proved by the similar manner and so we omit the proofs.

**Lemma 2.2.** Let \( h \in Y \). Then, for each \( i = 2, 4 \), \( T_i h \) satisfies
\[
\|T_i h\|_\infty \leq C\|h\|_1^{1/(p-1)}
\]
and
\[
\|\varphi_p^{-1}(w)(T_i h)'\|_\infty \leq \|h\|_1^{1/(p-1)}.
\]

**Lemma 2.3.** Let \( h \in Y \). Then \( T_3 h \) satisfies
\[
\|T_3 h\|_\infty \leq D\|h\|_1^{1/(p-1)}
\]
and
\[
\|\varphi_p^{-1}(w)(T_3 h)'\|_\infty \leq \|h\|_1^{1/(p-1)}.
\]
We define the Nemiskii operator \( N : X \to Y \) by
\[
(Nu)(t) = f(t, u(t), u'(t)), \quad t \in \mathbb{R}_+.
\]

It follows from \((F)\) that \( N \) maps bounded sets of \( X \) into bounded sets of \( Y \) and is continuous. For each \( i \in \{1, 2, 3, 4\} \), define \( L_i \triangleq T_i N : X \to X \). Then \( L_i \) is well defined and problem \((P)\), \((BC_i)\) has a solution \( u \) if and only if \( L_i \) has a fixed point \( u \) in \( X \).

To show the compactness of the operators \( L_i \) \( (i = 1, 2, 3, 4) \), we use the following compactness criterion.

**Theorem 2.4.** ([2]) Let \( Z \) be the space of all bounded continuous vector-valued functions on \( \mathbb{R}_+ \) and \( S \subset Z \). Then \( S \) is relatively compact in \( Z \) if the following conditions hold.

(i) \( S \) is bounded in \( Z \).

(ii) the functions from \( S \) are equicontinuous on any compact interval of \( \mathbb{R}_+ \).

(iii) the functions from \( S \) are equiuniform, that is, given \( \epsilon > 0 \), there exists a \( T = T(\epsilon) > 0 \) such that \( \| \phi(t) - \phi(\infty) \| \leq \epsilon \), for all \( t > T \) and all \( \phi \in S \).

**Lemma 2.5.** For each \( i \in \{1, 2, 3, 4\} \), the mapping \( L_i : X \to X \) is completely continuous.

**Proof.** We only prove that \( L_1 : X \to X \) is completely continuous since other cases can be proved by the similar manner.

First, we show that \( L_1 \) is compact. Let \( \Sigma \) be bounded in \( X \), i.e., there exists \( M > 0 \) such that \( \| u \| \leq M \) for all \( u \in \Sigma \). Then there exists \( h_M \in Y \) such that \( \| (Nu)(t) \| \leq h_M(t) \) for all \( t \in \mathbb{R}_+ \) and all \( u \in \Sigma \). By Lemma 2.4, \( L_1(\Sigma) \) is bounded in \( X \).

For \( t_1, t_2 \in \mathbb{R}_+ \) with \( t_1 < t_2 \), one has
\[
| (L_1u)(t_1) - (L_1u)(t_2) | = \left| \int_{t_1}^{t_2} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \int_s^\infty (Nu)(\tau) d\tau \right) ds \right| \leq \| h_M \|^{1/(p-1)} \int_{t_1}^{t_2} \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds
\]
and
\[
| (\varphi_p^{-1}(w)(L_1u)'(t_1) - (\varphi_p^{-1}(w)(L_1u)'(t_2)) | = \left| \varphi_p^{-1} \left( \int_{t_1}^\infty (Nu)(s) ds \right) - \varphi_p^{-1} \left( \int_{t_2}^\infty (Nu)(s) ds \right) \right|,
\]
which yield that \( L_1(\Sigma) \) and \( \{ \varphi_p^{-1}(w)(L_1u)' \mid u \in \Sigma \} \) are equicontinuous on \( \mathbb{R}_+ \) by the facts that \( \varphi_p^{-1} \) is uniformly continuous on \([-1, 1]\) and \( |(Nu)(t)| \leq h_M(t) \) for all \( t \in \mathbb{R}_+ \).

For \( u \in \Sigma \), one has
\[
\lim_{t \to \infty} (\varphi_p^{-1}(w)(L_1u)'(t)) = \lim_{t \to \infty} \varphi_p^{-1} \left( \int_t^\infty (Nu)(s) ds \right) = 0.
\]

Then
\[
| L_1u(t) - \lim_{t \to \infty} L_1u(t) | = \left| \int_t^\infty \varphi_p^{-1} \left( \frac{1}{|w(s)|} \int_s^\infty (Nu)(\tau) d\tau \right) ds \right| \leq \| h_M \|^{1/(p-1)} \int_t^\infty \varphi_p^{-1} \left( \frac{1}{|w(s)|} \right) ds
\]
and
\[
| (\varphi_p^{-1}(w)(L_1u)'(t) - \lim_{t \to \infty} (\varphi_p^{-1}(w)(L_1u)'(t)) | = \left| \varphi_p^{-1} \left( \int_t^\infty (Nu)(s) ds \right) \right| \leq \left( \int_t^\infty |h_M(s)| ds \right)^{1/(p-1)},
\]
which yield that \( L_1(\Sigma) \) and \( \{ \varphi_p^{-1}(w)(L_1u)' \mid u \in \Sigma \} \) are equiconvergent. By Theorem 2.4 we can conclude that \( T_1 \) is compact.

It follows from the Lebesgue dominated convergence theorem that \( L_1 : X \to X \) is continuous, and thus the proof is complete.

\[ \square \]

3. Main results

In this section, we give our main results.

**Theorem 3.1.** Assume \( B_p^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1 \). Then problem \((P),(BC_1)\) has at least one solution for every \( \gamma \in Y \).

**Proof.** Consider the differential equation, for \( \lambda \in [0,1] \),

\[
(w(t)\varphi_p(u'(t)))' + \lambda f(t, u(t), u'(t)) = 0, \ a.e. \ t \in \mathbb{R}_+,
\]

subject to the boundary condition \((BC_1)\).

Let \( u \) be any solution of \((3.1),(BC_1)\). Then, by \((F)\) and Lemma 2.1 one has

\[
\| (w\varphi_p(u'))' \|_1 = \lambda \| Nu \|_1
\leq \| \alpha \|_1 \| u \|_\infty^{-1} + \| \beta/w \|_1 \| \varphi_p^{-1}(w)u' \|_\infty^{-1} + \| \gamma \|_1
\leq B_p^{p-1}\| \alpha \|_1 \| (w\varphi_p(u'))' \|_1 + \| \beta/w \|_1 \| (w\varphi_p(u'))' \|_1 + \| \gamma \|_1,
\]

which yields

\[
\| (w\varphi_p(u'))' \|_1 \leq \frac{\| \gamma \|_1}{1 - (B_p^{p-1}\| \alpha \|_1 + \| \beta/w \|_1)}.
\]

It follows from Lemma 2.1 that the set of all possible solutions to problem \((3.1),(BC_1)\) is a priori bounded by a constant independent of \( \lambda \in [0,1] \). Thus the proof is complete in view of the Leray-Schauder continuation principle (see, e.g., [18, 21]).

\[ \square \]

Similarly, the following results are obtained.

**Theorem 3.2.** Assume \( C_p^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1 \). Then problems \((P),(BC_i)\) \( (i = 2,4) \) have at least one solution for every \( \gamma \in Y \).

**Theorem 3.3.** Assume \( D_p^{p-1}\|\alpha\|_1 + \|\beta/w\|_1 < 1 \). Then problem \((P),(BC_3)\) has at least one solution for every \( \gamma \in Y \).

Finally, we give an example to illustrate our results.

**Example 3.4.** In problems \((P),(BC_i)\) \( (i = 1,2,3,4) \), let \( p = 3, m = 3, a_1 = 1/2, \xi_1 = 1, \) and

\[
w(t) = \begin{cases} \varphi_3(-(1-t)^{1/2}), & 0 \leq t < 1, \\
\varphi_3((t-1)^{1/2}), & 1 \leq t < 2, \\
\varphi_3(\exp(t-2)), & t \geq 2. \end{cases}
\]

Then \( A = 1/2, B = 7, C = 11/2 \) and \( D = 8 \). For any \( \gamma \in Y \), we set

\[
f(t, u, v) = \frac{\sin t}{(t+70)^2} \varphi_3(u) + \frac{w(t)}{(t+70)^2} \varphi_3(v) + \gamma(t).
\]

Then \( \alpha(t) = \beta/w = 1/(t+70)^2 \), and \( \| \alpha \|_1 = \| \beta/w \|_1 = 1/70 \). Thus by Theorems 3.1, 3.2 and 3.3 problems \((P),(BC_i)\) \( (i = 1,2,3,4) \) has at least one solution for every \( \gamma \in Y \).

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References


