GLOBAL EXISTENCE AND $L^\infty$ ESTIMATES OF SOLUTIONS FOR A QUASILINEAR PARABOLIC SYSTEM

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Abstract. In this paper, we study the global existence, $L^\infty$ estimates and decay estimates of solutions for the quasilinear parabolic system $u_t = \nabla \cdot (|\nabla u|^m \nabla u) + f(u, v)$, $v_t = \nabla \cdot (|\nabla v|^n \nabla v) + g(u, v)$ with zero Dirichlet boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$.

1. Introduction

In this paper, we are concerned with the global existence, $L^\infty$ estimates and decay estimates of solutions for the quasilinear parabolic system

\begin{align*}
  u_t &= \nabla \cdot (|\nabla u|^m \nabla u) + f(u, v), \quad x \in \Omega, \ t > 0, \\
  v_t &= \nabla \cdot (|\nabla v|^n \nabla v) + g(u, v), \quad x \in \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \\
  u(x, t) &= v(x, t) = 0, \quad x \in \partial \Omega,
\end{align*}

where $\Omega$ is a bounded domain in $\mathbb{R}^N (N > 1)$ with smooth boundary $\partial \Omega$ and $m, n > 0$.

For $m = n = 0, f(u, v) = u^\alpha v^\beta, g(u, v) = u^\gamma v^\delta$ and $u_0(x), v_0(x) \geq 0$, the problem (1.1) has been investigated extensively and the existence and nonexistence of solutions for (1.1) are well understood (see [3, 5, 6, 13] and the references cited there). We summarize some of the results. Suppose that the initial data $u_0(x), v_0(x) \geq 0$ and $u_0, v_0 \in L^\infty(\Omega)$. Then

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(A1) let $\alpha > 1$ or $\beta > 1$ or $s_0 = (1 - \alpha)(1 - \beta) - pq < 0$. Problem (1.1) admits a global solution for small initial data and the solution for (1.1) must blow up in finite time for large initial data;

(A2) all solutions of (1.1) are global if $\alpha, \beta \leq 1$ and $s_0 \geq 0$.

The case $m > 0$ for the single equation

\[
\begin{align*}
  u_t &= \nabla \cdot (|\nabla u|^m \nabla u) + f(x, u), \quad x \in \Omega, \ t > 0, \\
  u(x, 0) &= u_0(x), \quad x \in \Omega, \\
  u(x, t) &= 0, \quad x \in \partial \Omega
\end{align*}
\]

has been widely investigated in [1, 2, 4, 7, 9, 11, 12] and the references therein. But the problem (1.1) is not considered sufficiently and there seems to be little results on global existence, $L^\infty$ estimates and blow-up of solutions for (1.1).

In this paper we are interested in extending the previous results A1 and A2 for $m = n = 0$ to $m, n > 0$. We consider problem (1.1) for general initial data (try to be more specific here) and obtain sufficient conditions for the global existence of solutions. Furthermore, we obtain $L^\infty$ and decay estimates for solutions of (1.1), that give the behavior of solutions as $t \to 0$ and $t \to \infty$. Our method, very different from that on the basis of comparison principle used in [3, 5, 6, 13, 14, 15, 16], is based on a priori estimates and an improved Moser’s technique as in [2, 10]. In contrast with other results (which results [2, 4, 7, 9, 11]), our initial data $u_0, v_0$ is neither restricted to be bounded nor nonnegative. To drive the $L^\infty$ estimates for solutions of (1.1), we must treat carefully the parameters $m, n, p, q, \alpha$ and $\beta$.

**Definition 1.1.** A pair of functions $(u(x, t), v(x, t))$ is a global weak solution of (1.1) if $(u(x, t), v(x, t)) \in (L^\infty_{loc}((0, \infty), W^{1,m+1}_0(\Omega)) \cap L^{m+1}_{loc}(R^+, W^{1,m+1}_0(\Omega))) \times (L^\infty_{loc}((0, \infty), W^{1,n+1}_0(\Omega)) \cap L^{n+1}_{loc}(R^+, W^{1,n+1}_0(\Omega)))$ and the following equalities

\[
\begin{align*}
  \int_0^t \int_{\Omega} \{-u \varphi_t + |\nabla u|^m \nabla u \nabla \varphi - f(u, v) \varphi\} \, dxdt &= \int_{\Omega} \{u_0(x) \varphi(x, 0) - u(x, t) \varphi(x, t)\} \, dx, \\
  \int_0^t \int_{\Omega} \{-v \varphi_t + |\nabla v|^n \nabla v \nabla \varphi - g(u, v) \varphi\} \, dxdt &= \int_{\Omega} \{v_0(x) \varphi(x, 0) - u(x, t) \varphi(x, t)\} \, dx
\end{align*}
\]

are valid for any $t > 0$ and $\varphi \in C^1(R^+, C^0_0(\Omega))$, where $R^+ = [0, \infty)$.

Our results read as follows.

**Theorem 1.2.** Suppose that

(H1) The functions $f(u, v), g(u, v) \in C^0(R^2) \cap C^1(R^2 \setminus (0, 0))$ and

\[
\begin{align*}
  |f(u, v)| &\leq K_1 |u|^\alpha |v|^p, \\
  |g(u, v)| &\leq K_2 |u|^q |v|^\beta, \quad (u, v) \in R^2,
\end{align*}
\]

(1.3)
where the parameters \( \alpha, \beta, p, q \) satisfy
\[
0 \leq \alpha < 1 + m, \quad 0 \leq \beta < 1 + n; \quad m, n, p, q > 0;
\]
\[
s = (m + 1 - \alpha)(n + 1 - \beta) - pq > 0.
\]

\((H_2)\) \( u_0(x) \in L^{p_0}(\Omega) \), \( v_0(x) \in L^{q_0}(\Omega) \) with
\[
p_0 > \max\{1, q + 1 - \alpha\}, \quad q_0 > \max\{1, p + 1 - \beta\}.
\]

Then problem (1.1) admits a global weak solution \( u(x, t), v(x, t) \) which satisfies
\[
u \in L^\infty \left( R^+, L^{p_0}(\Omega) \right), \quad v \in L^\infty \left( R^+, L^{q_0}(\Omega) \right)
\]
and the following estimates hold for any \( T > 0 \)
\[
\|u\|_\infty \leq Ct^{-\sigma}, \quad \|v\|_\infty \leq Ct^{-\sigma}, \quad 0 \leq t \leq T,
\]
\[
\|u\|_{m+2}^2 + \|v\|_{n+2}^2 \leq C (t^{-1-\sigma} + t^{1-2(p+\alpha)\sigma} + t^{1-2(q+\beta)\sigma}), \quad 0 \leq t \leq T,
\]
where \( C = C(T, \|u_0\|_{p_0}, \|v_0\|_{q_0}, \sigma, \sigma = \min\{N/(p_0 + mN), N/(q_0 + nN)\}) \).

**Theorem 1.3.** Suppose \( s < 0 \). Then there exist \( p_0, q_0 > 1, d_0 > 0 \) such that if \( u_0(x) \in L^{p_0}(\Omega), v_0(x) \in L^{q_0}(\Omega) \) and \( \|u_0\|_{p_0} + \|v_0\|_{q_0} < d_0 \) the problem (1.1) admits a global weak solution \( u(x, t), v(x, t) \) that
\[
\begin{align*}
 u(x, t) & \in L^\infty_\text{loc} \left( (0, \infty), W^{1,m+1}_{0}(\Omega) \right) \cap L^{m+1}_\text{loc} \left( R^+, W^{1,m+1}_{0}(\Omega) \right) \\
 v(x, t) & \in L^\infty_\text{loc} \left( (0, \infty), W^{1,n+1}_{0}(\Omega) \right) \cap L^{n+1}_\text{loc} \left( R^+, W^{1,n+1}_{0}(\Omega) \right)
\end{align*}
\]
satisfying
\[
\|u\|_{p_0} \leq C(1 + t)^{-\frac{1}{\theta}}, \quad \|v\|_{q_0} \leq C(1 + t)^{-\frac{1}{\theta}}, \quad t \geq 0,
\]
where \( \theta = \min\{m/p_0, n/q_0\} \).

To derive Theorem 1.2 and 1.3, we will use the following lemmas.

**Lemma 1.4.** [9] Let \( \beta \geq 0, N > p \geq 1, \beta + 1 \leq q, \) and \( 1 \leq r \leq q \leq (\beta + 1)Np/(N - p) \). Then for \( \|u\|^\beta u \in W^{1,p}(\Omega) \), we have
\[
\|u\|_q \leq C^{1/(\beta + 1)}\|u\|_{r}^{1-\theta}\|u\|_{1,p}^{\beta\theta/(\beta + 1)},
\]
with \( \theta = (\beta + 1)(r^{-1} - q^{-1})/(N^{-1} - p^{-1} + (\beta + 1)r^{-1})^{-1} \), where \( C \) is a constant depending only on \( N, p \) and \( r \).

**Lemma 1.5.** [11] Let \( y(t) \) be a nonnegative differentiable function on \((0, T]\) satisfying
\[
y'(t) + A t^{\lambda - 1}y^{1+\theta}(t) \leq Bt^{-k}y(t) + Ct^\delta
\]
with \( A, \theta > 0, \lambda \theta \geq 1, B, C \geq 0, k \leq 1 \). Then we have
\[
y(t) \leq A^{-1/\theta}(2A + 2BT^1-k)^{1/\theta}t^{-\lambda} + 2C(\lambda + BT^1-k)^{-1}t^{1-\delta} \quad 0 < t \leq T
\]

This paper is organized as follows. In Section 2, we apply Lemmas 1.4 and 1.5 to establish \( L^\infty \) estimates for solutions of problem (1.1). The proof of Theorem 1.3 will be given in Section 3.
2. Proof of Theorem 1.2

For \( j = 1, 2, \ldots \), we choose \( f_j(u, v), g_j(u, v) \in C^1 \) in such a way \( f_j(u, v) = f(u, v), g_j(u, v) = g(u, v) \) when \( u^2 + v^2 \geq j^{-2}, |f_j(u, v)| \leq \eta, |g_j(u, v)| \leq \eta \) when \( u^2 + v^2 \leq j^{-2} \) with some \( \eta > 0 \) and \( (f_j(u, v), g_j(u, v)) \to (f(u, v), g(u, v)) \) uniformly in \( R^2 \) as \( j \to \infty \).

Let \( (u_{0,j}, v_{0,j}) \in C^0(\Omega) \) and \( u_{0,j} \to u_0 \) in \( L^{p_0}(\Omega) \), \( v_{0,j} \to v_0 \) in \( L^q(\Omega) \) as \( j \to \infty \). We consider the approximate problem of (1.1)

\[
\begin{align*}
  u_t &= \nabla \cdot ([|\nabla u|^2 + j^{-1})^m/2 \nabla u] + f_j(u, v), & x \in \Omega, & t > 0, \\
  v_t &= \nabla \cdot ([|\nabla v|^2 + j^{-1})^{n/2} \nabla v] + g_j(u, v), & x \in \Omega, & t > 0, \\
  u(x, 0) &= u_{0,j}(x), & x \in \Omega, & t > 0, \\
  u(x, t) &= v(x, t) = 0, & x \in \partial \Omega.
\end{align*}
\]

(2.1)

The problem (2.1) is a standard quasilinear parabolic system and admits a unique smooth solution \( (u_j(x, t), v_j(x, t)) \) on \([0, T]\) for each \( j = 1, 2, \ldots \), see [7] [8]. Furthermore, if \( T < \infty \), then

\[
\limsup_{t \to T} (\|u_j(\cdot, t)\|_\infty + \|v_j(\cdot, t)\|_\infty) = +\infty.
\]

In the sequel, we will always write \((u, v)\) instead of \((u_j, v_j)\) and \((u^p, v^p)\) for \(|u|^{p-1}u, |v|^{p-1}v\) when \( p > 0 \). Also, let \( C \) and \( C_i \) be the generic constants independent of \( j \) and \( p \) changeable from line to line.

Lemma 2.1. Let \((H_1)\) and \((H_2)\) hold. If \((u(x, t), v(x, t))\) is the solution of problem (2.1). Then \( u \in \mathcal{L}^\infty (R^+, L^{p_0}(\Omega)), v \in \mathcal{L}^\infty (R^+, L^q(\Omega))\).

Proof. Let \( p_0, q_0 > 1 \). Multiplying the first equation in (2.1) by \(|u|^{p_0-2}u\), we obtain that

\[
\frac{1}{p_0} \frac{d}{dt} \|u\|_{p_0}^{p_0} + \frac{(p_0-1)(m+2)^{m+2}}{(p_0 + m)^{m+2}} \|\nabla u\|_{m+2}^{m+2} \leq \int_{\Omega} f_j(u, v) |u|^{p_0-2} u dx.
\]

(2.2)

Notice that

\[
\int_{\Omega} f_j(u, v) |u|^{p_0-2} u dx \leq \eta_j^{1-p_0} |\Omega| + C_1 \int_{\Omega} |u|^{\alpha + p_0 - 1} |v|^p dx.
\]

(2.3)

Similarly, we have

\[
\frac{1}{q_0} \frac{d}{dt} \|v\|_{q_0}^{q_0} + \frac{(q_0-1)(n+2)^{n+2}}{(q_0 + n)^{n+2}} \|\nabla v\|_{n+2}^{n+2} \leq \eta_j^{1-q_0} |\Omega| + C_2 \int_{\Omega} |v|^{\beta + q_0 - 1} |u|^q dx,
\]

(2.4)

with \( C_1, C_2 > 0 \).

By Young’s inequality, we obtain

\[
|u|^\gamma |v|^\rho + |u|^\delta |v|^\rho \leq \frac{|v|^{p_1}}{p_1} + \frac{|u|^{p_2}}{p_2} + \frac{|u|^{q_1}}{q_1} + \frac{|v|^{q_2}}{q_2},
\]

(2.5)

where \( \gamma = \alpha + p_0 - 1, \rho = \beta + q_0 - 1, t_0 = \gamma \rho - pq > 0 \) and

\[
p_1 = \frac{t_0}{p(\gamma - q)}, p_2 = \frac{t_0}{\gamma(\rho - p)}, q_1 = \frac{t_0}{q(\rho - p)}, q_2 = \frac{t_0}{\rho(\gamma - q)}.
\]

(2.6)
The assumption \((H_2)\) on \(p_0, q_0\) and \((1.3)\) imply that \(pp_1 < q_0 + n, qq_1 < p_0 + m.\) Thus we have from \((2.2)-(2.5)\) and a Sobolev’s inequality that
\[
\frac{d}{dt} \left( \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0} \right) + C_3 \left( p_0^{-m} \|u\|_{p_0^{p_0 + m}}^{p_0 + m} + q_0^{-n} \|v\|_{q_0^{q_0 + n}}^{q_0 + n} \right) \\
\leq \eta |\Omega| \left( p_0^{j - p_0} + q_0^{j - q_0} \right) + \int_{\Omega} \left( |u|^{\gamma_1} + |v|^{\gamma_2} \right) dx.
\]
By the Young’s inequality, we have
\[
|\Theta| \leq \eta |\Omega| \left( p_0^{j - p_0} + q_0^{j - q_0} \right) + C_4 \int_{\Omega} \left( |u|^{\gamma_1} + |v|^{\gamma_2} \right) dx.
\]
Using Young’s inequality and letting \(j \to \infty\) in \((2.7)\), we conclude that
\[
\frac{d}{dt} \left( \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0} \right) + C_5 \left( \|u\|_{p_0^{p_0 + m}}^{p_0 + m} + \|v\|_{q_0^{q_0 + n}}^{q_0 + n} \right) \leq C
\]
and
\[
\frac{d}{dt} \left( \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0} \right) + C_6 \left( \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0} \right)^{1+\varrho} \leq C
\]
with \(\varrho = \min\{m/p_0, n/q_0\}\). Thus \((2.9)\) implies that \(u(t) \in L^\infty(R^+, L^{p_0}(\Omega)), v(t) \in L^\infty(R^+, L^{q_0}(\Omega))\) if \(u_0 \in L^{p_0}(\Omega)\) and \(v_0 \in L^{q_0}(\Omega)\). The proof is completed.

**Lemma 2.2.** Under the assumptions of Lemma \([2.1]\) and for any \(T > 0\), the solution \((u(t), v(t))\) also satisfies
\[
\|u\|_\infty \leq Ct^{-a}, \quad \|v\|_\infty \leq Ct^{-b}, \quad 0 < t \leq T, \quad (2.10)
\]
\[
\|u\|_{m+2}^{m+2} + \|v\|_{n+2}^{n+2} \leq C \left( t^{-1 - \sigma} + t^{1 - 2(p + a)\sigma} + t^{1 - 2(q + b)\sigma} \right), \quad 0 < t \leq T, \quad (2.11)
\]
where the constant \(C\) depends on \(T, \|u_0\|_{p_0}, \|v_0\|_{q_0}\) and \(a = N/(p_0(m + 2) + mN), b = N/(q_0(n + 2) + nN), \sigma = \min\{a, b\}\).

**Proof.** We only consider \(N > \max\{m + 2, n + 2\}\) and the other cases can be treated in a similar way.

Multiplying the first equation and the second equation in \((2.1)\) by \(|u|^{\lambda - 2} u\) and \(|v|^{\mu - 1} v\) respectively, we obtain
\[
\frac{d}{dt} \left( \|u\|_\lambda^{\lambda} + \|v\|_\mu^{\mu} \right) + C_1 \left( \lambda^{-m} \|\nabla u\|_{m+2}^{m+2} \|\nabla v\|_{n+2}^{n+2} \right) \\
\leq C_2(\lambda + \mu) \left( 1 + \int_{\Omega} |u|^{\alpha + \lambda - 1} |v|^p + |u|^q |v|^{\beta + \mu - 1} \right) dx.
\]
By the Young’s inequality, we have
\[
|u|^{\gamma_1} |v|^p + |u|^q |v|^{\gamma_2} \leq \frac{|v|^{\gamma_1 \epsilon_2}}{\epsilon_1} + \frac{|u|^{\gamma_2 \epsilon_1}}{\epsilon_2} + \frac{|u| q \eta_1}{\eta_2} + \frac{|v|^{\gamma_2 \eta_1}}{\eta_2},
\]
with \(\gamma_1 = \alpha + \lambda - 1, \gamma_2 = \beta + \mu - 1\) and \(p \epsilon_1 = \gamma_2 \eta_2, \gamma_1 \epsilon_2 = q \eta_1, \epsilon_1^{-1} + \epsilon_2^{-1} = 1, \eta_1^{-1} + \eta_2^{-1} = 1\).

The direct computation shows that
\[
\eta_1 = \frac{\tau}{q(\gamma_2 - p)}, \quad \eta_2 = \frac{\tau}{\gamma_2(\gamma_1 - q)}, \quad \epsilon_1 = \frac{\tau}{\gamma_1 (\gamma_2 - q)}, \quad \epsilon_2 = \frac{\tau}{\gamma_1 (\gamma_2 - p)}.
\]
where \( \tau = \gamma_1 \gamma_2 - pq > 0 \), \( \lambda, \mu \) are chosen properly so that \( 0 < p \varepsilon_1 < \mu + n \) and \( 0 < q \eta_1 < \lambda + m \). We take two sequences of \( \{ \lambda_k \} \) and \( \{ \mu_k \} \) as follows

\[
\lambda_1 = p_0, \quad \lambda = \lambda_k = b_1 + b_{12} R^{k-1}; \\
\mu_1 = q_0, \quad \mu = \mu_k = b_2 + b_{22} R^{k-1}, \quad k = 2, 3, \ldots
\]

where \( b_1 = q + 1 - \alpha, \ b_{12} = (b_1 + m) / s, \ b_2 = p + 1 - \beta, \ b_{22} = (b_2 + n) / s \) and \( R \) is chosen so that \( R > 1, \ \lambda_2 > p_0, \ \mu_2 > q_0 \). Notice that \( \lambda_k \sim \mu_k \) as \( k \to \infty \).

We now derive the estimates for the integrals \( \int_{\Omega} |v|^p \varepsilon_1 dx \) and \( \int_{\Omega} |u|^q \eta dx \). If \( p \varepsilon_1 \leq \mu \) and \( q \eta_1 \leq \lambda \), then we have

\[
\int_{\Omega} |v|^p \varepsilon_1 dx \leq C \left( 1 + \int_{\Omega} |v|^\mu dx \right), \quad \int_{\Omega} |u|^q \eta dx \leq C \left( 1 + \int_{\Omega} |u|^\lambda dx \right). \tag{2.15}
\]

Without loss of generality, we suppose \( \mu < p \varepsilon_1 < \mu + n, \ \lambda < q \eta_1 < \lambda + m \) and \( r = \tau / (\gamma_1 - q) - \mu > 0, \ h = \tau / (\gamma_2 - p) - \lambda > 0 \). Then from (2.12) and (2.13), we have

\[
\frac{d}{dt} \left( \|u\|_\lambda^2 + \|v\|_\mu^2 \right) + 2 C_1 \left( \lambda^{-m} \|\nabla u^\frac{\lambda}{m+2}\|_{m+2} + \mu^{-n} \|\nabla v^\frac{\mu}{n+2}\|_{n+2} \right) \tag{2.16}
\]

\[
\leq C_2 \lambda \left( 1 + \|u\|^{\lambda+\delta}_{\lambda+\delta} \right) + C_2 \mu \left( 1 + \|v\|^{\mu+\tau}_{\mu+\tau} \right).
\]

where the constants \( C_1, C_2 \) are independent of \( \lambda \) and \( \mu \). Furthermore, we have following by Hölder’s and Sobolev’s inequalities

\[
\int_{\Omega} |u|^{\lambda+h} dx \leq \|u\|^{\theta_1}_{\theta_0} \|u\|^{\theta_2}_{\theta_0} \|u\|^{\theta_3}_{\mu} \leq C \|u\|^{\theta_1}_{\lambda} \|\nabla u^\frac{\lambda}{m+2}\|_{m+2} + \lambda \tag{2.17}
\]

\[
C_1 C_2^{-1} \lambda^{-1-m} \|\nabla u^\frac{\lambda}{m+2}\|_{m+2} + C_3 \lambda^{m-1} \|u\|_{\lambda}^m
\]

with

\[
\lambda^* = \frac{N(\lambda + m)}{N - m - 2}, \quad \theta_1 = \lambda \left( 1 - \frac{hN}{p_0(m + 2) + m N^2}, \quad \theta_2 = \frac{hN}{p_0(m + 2) + m N^2}, \quad \theta_3 = \frac{hN(\lambda + m)}{p_0(m + 2) + m N} \right)
\]

\[
\sigma_1 = \frac{(m + 1)(p_0(m + 2) + N(m - h))}{h N} > 0.
\]

Similarly, we can derive that

\[
\int_{\Omega} |v|^{\mu+\tau} dx \leq C_1 C_2^{-1} \mu^{-1-n} \|\nabla v^\frac{\mu}{n+2}\|_{n+2} + C_3 \mu^{\sigma_2} \|v\|_{\mu}^\mu, \tag{2.18}
\]

with \( \sigma_2 = (n + 1)(q_0(n + 2) + N(n - r))/r N \). Hence it follows from (2.16)-(2.18) that

\[
\frac{d}{dt} \left( \|u\|_\lambda^2 + \|v\|_\mu^2 \right) + C_1 \left( \lambda^{-m} \|\nabla u^\frac{\lambda}{m+2}\|_{m+2} + \mu^{-n} \|\nabla v^\frac{\mu}{n+2}\|_{n+2} \right) \tag{2.19}
\]

\[
\leq C_2 \lambda \left( 1 + \|u\|_{\lambda}^2 \right) + C_3 \mu \left( 1 + \|v\|_{\mu}^{\mu_2} \right).
\]

Now we employ an improved Moser’s technique as in [2, 10]. Let \( \{ \lambda_k \}, \ \{ \mu_k \} \) be two sequences as defined in (2.14). From Lemma 1.4 we see that

\[
\|u\|_{\lambda_k} \leq C \frac{m+2}{m+2} \|u\|_{\lambda_{k-1}}^\frac{1-\theta_k}{\lambda_{k-1}} \|\nabla u^\frac{\lambda_k}{m+2}\|_{m+2}^\frac{(m+2)\theta_k}{\lambda_k}, \tag{2.20}
\]
where the constant $C$ is independent of $\lambda_k$ and $\mu_k$, and
\[
\theta_k = \frac{\lambda_k + m}{m + 2} \left( \frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k} \right) \left( \frac{1}{N} - \frac{1}{m + 2} + \frac{\lambda_k + m}{(m + 2)\lambda_{k-1}} \right)^{-1},
\]
\[
\overline{\theta}_k = \frac{\mu_k + n}{n + 2} \left( \frac{1}{\mu_{k-1}} - \frac{1}{\mu_k} \right) \left( \frac{1}{N} - \frac{1}{n + 2} + \frac{\mu_k + n}{(n + 2)\mu_{k-1}} \right)^{-1}.
\]

Let $t_k = \frac{\lambda_k + m}{\mu_k} - \lambda_k$, $s_k = \frac{\mu_k + n}{\mu_k} - \mu_k$. Then (2.20) and (2.21) give
\[
\lambda_k^{-m} \| \nabla u \|_\infty^{m+2} \| u \|_{\lambda_k} \| u \|_{\lambda_k-1} \geq C^{-\frac{m+2}{\mu_k}} \| \lambda_k \| u \|_{\lambda_k} \| u \|_{\lambda_k-1}^{m+2}, \tag{2.22}
\]
\[
\mu_k^{-n} \| \nabla v \|_\infty^{n+2} \| v \|_{\mu_k} \| v \|_{\mu_k-1} \geq C^{-\frac{n+2}{\mu_k}} \| \mu_k \| v \|_{\mu_k} \| v \|_{\mu_k-1}^{n+2}. \tag{2.23}
\]

Denote
\[
y_k(t) = \| u \|_{\lambda_k} \| v \|_{\mu_k}, \quad t \geq 0.
\]

Then inserting (2.22) - (2.23) into (2.19) ($\lambda = \lambda_k, \mu = \mu_k$), we find that
\[
y_k'(t) + C_1 C^{-\frac{m+2}{s_k}} \| u \|_{\lambda_k} \| u \|_{\lambda_k-1}^{m-t} + C_1 C^{-\frac{n+2}{\mu_k}} \| v \|_{\mu_k} \| v \|_{\mu_k-1}^{n-s_k} \geq C_3 (\lambda_k + \mu_k) + C \lambda_k^{s+1} \| u \|_{\lambda_k} + C \mu_k^{s+1} \| v \|_{\mu_k}. \tag{2.24}
\]

We claim that there exist the bounded sequence $\{\xi_k\}, \{\eta_k\}, \{m_k\}, \{r_k\}$ such that
\[
\| u \|_{\lambda_k} \leq \xi_k t^{-m_k}, \quad \| v \|_{\mu_k} \leq \eta_k t^{-r_k}, \quad 0 < t \leq T. \tag{2.25}
\]

Without loss of generality, we suppose that $\xi_k, \eta_k \geq 1$. By Lemma 2.1 (2.25) holds for $k = 0$ if we take $m_0 = r_0 = 0$ and $\xi_0 = \sup_{t \geq 0} \| u \|_{\lambda_0}, \eta_0 = \sup_{t \geq 0} \| v \|_{\mu_0}$. If (2.25) is true for $k - 1$, then we have from (2.24) that
\[
y_k'(t) + C_1 \| u \|_{\lambda_k} (\xi_{k-1} t^{-m_{k-1}})^{m-t} + C_1 \| v \|_{\mu_k} (\eta_{k-1} t^{-r_{k-1}})^{n-s_k} \leq C (\lambda_k + \mu_k) \left( \lambda_k^{s+1} \| u \|_{\lambda_k} + \mu_k^{s+1} \| v \|_{\mu_k} \right). \tag{2.26}
\]

We take $\sigma_0 = \max\{\sigma_1, \sigma_2\}, \tau_k = \min\{t_k/\lambda_k, s_k/\mu_k\}, \alpha_k = \min\{m - t_k, n - s_k\}$ and $A_{k-1} = \max\{\xi_{k-1}, \eta_{k-1}\}, \beta_k = \max\{t_k - m, s_k - n\}$. Then we have from (2.26) that
\[
y_k'(t) + C_3 A_k \alpha_k t^{\beta_k} y_k(t) \leq CA_k C \lambda_k^{s+1} y_k(t) + CA_k \lambda_k^{s+1} T^{\beta_k}, \quad 0 < t < T. \tag{2.27}
\]

Applying Lemma 1.5 to (2.27), we get
\[
y_k(t) \leq B_k t^{-(1+\beta_k)/\tau_k}, \quad 0 < t < T, \tag{2.28}
\]

where
\[
B_k = 2 \left( C_3 A_k^{\alpha_k} \right)^{-\frac{1}{\tau_k}} \left( C_3 \lambda_k^{s+1} + \frac{1 + \beta_k}{\tau_k} \right)^{\frac{1}{\tau_k}} + 2C \lambda_k \left( C \lambda_k^{s+1} + \frac{1 + \beta_k}{\tau_k} \right)^{-1}.
\]

Moreover, (2.28) implies that
\[
\| u \|_{\lambda_k} \leq B_k^{\frac{1}{\tau_k}} t^{1+\beta_k/\tau_k}, \quad \| v \|_{\mu_k} \leq B_k^{\frac{1}{\tau_k}} t^{1+\beta_k/\tau_k}, \quad 0 < t \leq T. \tag{2.29}
\]
We take
\[ \xi_k = B_k^{\frac{1}{\tau_k}}, \quad \eta_k = B_k^{\frac{1}{\alpha_k}}, \quad m_k = \frac{1 + \beta_k}{\lambda_k \tau_k}, \quad r_k = \frac{1 + \beta_k}{\mu_k \tau_k}. \]

By a similar argument in [2, 10], we know that \( \{\xi_k\}, \{\eta_k\} \) are bounded and there exist two subsequences \( \{m_{kl}\} \subset \{m - k\} \) and \( \{r_{kl}\} \subset \{r_k\} \) such that
\[ m_{kl} \to a = \frac{N}{p_0(m + 2) + mN}, \quad r_{kl} \to b = \frac{N}{q_0(n + 2) + nN}, \quad (as \ l \to \infty). \]

Therefore, letting \( l \to \infty \) in \( (2.28) \), we obtain
\[ \|u\|_\infty \leq Ct^{-a}, \quad \|v\|_\infty \leq Ct^{-b}, \quad 0 < t < T, \quad (2.30) \]
This yields \( (2.10) \).

It remains to prove the estimate \( (2.11) \). In order to derive \( (2.11) \), we use a similar argument in [10]. We first choose \( \mu > \max \{\sigma, 2(p + \alpha)\sigma - 2, 2(q + \beta)\sigma - 2\} \) and \( h(t) \in C([0, \infty) \cap \mathbb{R}(0, \infty) \) such that \( h(t) = \vartheta^t, \ 0 \leq t \leq 1; h(t) = 2, t \geq 2 \) and \( h(t), h'(t) \geq 0 \) in \( (0, \infty) \). Then multiplying the first equation by \( h(t)u \) and the second equation by \( h(t)v \) in \( (2.1) \), and letting \( j \to \infty \), we obtain
\[ \int_0^t h(s)g(s)ds + \frac{1}{2}h(t) \int_\Omega (|u|^2 + |v|^2)dx \leq \int_\Omega (u|^\mu + |v|^\mu)dx \]
\[ \leq \frac{1}{2} \int_0^t \int_\Omega h'(s)(|u|^2 + |v|^2)dxds + C \int_0^t \int_\Omega h(s)(|u|^{1+\alpha}|v|^p + |u|^q|v|^{1+\beta})dxds \]
with \( g(t) = \|\nabla u\|_{m+2}^2 + \|\nabla v\|_{n+2}^2, \ t \geq 0. \)

By Young's inequality and the assumption \( (1.4) \), we obtain
\[ C \int_\Omega (|u|^{1+\alpha}|v|^p + |u|^q|v|^{1+\beta})dx \leq \int_\Omega (|u|^\tau_1 + |v|^\tau_2)dx \]
\[ \leq \varepsilon \int_\Omega (|u|^{m+2} + |v|^{n+2})dx + C_\varepsilon |\Omega| \leq C(|\nabla u|_{m+2}^2 + |\nabla v|_{n+2}^2) + C_\varepsilon |\Omega| \]
for any \( \varepsilon > 0 \) and \( \tau_1 = ((\alpha + 1)\beta + 1) - pq)/(\beta + 1 - p) \leq m + 2, \tau_2 = ((\alpha + 1)(\beta + 1) - pq)/(\alpha + 1 - q) < n + 2. \) Furthermore, we take \( \varepsilon = 1/2. \) Then \( (2.31) \) \( (2.32) \) yields
\[ \int_0^t h(s)g(s)ds + h(t)(\|u\|_2^2 + \|v\|_2^2) \leq Ct^{\mu - \sigma} \quad (2.33) \]

Next, let \( \rho(t) = \int_0^t h(s)ds, \ t \geq 0. \) Similarly, multiplying the first equation in \( (2.1) \) by \( \rho(t)u \) and the second equation by \( \rho(t)v \), and letting \( j \to \infty \), we have from \( (2.30) \) \( (2.31) \) that
\[ \int_0^t \rho(s)(\|u\|^2 + |v|^2)ds + \rho(t)g(t) \leq C \int_0^t \int_\Omega \rho(s)(|u|^{2\alpha}|v|^{2p} + |u|^{2q}|v|^{2\beta})dxds \]
\[ + \int_0^t \rho'(s)g(s)ds \leq C \int_0^t \rho(s) \left( s^{-(\alpha + p)\sigma} + s^{-(\beta + q)\sigma} \right)ds + Ct^{\mu - \sigma} \]
\[ \leq C \left( t^{\mu - \sigma} + t^{\mu + 2 - (p + \alpha)\sigma} + t^{\mu + 2 - (q + \beta)\sigma} \right), \ 0 < t < T. \quad (2.34) \]
Thus (2.34) implies
\[ g(t) \leq C \left( t^{-1-\sigma} + t^{1-2(p+\alpha)} + t^{1-2(q+\beta)} \right), \quad 0 < t \leq T, \] (2.35)
and (2.11) is proved. The proof is completed. \( \square \)

**Proof of Theorem 1.2** We notice that the estimate constant \( C \) in (2.30) and (2.35) is independent of \( j \), we may obtain the desired solution \((u, v)\) as limit of \( \{(u_j, v_j)\} \) (or a subsequence) by the standard compact argument as in [6, 8, 9, 10]. The solution \((u, v)\) of problem (1.1) also satisfies (1.5)-(1.6). The proof is completed. \( \square \)

**Remark:**
- From the proof of Theorem 1.2, we see that if the assumption (1.3) is replaced by
\[ |f(u, v)| \leq K_1(1 + |u|^a|v|^p), \quad |g(u, v)| \leq K_2(1 + |u|^q|v|^\beta), \]
the conclusions in Theorem 1.2 still hold.

### 3. Proof of Theorem 1.3

By the standard compact argument as in [2, 7, 9, 10], we only consider the estimate (1.8) and show that \((u, v) \in L_{loc}^{1,m+1}(R^+, W_0^{1,m+1}(\Omega)) \cap L_{loc}^{1,n+1}(R^+, W_0^{1,n+1}(\Omega))\) for the solution of (2.1).

**Proof of Theorem 1.3** Suppose that \( s < 0 \) holds. Let
\[ p_0 = b_1 + b_{12}\varepsilon > 1, \quad q_0 = b_2 + b_{22}\varepsilon > 1, \] (3.1)
with \( b_1 = q+1-\alpha, b_2 = p+1-\beta, b_{12} = -(q+m+1-\alpha)/s, b_{22} = -(p+n+1-\beta)/s. \) Since \( s < 0 \), we can take \( \varepsilon > 0 \) such that \( p_0 \geq \max\{4q, 4\alpha, 2 + 2\alpha\}, q_0 \geq \max\{4p, 4\beta, 2 + 2\beta\}, S_0 = (\alpha + p_0 - 1)(\beta + q_0 - 1) - pq > 0. \) Then it follows from (2.5) and (2.7) that
\[
\frac{d}{dt}\left( \|u\|_{p_0}^{q_0} + \|v\|_{q_0}^{p_0} \right) + C_1 \left( \|\nabla u\|_{m+2}^{q_0 + m} + \|\nabla v\|_{m+2}^{q_0 + n} \right) \leq C \int |u|^{q_1} + |v|^{p_1} \, dx, \tag{3.2}
\]
where \( q_1 = S_0/(q_0 + \beta - 1 - p) > p_0 + m, pp_1 = S_0/(\alpha + p_0 - q - 1) > q_0 + n. \) We now estimate the right-hand side of (3.2). Let \( q_1 = p_0 + \theta, pp_1 = q_0 + \tau \) and \( \theta > m, \tau > n. \) Then
\[
\int_\Omega |u|^{q_1} \, dx = \|u\|_{p_0}^{q_0 + \theta} \leq C_2 \|u\|_{p_0}^{\theta - m} \|\nabla u\|_{m+2}^{q_0 + m}, \tag{3.3}
\]
\[
\int_\Omega |v|^{p_1} \, dx = \|v\|_{q_0}^{p_0 + \tau} \leq C_2 \|v\|_{q_0}^{\tau - n} \|\nabla v\|_{m+2}^{q_0 + n}, \tag{3.4}
\]
Denote
\[ \phi(t) = \|u\|_{p_0}^{p_0} + \|v\|_{q_0}^{q_0}, \quad f(t) = \|\nabla u\|_{m+2}^{m+2} + \|\nabla v\|_{m+2}^{m+2}, \]
then (3.2) becomes
\[
\phi'(t) + C_1 f(t) \leq C_2 \left( \|u\|_{p_0}^{\theta-m} \|\nabla u\|_{m+2}^{m+2} + \|v\|_{q_0}^{r-n} \|\nabla v\|_{n+2}^{q_0+n} \right) \tag{3.5}
\]
with \( \alpha_0 = \min\{(\theta - m)/p_0, (r - n)/q_0\} > 0 \).

(3.5) implies that there is \( C_0 > 0 \) such that
\[
\phi'(t) + C_0 f(t) \leq 0 \quad \text{if} \quad C_3 \phi^{\alpha_0}(0) = C_3 \left( \|u_0\|_{p_0}^{\rho_0} + \|v_0\|_{q_0}^{\nu_0} \right)^{\alpha_0} < C_1. \tag{3.6}
\]
Furthermore, we have from Sobolev embedding theorems that
\[
\|\nabla u\|_{m+2}^{m+2} \geq d_1 \|u\|_{p_0}^{p_0+m} \geq d_2 \|u\|_{p_0}^{p_0+m}, \quad \|\nabla v\|_{n+2}^{q_0+n} \geq d_2 \|v\|_{q_0}^{q_0+n},
\]
for some \( d_2 > 0 \). Hence,
\[
f(t) \geq d_2 \left( \|u\|_{p_0}^{p_0+m} + \|v\|_{q_0}^{q_0+m} \right) \geq d_2 \phi^{1+\vartheta}, \quad \vartheta = \min\{m/p_0, n/q_0\}.
\]

Now (3.6) gives
\[
\phi'(t) + d_2 \phi^{1+\vartheta} \leq 0, \quad t \geq 0. \tag{3.7}
\]
This implies that
\[
\phi(t) \leq C (1 + t)^{-\frac{\vartheta}{\vartheta}}. \tag{3.8}
\]

Next, we show that \( (u, v) \in L^{1+m+1}_{\text{loc}} \left( R^+, W^{1,m+1}_0 \right) \cap L^{1,n+1}_{\text{loc}} \left( R^+, W^{1,n+1}_0 \right) \). By the definition of \( p_0 \) and \( q_0 \), we have from (3.8) that for any \( t \geq 0 \),
\[
\int_{\Omega} |u|^{1+\alpha}|v|^p dx \leq C_1 \|u\|_{p_0}^{1+\alpha} \|v\|_{q_0}^p, \quad \int_{\Omega} |u|^q |v|^{1+\beta} dx \leq C_1 \|u\|_{p_0}^q \|v\|_{q_0}^{1+\beta}
\]
Here \( C_1 \) is a constant independent of \( t \). Thus (2.31) yields that
\[
\int_0^t h(s) g(s) ds \leq C \left( h(t) + \int_0^t g(s) ds \right) \leq C (h(t) + \rho(t)), \quad t \geq 0. \tag{3.9}
\]
Similarly, we have
\[
\int_{\Omega} |u|^{2\alpha}|v|^{2p} dx \leq \|u\|_{p_0}^{2\alpha} \|v\|_{q_0}^{2p}, \quad \int_{\Omega} |u|^{2q}|v|^{2\beta} dx \leq \|u\|_{p_0}^{2q} \|v\|_{q_0}^{2\beta} \leq C_2.
\]
Then from (2.34) and (3.9), we obtain
\[
\rho(t) g(t) \leq C_3 \left( \int_0^t \rho(s) ds + \int_0^t h(s) g(s) ds \right) \leq C_3 \left( \int_0^t \rho(s) ds + h(t) + \rho(t) \right) \tag{3.10}
\]
It implies
\[
g(t) \leq C_4 (t + t^{-1} + 1), \quad 0 \leq t \leq T, \tag{3.11}
\]
and \( (u, v) \in L^{1,m+1}_{\text{loc}} \left( R^+, W^{1,m+1}_0 \right) \cap L^{1,n+1}_{\text{loc}} \left( R^+, W^{1,n+1}_0 \right) \). This completes the proof of Theorem 1.2. The proof is completed. \( \square \)

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REFERENCES


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