SOME PROPERTIES OF $L_{p,w}(0 < p \leq 1)$

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Abstract. In this article we explain some properties of $L_{p,w}$ when $0 < p \leq 1$ and $w$ is weight. These properties are general and we derive them from $L_p$ spaces.

1. Introduction and preliminaries

The concept of coorbit spaces theory was originally developed by Feichtinger and Gröchenig [5,6,7] in the late 1980’s with the aim to provide a unified and group theoretical approach to function spaces and their atomic decompositions. After that S. Dahlke, G. Steidl and G. Teschke have studied coorbit spaces in [2,3,4]. We should know about $L_{p,w}$ for concept of coorbit spaces. Then if we introduce some properties of $L_{p,w}$ so it will be useful for coorbit spaces. Really the idea of this article has made when we was researching about coorbit spaces.

Definition 1.1. Let $G$ be a separable, locally compact, topological Hausdorff group, then $X = \frac{G}{P}$ is a homogeneous space, where $P$ is a closed subgroup of $G$.

Definition 1.2. Let $G$ be a separable, locally compact, topological Hausdorff group with right Haar measure $v$. A unitary representation of $G$ in a Hilbert space $H$ is defined as a mapping $U$ of $G$ into the space of unitary operators on $H$ such that $U(gg') = U(g)U(g')$ for all $g, g' \in G$ and $U(e) = \text{Id}$ which $e$ is identity element in $G$.

Definition 1.3. If the right and the left Haar measure coincide, Simply it is called Haar measure, and $G$ is said to be unimodular.

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Definition 1.4. A unitary representation \( U \) is called irreducible, if the only closed subspaces of \( H \) which are invariant under all operators \( U(g) \) (\( g \in G \)) are \{0\} and \( H \).

Definition 1.5. The representation \( U \) is said to be square integrable, if there exists a nonzero vector \( \psi \in H \) which fulfills the admissibility condition

\[
\int_G |\langle \psi, U(g)\psi \rangle_H|^2 dv(g) < \infty
\]

Let \( X \) is a homogeneous space. Because \( U \) is not directly defined on \( X \), it is necessary to embed \( X \) in \( G \). This can be realized by using the canonical fiber bundle structure of \( G \) with projection \( \Pi : G \to X \). Let \( \sigma : X \to G \) be a Borel section of this fiber bundle, i.e., \( \Pi \circ \sigma(h) = h \) for all \( h \in H \). In this article, we always assume that \( X \) is homogeneous space, and carries a \( G \)-invariant measure \( \mu \), i.e., a measure invariant under the action \( h \to hg(h \in X, g \in G) \) and \( < ., . > \) always denotes the \( L^2 \)-inner product

\[
< F, K > = \int_X F(x)\overline{K(x)}d\mu(x)
\]

whenever integral is defined.

2. Main results

Proposition 2.1. Let \( G \) be a locally compact group with left Haar measure \( \mu \), and assume that \( \Pi \) is a square integrable representation of \( G \) on \( H \), then there exists a unique positive self-adjoint operator \( U \) with domain \( D(U) \), such that

1. \( V_g(g) \in L^2(G) \iff g \in D(U) \)
2. For all \( g_1, g_2 \in D(U) \) and \( f_1, f_2 \in H \)

\[
\int_G < f_1, \Pi(x)g_1 > < f_2, \Pi(x)g_2 > d\mu(x) = < U(g_1)U(g_2), f_1 >
\]

\( D(U) \) is dense in \( H \). If \( G \) is unimodular, then \( D(U) = H \) and \( U \) is a multiple of the identity on \( H \).

Proof. We refer the readers to[1, Theorem 17.1.4].

Definition 2.2. An irreducible, unitary representation \( U \) of \( G \) on \( H \) is called square integrable mod \((P, \sigma)\), if there exists \( \psi \in H \) such that the integral

\[
\int_X < f, U(\sigma(h)^{-1}\psi) >_H U(\sigma(h)^{-1}\psi)d\mu(h)
\]

converges weakly to a positive, bounded operator \( A_\sigma \) (dependent on \( \sigma \) and \( \psi \)) which has a bounded inverse \( A_\sigma^{-1} \), in the sense that

\[
< A_\sigma f, g >_H = \int_X < f, U(\sigma(h)^{-1}\psi) >_H < g, U(\sigma(h)^{-1}\psi) >_H d\mu(h).
\]

Definition 2.3. If \( A_\sigma = \lambda Id \) for some \( \lambda > 0 \), then we call \( U \) strictly square integrable mod \((P, \sigma)\) and \((\psi, \sigma)\) is a strictly admissible pair.
Theorem 2.4. Let $X = \frac{G}{P}$ be a homogeneous space and unimodular. If $\Pi$ is square integrable mod $(P, \sigma)$ which $\sigma$ is a section from $X$ to $G$, then $\Pi$ is strictly square integrable.

Proof. By use of proposition 2.1, there exists positive, self-adjoint operator $U$. Suppose that $f_2 = f$, $f_1 = g$, and, $g_1 = g_2 = \psi \neq 0$. We have

$$\int_X < g, \Pi(h)\psi > < f, \Pi(h)\psi > d\mu(h) = < U\psi, U\psi > < f, g >$$

Since $\Pi$ is square integrable mod $(P, \sigma)$ then there exists $A_\sigma$ that is invertible, bounded with bounded inverse $A_\sigma^{-1}$. Hence, for all $g \in H$ we have

$$< A_\sigma f, g > = \int_X < f, \Pi(h)\psi > < g, \Pi(h)\psi > d\mu(h) = < U\psi, U\psi > < f, g >$$

$$\Rightarrow < A_\sigma f, g > = \|U\psi\|^2 < f, g >$$

$$\Rightarrow < A_\sigma f, g > = <\|U\psi\|^2 f, g >$$

Because $g$ is an arbitrary element in $H$ so we have $A_\sigma = \|U\psi\|^2Id$. If $\lambda = \|U\psi\|^2$, then $\Pi$ is strictly square integrable. $\square$

Definition 2.5. If $w$ be positive, continuous function on $G$ and for all $g \in G$, $0 < w(g) \leq 1$ so we say that $w$ is weight function on $G$.

Definition 2.6. Let $X$ be homogeneous space, Similar to [3] we introduce weighted $L_p$ spaces on $X$ for $0 < p \leq 1$ by

$$L_{p,w} = \{ f \text{ measurable on } X : \|f\|_{L_{p,w}} = \left( \int_X |f(h)|^p w^p(\sigma(h))d\mu(h) \right)^{\frac{1}{p}} < \infty \}.$$ 

Theorem 2.7. Suppose that $X$ is homogeneous and topological vector space, then $L_{p,w}(X)$ for $0 < p < 1$ is a locally bounded $F$-space.

Proof. In the beginning we define

$$\Delta(f) = \int_X |f(h)|^p w^p(\sigma(h))d\mu(h)$$

For more details we refer the readers to [8]. $\square$

Corollary 2.8. Let $X$ be measurable, homogeneous and topological vector space, $r > 0$ and $0 < p < 1$. We know that there exists $n$ belong to natural numbers such that $n^{p-1}\Delta(f) < r$. If $X = \bigcup_{i=1}^n A_i$ such that $A_i$ for $1 \leq i \leq n$ are measurable sets and for $i \neq j$, $A_i \cap A_j = \emptyset$ and \[\int_A |f(h)|^p w^p(\sigma(h))d\mu(h) = \frac{\Delta(f)}{n},\] then $L_{p,w}(X)$ contains no convex open sets, other $\emptyset$ and $L_{p,w}(X)$.

Proof. Suppose $V \neq \emptyset$ is open and convex in $L_{p,w}$. Assume $0 \in V$, without loss of generality. Then $B_r \subset V$, for some $r > 0$. Define $g_i(h) = n f(h)$ if $h \in A_i$, $g_i(h) = 0$ otherwise. Then by use of hypothesis $\Delta(g_i) = n^{p-1}\Delta(f) < r$ for $1 \leq i \leq n$ we have $g_i \in V$. Since $V$ is convex and $f = \frac{1}{n}(g_1 + ... + g_n)$ is follows that $f \in V$. Hence $V = L_{p,w}$. $\square$
Corollary 2.9. Suppose that hypothesis in previous corollary are true, then $(L_{p,w}(X))^* = 0$.

Proof. Suppose that $\Lambda : L_{p,w} \to Y$ is a continuous linear mapping of $L_{p,w}$ into some locally convex space $Y$. Let $\beta$ be a convex local base for $Y$ and $V \in \beta$, then $\Lambda^{-1}(V)$ is convex, open and not empty. Hence by use of previous corollary $\Lambda^{-1}(V) = L_{p,w}$. Consequently $\Lambda(L_{p,w}) \subset V$ for every $V \in \beta$ we conclude that $\Lambda f = 0$ for every $f \in L_{p,w}$. Thus 0 is the only continuous linear mapping of $L_{p,w}$ into any locally convex space $Y$. If $Y$ be complex scalers then $(L_{p,w}(X))^* = 0$. □

Definition 2.10. Let $U$ is strictly square integrable mod $(P, \sigma)$, then for $\psi \in H$

\[ V \psi : H \to L_2(X) \]

\[ V \psi f(h) := < f, U(\sigma(h)^{-1})\psi >_H \]

\[ H_{1,w} := \{ f \in H : V \psi f \in L_{1,w}(X) \}. \]

Corollary 2.11. $H_{1,w}$ is dense in $H$.

Proof. We refer readers to [3, Lemma 3.1]. □

Theorem 2.12. Let $\Lambda : H_{1,w} \to L_{p,w}$ is continuous(relative to the topology that $H_{1,w}$ inherits from $H$)and linear, then $\Lambda$ has a continuous linear extension $\tilde{\Lambda}$ so that $\tilde{\Lambda} : H \to L_{p,w}$.

Proof. $H$ is a Hilbert space, hence, it is topological vector space. Now suppose that $V_n$ be balanced neighborhoods of 0 in $H$ such that $V_n + V_n \subset V_{n-1}$. Now $\Lambda : H_{1,w} \to L_{p,w}$ is continuous, then by the use of continuity definition we know that $\Lambda$ is continuous in 0. Hence for $\varepsilon = 2^{-n}$ there exists $V_n \cap H_{1,w}$ so that

\[ \forall x \in V_n \cap H_{1,w} : d(0, \Lambda x) < \varepsilon = 2^{-n} \]

It is remarkable that $V_n \cap H_{1,w}$ is neighborhoods of 0 relative to the topology that $H_{1,w}$ inherits from $H$. By corollary 2.11, $H_{1,w}$ is dense in $H$. Then we have two choices for all $x \in H$

1. $x \in H_{1,w}$. In this form for all $n$ we define $x_n = x$ and

\[ \tilde{\Lambda}x = \lim_{n \to \infty} \Lambda x_n = \Lambda x \]

2. $x$ is a limit point for $H_{1,w}$. Then

\[ (x + V_n) \cap H_{1,w} \neq \emptyset \implies x_n \in (x + V_n) \cap H_{1,w}. \]

We intend to show $\{\Lambda x_n\}$ is Cauchy sequence in $L_{p,w}$. $d$ is invariant so

\[ d(0, \Lambda(x_n - x)) < 2^{-n} \implies d(\Lambda x, \Lambda x_n) < 2^{-n} \]

Hence

\[ d(\Lambda x_n, \Lambda x_m) \leq d(\Lambda x_n, \Lambda x) + d(\Lambda x, \Lambda x_m) \leq 2^{-n} + 2^{-m} \]
For $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \varepsilon$. If $n, m \geq k$, then we have

$$2^n, 2^m \geq 2^k \implies d(\Lambda x_n, \Lambda x_m) \leq \frac{1}{2^n} + \frac{1}{2^m} \leq 2\varepsilon.$$

We know that $L_{p,w}$ is F-space and $d$ is invariant and complete, then limit for \{\Lambda x_n\} exists. If we show this limit with $\tilde{\Lambda}x$, it will be well-defined, linear and continuous.

**Theorem 2.13.** Let $\mu$ is positive, unimodular and $\sigma$-finite measure with $\sigma$-algebra $\Sigma$. If $w$ be weight function such that $\int_X w(\sigma(h))d\mu(h) \leq 1$, then there will exist $S(x) \in L_{1,w}$ so that $0 < S(x) < 1$.

**Proof.** $\mu$ is $\sigma$-finite, then there exists $\{E_i\}_{i \in I} \subseteq \Sigma$ such that $X = \bigcup_{i=1}^\infty E_i$ and $\mu(E_i) < \infty$.

Now, we define $S_i(x) = \frac{2^{-i}}{(1 + \mu(E_i))(1 + w(\sigma(x)))}$ for $x \in E_i$ and $S_i(x) = 0$ otherwise. We define $S(x) = \sum_{n=1}^\infty S_n(x)$. Then

$$0 < S(x) = \sum_{n=1}^\infty S_n(x) \leq \sum_{n=1}^\infty \frac{2^{-n}}{(1 + \mu(E_n))(1 + w(\sigma(x)))} \leq 1$$

If $x \in X$ there exists $i \in I$ such that $x \in E_i$, then $S(x) > 0$. Furthermore, there exists $n$ such that $\mu(E_n) \neq 0$ and

$$\sum_{n=1}^\infty \frac{2^{-n}}{(1 + \mu(E_n))(1 + w(\sigma(x)))} < \sum_{n=1}^\infty 2^{-n} < 1$$

So, $0 < S < 1$. Now we should show $S \in L_{1,w}$. Hence

$$\int_X S(x)w(\sigma(x))d\mu(x) = \int_X \left(\sum_{n=1}^\infty S_n(x)\right)w(\sigma(x))d\mu(x) = \sum_{n=1}^\infty \int_X S_n(x)w(\sigma(x))d\mu(x) = \sum_{n=1}^\infty \sum_{i=1}^\infty \int_{E_i} S_i(x)w(\sigma(x))d\mu(x) = \sum_{n=1}^\infty \int_{E_n} S_n(x)w(\sigma(x))d\mu(x) = \sum_{n=1}^\infty \frac{2^{-n}}{(1 + \mu(E_n))(1 + w(\sigma(x)))} \int_{E_n} w(\sigma(x))d\mu(x) \leq \sum_{n=1}^\infty \frac{2^{-n}}{1 + \mu(E_n)} \int_{E_n} w(\sigma(x))d\mu(x) \leq \sum_{n=1}^\infty 2^{-n} = 1$$

Then $S(x) \in L_{1,w}$.  \qed
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