CONTRACTIONS OVER GENERALIZED METRIC SPACES

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ABSTRACT. A generalized metric space (g.m.s) has been defined as a metric space in which the triangle inequality is replaced by the ‘Quadrilateral inequality’, \( d(x, y) \leq d(x, a) + d(a, b) + d(b, y) \) for all pairwise distinct points \( x, y, a \) and \( b \) of \( X \). \((X, d)\) becomes a topological space when we define a subset \( A \) of \( X \) to be open if to each \( a \) in \( A \) there corresponds a positive number \( r_a \) such that \( b \in A \) whenever \( d(a, b) < r_a \). Cauchyness and convergence of sequences are defined exactly as in metric spaces and a g.m.s \((X, d)\) is called complete if every Cauchy sequence in \((X, d)\) converges to a point of \( X \). A. Branciari [1] has published a paper purporting to generalize Banach’s Contraction principle in metric spaces to g.m.s. In this paper we present a correct version and proof of the generalization.

1. Main result

In what follows \( \mathbb{N} \) denotes the set of natural numbers. The basic terms are already defined in the abstract. We denote \( \{ y \in X : d(x, y) < r \} \) for \( x \) in a g.m.s \((X, d)\) by \( B_r(x) \). In [1], the following were taken for granted and used:

1. \( \{ B_r(x) : r > 0, x \in X \} \) is a basis for a topology on \( X \)
2. \( d \) is continuous in each of the coordinates and
3. a g.m.s is a Hausdorff space.

The following examples shows that (1), (2) and (3) are false.

**Example 1.1.** Let \( A = \{0, 2\} \), \( B = \{\frac{1}{n} : n \in \mathbb{N}\} \), \( X = A \cup B \). Define \( d \) on \( X \times X \) as follows: \( d(x, y) = 0 \) if \( x = y \), \( d(x, y) = 1 \) if \( x \neq y \) and \( \{x, y\} \subseteq A \) or \( \{x, y\} \subseteq B \), \( d(x, y) = d(y, x) = y \) if \( x \in A \) and \( y \in B \). Then \((X, d)\) is a complete g.m.s in which

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(a) the sequence \((\frac{1}{n})_{n \in \mathbb{N}}\) converges to both 0 and 2 and it is not a Cauchy sequence, 

(b) there does not exist \(r > 0\) such that \(B_r(0) \cap B_r(2) = \emptyset\),

(c) \(B_{\frac{2}{3}}(\frac{1}{3}) = \{0, 2, \frac{1}{3}\}\) and there does not exist \(r > 0\) such that \(B_r(0) \subseteq B_{\frac{2}{3}}(\frac{1}{3})\),

(d) \(\lim d(\frac{1}{n}, \frac{1}{2}) \neq d(0, \frac{1}{2})\).

**Remark 1.2.** Even though the sets \(B_r(x)\) do not form an open basis for a topology on a g.m.s \(X\), the subsets \(A\) of \(X\) satisfying the following condition form a topology on \(X\): To each \(a\) in \(A\) there corresponds \(r > 0\) such that \(B_r(a) \subseteq A\).

**Theorem 1.3.** (Banach’s Contraction principle in a g.m.s.) Let \((X, d)\) be a Hausdorff and complete g.m.s and let \(f : X \to X\) be a mapping and \(0 < \lambda < 1\) satisfying the inequality \(d(fx, fy) \leq \lambda d(x, y)\) for all \(x, y\) in \(X\)(such a mapping is called a contraction mapping on \(X\) and \(\lambda\) is called the contractive constant of \(f\)). Then there is a unique point \(x \in X\) satisfying \(f(x) = x\) (such a point is called a fixed point of \(f\)).

**Proof.** Let \(x \in X\), \(a_n = f^n(x)\) for \(n \geq 0\) and \(c = \inf S\) where \(S = \{d(a_{n-1}, a_n) : n \in \mathbb{N}\}\). We claim that \(c = 0\). If \(c \neq 0\) then \(c < \frac{c}{X}\) and hence there is a positive integer \(n\) such \(d(a_{n-1}, a_n) < \frac{c}{X}\) so that \(\lambda d(a_{n-1}, a_n) < c\). By Contractive property of \(f\) we have \(d(f^n x, f^{n+1} x) < c\) a contradiction to the minimality of \(c\). Hence \(c = 0\). The monotonically decreasing property of the sequence \(d(a_n, a_{n+1})\) implies that \(d(a_n, a_{n+1})\) converges to 0 ............(∗).

We claim that \(f\) has a periodic point. Suppose, to obtain a contradiction, \(f\) has no periodic point. Then \(\{a_n\}\) is a sequence of distinct points and for \(m > n + 1\), we have

\[
\begin{align*}
d(a_n, a_m) &= d(f^n x, f^m x) \\
&\leq d(f^n x, f^{n+1} x) + d(f^{n+1} x, f^{m+1} x) + d(f^{m+1} x, f^m x) \\
&\leq (\lambda^n + \lambda^m)d(x, f x) + \lambda d(f^n x, f^m x) \\
&\quad \text{(By Quadrilateral inequality)}
\end{align*}
\]

which implies \((1 - \lambda)d(a_n, a_m) \leq (\lambda^n + \lambda^m)d(x, f x)\) and hence \(\{a_n\}\) is a Cauchy sequence in \((X, d)\) (in view of (∗)). By Completeness, \(a_n \to a\) for some \(a\) in \(X\). Also \(d(fa_n, fa) \leq \lambda d(a_n, a)\) and \(d(a_n, a) \to 0\). So \(d(fa_n, fa) = d(a_{n+1}, fa) \to 0\). Hence \(a_n \to a\) and \(a_{n+1} \to fa\). Since \((X, d)\) is Hausdorff it follows that \(a = fa\), a contradiction to the assumption that \(f\) has no periodic point. Thus \(f\) has a periodic point say \(a\) of period \(n\). Suppose if possible \(n > 1\). Then \(d(a, fa) = d(f^n a, f^{n+1}a) < \lambda^n d(a, fa)\), a contradiction. So \(n = 1\) and \(a\) is a fixed point of \(f\). If \(a, b\) are fixed points of \(f\) then \(d(a, b) = d(fa, fb) \leq \lambda d(a, b)\). Since \(0 < \lambda < 1\), we have \(a = b\). \(\square\)

**Remark 1.4.** Several publications attempting to generalize fixed point theorems in metric spaces to g.m.s are plagued by the use of (1), (2), and (3) above (see for example [2], [3], [4], [5] and [6]). Valid proofs for many of them can be offered as in theorem1.3 which will be communicated soon by the authors for publication.
Further general topological properties of a g.m.s have been extensively studied by us and will be communicated in a forthcoming paper.

REFERENCES


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