B.Y. CHEN INEQUALITIES FOR BI-SLANT SUBMANIFOLDS IN GENERALIZED COMPLEX SPACE FORMS

S.S. SHUKLA¹ AND PAWAN KUMAR RAO²*

Abstract. The aim of the present paper is to study Chen inequalities for slant, bi-slant and semi-slant submanifolds in generalized complex space forms.

1. Introduction

In [7] B.Y. Chen recalls one of the basic problems in submanifold theory as to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. In [5] he established a sharp inequality for the sectional curvature of a submanifold in a real space forms in terms of the scalar curvature and squared mean curvature. Afterward several geometers [16],[20],[23] obtained similar inequalities for submanifolds in generalized complex space forms. Many geometers also studied contact version of above inequalities [1],[13],[15]. In this article, we establish Chen inequalities for bi-slant and semi-slant submanifolds in generalized complex space forms.

2. Preliminaries

Let $\tilde{M}$ be an almost Hermitian manifold with an almost complex structure $J$ and Riemannian metric $g$. If $J$ is integrable, i.e. the Nijenhuis tensor $[J, J]$ of $J$ vanishes, then $\tilde{M}$ is called a Hermitian manifold. The fundamental 2-form $\Omega$ of $\tilde{M}$ is defined by

$$\Omega(X, Y) = g(X, JY), \text{ for all, } X, Y \in T\tilde{M}.$$
An almost Hermitian manifold $\tilde{M}$ is called an almost Kaehler manifold if the fundamental 2-form $\Omega$ is closed and it becomes Kaehler manifold if $\tilde{\nabla}J = 0$, where $\tilde{\nabla}$ denotes the operator of covariant differentiation with respect to $g$ on $\tilde{M}$.

If an almost complex structure $J$ satisfies
\begin{equation}
(\tilde{\nabla}_X J)Y + (\tilde{\nabla}_Y J)X = 0,
\end{equation}
for any vector fields $X$ and $Y$ on $\tilde{M}$, then the manifold is called a nearly Kaehler manifold.

A. Gray [14] introduced the notion of constant type for a nearly Kaehler manifold, which led to the definition of RK-manifolds. An RK-manifold $\tilde{M}$ is an almost Hermitian manifold for which the curvature tensor $\tilde{R}$ is $J$-invariant, i.e.
\begin{equation}
\tilde{R}(JX, JY, JZ, JW) = \tilde{R}(X, Y, Z, W),
\end{equation}
for all vector fields $X, Y, Z, W \in T\tilde{M}$.

An almost Hermitian manifold $\tilde{M}$ is said to have (pointwise) constant type if for each $p \in \tilde{M}$ and for all vector fields $X, Y, Z \in T_p\tilde{M}$ such that
\begin{equation}
g(X, Y) = g(X, Z) = g(X, JY) = g(X, JZ) = 0,
\end{equation}
we have
\begin{equation}
\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \tilde{R}(X, Z, X, Z) - \tilde{R}(X, Z, JX, JZ).
\end{equation}

An RK-manifold $\tilde{M}$ has (pointwise) constant type if and only if there is a differentiable function $\alpha$ on $\tilde{M}$ such that
\begin{equation}
\tilde{R}(X, Y, X, Y) - \tilde{R}(X, Y, JX, JY) = \alpha \{g(X, X)g(Y, Y) - g^2(X, Y)
- g^2(X, JY)\},
\end{equation}
for all vector fields $X, Y \in T\tilde{M}$.

Furthermore, $\tilde{M}$ has global constant type if $\alpha$ is constant. The function $\alpha$ is called the constant type of $\tilde{M}$. An RK-manifold of constant holomorphic sectional curvature $c$ and constant type $\alpha$ is called a generalized complex space form, denoted by $\tilde{M}(c, \alpha)$. The curvature tensor $\tilde{R}$ of $\tilde{M}(c, \alpha)$ has the following expression:
\begin{equation}
\tilde{R}(X, Y, Z, W) = c^3 \alpha^2 \left\{ \frac{1}{4} \left( g(X, Z)g(Y, W) - g(X, W)g(Y, Z) \right)
+ \frac{1}{2} \left( g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z) \right) + 2g(X, JY)g(Z, JW) \right\},
\end{equation}
for all vector fields $X, Y, Z, W \in T\tilde{M}$.
If $c = \alpha$, then $\tilde{M}(c, \alpha)$ is a space of constant curvature. A complex space form $\tilde{M}(c)$ (i.e., a Kaehler manifold of constant holomorphic sectional curvature $c$) belongs to the class of almost Hermitian manifold $\tilde{M}(c, \alpha)$ (with constant type zero).

Let $M$ be a Riemannian manifold and $K(\pi)$ the sectional curvature of $M$ associated with a plane section $\pi \subset T_pM$, $p \in M$.

For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_pM$, the scalar curvature $\tau$ at $p$ is defined by

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j).$$

We denote by

$$\inf K(p) = \inf \{K(\pi) : \pi \subset T_pM, \dim \pi = 2\}.$$

The first Chen invariant $\delta_M(p)$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p).$$

Let $L$ be a subspace of $T_pM$ of dimension $k \geq 2$ and $\{e_1, \ldots, e_k\}$ an orthonormal basis of $L$. Define $\tau(L)$ be the scalar curvature of the $k$-plane section $L$ by

$$\tau(L) = \sum_{i<j} K(e_i \wedge e_j), \quad i, j = 1, \ldots, k.$$

Given an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_pM$, we denote by $\tau_1, \ldots, \tau_k$ the scalar curvature of $k$-plane section spanned by $e_1, \ldots, e_k$. The scalar curvature $\tau(p)$ of $M$ at $p$ is the scalar curvature of the tangent space of $M$ at $p$. If $L$ is a 2-plane section, then $\tau(L)$ reduces to the sectional curvature $K(L)$ of the plane section $L$. If $K(\pi)$ is the sectional curvature of $M$ for a plane section $\pi$ in $T_pM$, $p \in M$, then scalar curvature $\tau(p)$ at $p$ is given by

$$\tau(p) = \sum_{i<j} K_{ij},$$

where $\{e_1, \ldots, e_n\}$ is an orthonormal basis for $T_pM$ and $K_{ij}$ is the sectional curvature of the plane section spanned by $e_i$ and $e_j$ at $p \in M$.

We recall the following Lemma of Chen [6].

**Lemma 2.1.** Let $n \geq 2$ and $a_1, \ldots, a_n, b$ be $(n+1)$-real numbers, such that

$$(\sum_{i=1}^n a_i)^2 = (n - 1)(\sum_{i=1}^n a_i^2 + b).$$

Then $2a_1a_2 \geq b$ with equality holding if and only if $a_1 + a_2 = a_3 = \ldots = a_n$. 

Let $M$ be an $n$-dimensional submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$ and we denote by $h$, $\nabla$ and $\nabla^\perp$ the second fundamental form of $M$, the induced connection on $M$ and the normal bundle $T^\perp M$. Then, the Gauss and Weingarten formulae are given respectively

(2.14) $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$

and

(2.15) $\tilde{\nabla}_X V = -A_V X + \nabla^\perp_X V$,

for all vector fields $X, Y$ tangent to $M$ and vector field $V$ normal to $M$, where $A_V$ is the shape operator in the direction of $V$. The second fundamental form and the shape operator are related by

(2.16) $g(h(X, Y), V) = g(A_V X, Y)$.

Let $R$ be the Riemannian curvature tensor of $M$, then the equation of Gauss is given by,

(2.17) $\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W))$,

for any vector fields $X, Y, Z, W$ tangent to $M$.

Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of the tangent space $T_p M$. We denote by $H(p)$ the mean curvature vector at $p$, that is

(2.18) $H(p) = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i)$.

Also, we set

(2.19) $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j \in \{1, \ldots, n\}$, $r \in \{n+1, \ldots, 2m\}$,

and

(2.20) $||h||^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j))$.

For any $p \in M$ and $X \in T_p M$, we put

(2.21) $JX = PX + FX$,

where $PX$ and $FX$ are the tangential and normal components of $JX$ respectively.

Let us denote

(2.22) $||P||^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j)$. 
Now, we recall that for a submanifold \( M \) in a Riemannian manifold, the relative null space of \( M \) at a point \( p \) is defined by

\[
N_p = \{ X \in T_p M \mid h(X, Y) = 0, \text{ for all } Y \in T_p M \}.
\]

**Definition (2.1)**[2]. A differential distribution \( D \) on \( M \) is called a slant distribution if for each \( p \in M \) and each non-zero vector \( X \in D_p \), the angle \( \theta_D(X) \) between \( JX \) and the vector subspace \( D_p \) is constant, which is independent of the choice of \( p \in M \) and \( X \in D_p \). In this case, the constant angle \( \theta_D \) is called the slant angle of the distribution \( D \).

**Definition (2.2)**[2]. A submanifold \( M \) is said to be a slant submanifold if for any \( p \in M \) and \( X \in T_p M \), the angle between \( JX \) and \( T_p M \) is constant, i.e., it does not depend on the choice of \( p \in M \) and \( X \in T_p M \). The angle \( \theta \in [0, \frac{\pi}{2}] \) is called the slant angle of \( M \) in \( \tilde{M} \).

Invariant and anti-invariant submanifolds are slant submanifolds with slant angle \( \theta = 0 \) and \( \theta = \frac{\pi}{2} \), respectively. A slant submanifold which is neither invariant nor anti-invariant is called a proper slant submanifold.

**Definition (2.3)**[3]. A submanifold \( M \) is called a bi-slant submanifold of \( \tilde{M} \) if there exist two orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \), such that

(i) \( TM \) admits the orthogonal direct decomposition \( TM = D_1 \oplus D_2 \),
(ii) for any \( i = 1, 2 \), \( D_i \) is slant distribution with slant angle \( \theta_i \).

On the other hand, CR-submanifolds of \( \tilde{M} \) are bi-slant submanifolds with \( \theta_1 = 0 \) and \( \theta_2 = \frac{\pi}{2} \).

Let \( 2d_1 = \text{dim} D_1 \) and \( 2d_2 = \text{dim} D_2 \).

If either \( d_1 \) or \( d_2 \) vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds are particular cases of bi-slant submanifolds.

**Definition (2.4)**[3]. A submanifold \( M \) is said to be a semi-slant submanifold of \( \tilde{M} \) if there exist two orthogonal distributions \( D_1 \) and \( D_2 \) on \( M \), such that

(i) \( TM \) admits the orthogonal direct decomposition \( TM = D_1 \oplus D_2 \),
(ii) the distribution \( D_1 \) is an invariant distribution, that is, \( J(D_1) = D_1 \),
(iii) the distribution \( D_2 \) is slant with angle \( \theta \neq 0 \).

The invariant distribution of a semi-slant submanifold is a slant distribution with zero slant angle. Thus, it is obvious that, semi-slant submanifolds are particular cases of bi-slant submanifolds. However if \( 2d_1 = \text{dim} D_1 \) and \( 2d_2 = \text{dim} D_2 \)

(a) \( d_2 = 0 \), then \( M \) is an invariant submanifold.
(b) \( d_1 = 0 \) and \( \theta = \frac{\pi}{2} \), then \( M \) is an anti-invariant submanifold.
(c) \( d_1 = 0 \) and \( \theta \neq \frac{\pi}{2} \), then \( M \) is a proper slant submanifold, with slant angle \( \theta \).

A semi-slant submanifold is proper if \( d_1d_2 \neq 0 \) and \( \theta \neq \frac{\pi}{2} \).
3. B.Y. Chen Inequalities

In this section, we establish Chen inequalities for proper bi-slant submanifolds in a generalized complex space form. We consider a plane section $\pi$ invariant by $P$ and denote $\dim D_1 = 2d_1$ and $\dim D_2 = 2d_2$.

**Theorem 3.1.** Let $M$ be an $n$-dimensional proper bi-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then

(I) For any plane section $\pi$ invariant by $P$ and tangent to $D_1$,

\[
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c+3\alpha}{4}(n+1) \right\} + \frac{(c-\alpha)}{4}\{3(d_1 - 1)\cos^2\theta_1 + 3d_2\cos^2\theta_2\}
\]

and

(II) For any plane section $\pi$ invariant by $P$ and tangent to $D_2$,

\[
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c+3\alpha}{4}(n+1) \right\} + \frac{c-\alpha}{4}\{3d_1\cos^2\theta_1 + 3(d_2 - 1)\cos^2\theta_2\}.
\]

The equality case of inequalities (3.1) and (3.2) hold at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c, \alpha)$ at $p$ have the following forms:

\[
A_{n+1} = \begin{pmatrix}
a & 0 & 0 & \ldots & 0 \\
0 & b & 0 & \ldots & 0 \\
0 & 0 & \mu & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \mu 
\end{pmatrix}, \quad a + b = \mu,
\]

\[
A_r = \begin{pmatrix}
h_{11}^r & h_{12}^r & 0 & \ldots & 0 \\
h_{12}^r & -h_{11}^r & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 
\end{pmatrix}
\]

where

\[
A_r = A_{e_r}, \quad r = n + 1, \ldots, 2m.
\]

\[
h_{ij}^r = g(h(e_i, e_j), e_r), \quad r = n + 1, \ldots, 2m.
\]

**Proof.** The Gauss equation for the submanifold $M$ is given by

\[
\hat{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),
\]
for all vector fields $X, Y, Z, W \in TM$, where $\tilde{R}, R$ denote the curvature tensors of $\tilde{M}(c, \alpha)$ and $M$ respectively.

The curvature tensor $\tilde{R}$ of $\tilde{M}(c, \alpha)$ has the following expression [20]:

$$\tilde{R}(X, Y, Z, W) = \frac{c+3\alpha}{4}\{g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\} + \frac{c-\alpha}{4}\{g(JX, Z)g(JY, W) - g(JX, W)g(JY, Z) + 2g(X, JY)g(Z, JW)\},$$

for any vector fields $X, Y, Z, W \in TM$.

Let $p \in M$, we choose an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of $T_pM$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2n}\}$ of $T^\perp_pM$. By substituting $X = Z = e_i$, $Y = W = e_j$ in equation (3.8), we have

$$\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c+3\alpha}{4}\{n^2 - n\} + \frac{c-\alpha}{4}\{g(Je_i, e_j)g(Je_j, e_i) + 2g(e_i, Je_j)g(e_i, Je_j)\}
= \frac{c+3\alpha}{4}\{n^2 - n\} + \frac{c-\alpha}{4}\{\sum_{i,j=1}^{n} g^2(Je_i, e_j)\}.$$

Let $M$ be a proper bi-slant submanifold of $\tilde{M}(c, \alpha)$ and $\dim M = n = 2d_1 + 2d_2$. We consider an adapted bi-slant orthonormal frames

$$e_1, e_2 = \frac{1}{\cos \theta_1} Pe_1, \ldots, e_{2d_1-1}, e_{2d_1} = \frac{1}{\cos \theta_1} Pe_{2d_1-1},
\quad e_{2d_1+1}, e_{2d_1+2} = \frac{1}{\cos \theta_2} Pe_{2d_1+1},
\quad \ldots, \quad e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \frac{1}{\cos \theta_2} Pe_{2d_1+2d_2-1}.$$

Obviously, we have

$$g^2(Je_i, e_{i+1}) = \cos^2 \theta_1, \text{ for } i \in \{1, \ldots, 2d_1 - 1\} \text{ and }
= \cos^2 \theta_2, \text{ for } i \in \{2d_1 + 1, \ldots, 2d_1 + 2d_2 - 1\}.$$

Then, we have

$$\sum_{i,j=1}^{n} g^2(Je_i, e_j) = 2(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2).$$

Substituting (3.12) into (3.9), we have

$$\tilde{R}(e_i, e_j, e_i, e_j) = \frac{c+3\alpha}{4}\{n^2 - n\} + \frac{c-\alpha}{4}\{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)\}.$$

The equation (3.7) gives

$$\tilde{R}(e_i, e_j, e_i, e_j) = 2\tau + ||h||^2 - n^2 ||H||^2.$$

By using equations (3.13) and (3.14), we get
From the Gauss equation for (3.20) 2h

Using the Lemma (2.1) and equation (3.19), we obtain

(3.19) \( (3.18) \sum_{i,j=1}^{n} (h'_{ij})^2 + \epsilon \).

The above equation implies

(3.17) \( n^2||H||^2 = (n - 1)(\epsilon + ||h||^2) \).

Let \( p \in M, \pi \subset T_pM, \dim \pi = 2 \) and \( \pi \) invariant by \( P \).

Now, we consider two cases:

Case (a): The plane section \( \pi \) is tangent to \( D_1 \).

We may assume that \( \pi = sp\{e_1, e_2\} \). We choose \( e_{n+1} = \frac{H}{||H||} \).

From the equation (3.17) becomes,

(3.18) \( (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n - 1)\sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \).

The above equation implies

(3.19) \( (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n - 1)\sum_{i,j=1}^{n} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \).

Using the Lemma (2.1) and equation (3.19), we obtain

(3.20) \( 2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \).

From the Gauss equation for \( X = Z = e_1 \) and \( Y = W = e_2 \), we get

(3.21) \( K(\pi) = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4} \cos^2 \theta_1 + \sum_{r=n+1}^{2m} [h_{11}^r h_{22}^r - (h_{12}^r)^2] \)

\[ \geq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4} \cos^2 \theta_1 + \frac{1}{2} \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \]

\[ + \sum_{r=n+1}^{2m} h_{11}^r h_{22}^r - \sum_{r=n+1}^{2m} (h_{12}^r)^2 \]

\[ = \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4} \cos^2 \theta_1 + \frac{1}{2} \sum_{i\neq j}(h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j>2} (h_{ij}^r)^2 \]

\[ + \frac{1}{2} \sum_{r=n+2}^{2m} (h_{11}^r + h_{22}^r)^2 + \sum_{j>2} [(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{\epsilon}{2} \]

\[ \geq \frac{c+3\alpha}{4} + 3\frac{c-\alpha}{4} \cos^2 \theta_1 + \frac{\epsilon}{2} \].
From the equations (3.16), (3.21) and (2.9), it follows that

\[(3.22) \quad \inf K \geq \frac{c_{+}3\alpha}{4} + 3 \frac{c_{-}\alpha}{4} \cos^2 \theta_1 + \tau - \frac{n^2}{2(n-1)} (n-2) \|H\|^2 \]

\[- \frac{c_{+}3\alpha}{8} \{n(n-1)\} - \frac{c_{-}\alpha}{8} \{6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2)\}.

From the equations (3.22) and (2.10), we get

\[(3.23) \quad \delta_M \leq \frac{n-2}{2} \{\frac{n^2}{n-1} \|H\|^2 + \frac{c_{+}3\alpha}{4} (n+1)\}

+ \frac{c_{-}\alpha}{4} \{3(d_1 - 1) \cos^2 \theta_1 + 3d_2 \cos^2 \theta_2\},

where \(\delta_M\) is Chen invariant. This proves the inequality (3.1).

Case (b): The plane section \(\pi\) is tangent to \(D_2\).

From the equation (3.17), we have

\[(\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1) \{ \sum_{r=n+1}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \}.

The above equation implies

\[(3.24) \quad (\sum_{i=1}^{n} h_{ii}^{n+1})^2 = (n-1) \{ \sum_{i=1}^{n} (h_{ii}^{n+1})^2 + \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon \}.

Using the Lemma (2.1) and equation (3.24), we obtain

\[(3.25) \quad 2h_{11}^{n+1}h_{22}^{n+1} \geq \sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon.

From the Gauss equation for \(X = Z = e_1\) and \(Y = W = e_2\), we get

\[(3.26) \quad K(\pi) = \frac{c_{+}3\alpha}{4} + 3 \frac{c_{-}\alpha}{4} \cos^2 \theta_2 + \sum_{r=n+1}^{2m} [h_{11}^r h_{22}^r - (h_{12}^r)^2]

\[\geq \frac{c_{+}3\alpha}{4} + 3 \frac{c_{-}\alpha}{4} \cos^2 \theta_2 + \frac{1}{2} \{\sum_{i\neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 + \epsilon\}

\[\geq \frac{c_{+}3\alpha}{4} + 3 \frac{c_{-}\alpha}{4} \cos^2 \theta_2 + \frac{1}{2} \{\sum_{i\neq j} (h_{ij}^{n+1})^2 + \frac{1}{2} \sum_{r=n+2}^{2m} \sum_{i,j=2}^{n} (h_{ij}^r)^2

\[+ \frac{1}{2} \sum_{r=n+2}^{2m} (h_{11}^r + h_{22}^r)^2 + \sum_{j>2} [(h_{1j}^{n+1})^2 + (h_{2j}^{n+1})^2] + \frac{\epsilon}{2}\]

\[\geq \frac{c_{+}3\alpha}{4} + 3 \frac{c_{-}\alpha}{4} \cos^2 \theta_2 + \frac{\epsilon}{2}.

From the relations (3.16), (3.26) and (2.9), it follows that
\[ (3.27) \quad \inf K \geq \frac{c + 3\alpha}{4} + 3\frac{c - \alpha}{4} \cos^2 \theta_2 + \tau - \frac{n^2}{2(n-1)} (n-2) ||H||^2 \]
\[ \quad - \frac{c + 3\alpha}{8} \left\{ n(n-1) \right\} - \frac{c - \alpha}{8} \left\{ 6(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) \right\}. \]

From the equations (3.27) and (2.10), we get
\[ (3.28) \quad \delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c + 3\alpha}{4} (n + 1) \right\} \]
\[ \quad + \frac{c - \alpha}{4} \left\{ 3d_1 \cos^2 \theta_1 + 3(d_2 - 1) \cos^2 \theta_2 \right\}. \]

This proves the inequality (3.2).

The equality case at a point \( p \) holds, if and only if equality holds in each of inequalities (3.20), (3.23) and (3.28) and Lemma (2.1). So we have
\[ h_{ij}^{n+1} = 0, \ \forall \ i \neq j, \ i, j > 2, \]
\[ h_{ij}^r = 0, \ \forall \ i \neq j, \ i, j > 2, \ r = n + 1, \ldots, 2m, \]
\[ h_{11}^r + h_{22}^r = 0, \ \forall \ r = n + 2, \ldots, 2m, \]
\[ h_{ij}^{n+1} = h_{n+1}^{n+1} = 0, \ \forall \ j > 2, \]
\[ h_{11}^{n+1} + h_{22}^{n+1} = h_{33}^{n+1} = \ldots = h_{nn}^{n+1}. \]

We may choose \( \{e_1, e_2\} \) such that \( h_{12}^{n+1} = 0 \) and we denote by \( a = h_{11}, \ b = h_{22}, \ \mu = h_{33}^{n+1} = \ldots = h_{nn}^{n+1} \). Then the shape operators take the desired forms.

Now, we can state the following:

**Corollary 3.2.** Let \( M \) be an \( n \)-dimensional proper semi-slant submanifold of a \( 2m \)-dimensional generalized complex space form \( \tilde{M}(c, \alpha) \). Then

(I) For any plane section \( \pi \) invariant by \( P \) and tangent to \( D_1 \),
\[ (3.29) \quad \delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c + 3\alpha}{4} (n + 1) \right\} \]
\[ \quad + \frac{(c - \alpha)}{4} \left\{ 3(d_1 - 1) + 3d_2 \cos^2 \theta \right\} \]

and

(II) For any plane section \( \pi \) invariant by \( P \) and tangent to \( D_2 \),
\[ (3.30) \quad \delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c + 3\alpha}{4} (n + 1) \right\} \]
\[ \quad + \frac{c - \alpha}{4} \left\{ 3d_1 + 3(d_2 - 1) \cos^2 \theta \right\}. \]

The equality case of inequalities (3.29) and (3.30) holds at a point \( p \in M \) if and only if there exists an orthonormal basis \( \{e_1, e_2, \ldots, e_n\} \) of \( T_p M \) and an orthonormal basis \( \{e_{n+1}, \ldots, e_{2m}\} \) of \( T_p^\perp M \) such that the shape operators of \( M \) in \( \tilde{M}(c, \alpha) \) at \( p \) have the forms (3.3) and (3.4).
Corollary 3.3. Let $M$ be an $n$-dimensional $\theta$-slant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then
\begin{equation}
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c+3\alpha}{4} (n + 1) + 3\frac{\alpha}{4} \cos^2 \theta \right\}.
\end{equation}

The equality case of the inequality (3.31) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c, \alpha)$ at $p$ have the forms (3.3) and (3.4).

Corollary 3.4. Let $M$ be an $n$-dimensional invariant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then
\begin{equation}
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c+3\alpha}{4} (n + 1) + 3\frac{\alpha}{4} \right\}.
\end{equation}

The equality case of the inequality (3.32) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c, \alpha)$ at $p$ have the forms (3.3) and (3.4).

Corollary 3.5. Let $M$ be an $n$-dimensional anti-invariant submanifold of a $2m$-dimensional generalized complex space form $\tilde{M}(c, \alpha)$. Then
\begin{equation}
\delta_M \leq \frac{n-2}{2} \left\{ \frac{n^2}{n-1} ||H||^2 + \frac{c+3\alpha}{4} (n + 1) \right\}.
\end{equation}

The equality case of the inequality (3.33) holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_p M$ and an orthonormal basis $\{e_{n+1}, \ldots, e_{2m}\}$ of $T_p^\perp M$ such that the shape operators of $M$ in $\tilde{M}(c, \alpha)$ at $p$ have the forms (3.3) and (3.4).

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1 Department of Mathematics, University of Allahabad, Allahabad, U.P., India-211002
E-mail address: sshukla_au@rediffmail.com

2 Department of Mathematics, University of Allahabad, Allahabad, U.P., India-211002
E-mail address: babapawanrao@rediffmail.com