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Abstract. Using Dotson’s convexity structure, the authors in [16, 17, 18] established some deterministic and random common fixed point results. In this note, we comment that the proofs of the results in [16, 17, 18] are incomplete and incorrect.

1. Introduction and preliminaries

Let $X$ be a linear space. A $p$-norm on $X$ is a real-valued function ($0 < p \leq 1$), satisfying the following conditions:

(i) $\|x\|_p \geq 0$ and $\|x\|_p = 0 \iff x = 0$

(ii) $\|\alpha x\|_p = |\alpha|^p \|x\|_p$

(iii) $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

for all $x, y \in X$ and all scalars $\alpha$. The pair $(X, \|\cdot\|_p)$ is called a $p$-normed space. It is a metric linear space with a translation invariant metric $d_p$ defined by $d_p(x, y) = \|x - y\|_p$ for all $x, y \in X$. If $p = 1$, we obtain the concept of a normed space. It is well-known that the topology of every Hausdorff locally bounded
topological linear space is given by some $p$-norm, $0 < p \leq 1$(see [7, 14, 19]).

Let $X$ be a metric linear space and $M$ a nonempty subset of $X$. Let $I : M \rightarrow X$ be a mapping. A mapping $T : M \rightarrow X$ is called $I$-Lipschitz if there exists $k \geq 0$ such that $d(Tx, Ty) \leq kd(Ix, Iy)$ for any $x, y \in M$. If $k < 1$ (respectively, $k = 1$), then $T$ is called $I$-contraction (respectively, $I$-nonexpansive). The map $T : M \rightarrow X$ is said to be completely continuous if $\{x_n\}$ converges weakly to $x$ implies that $\{Tx_n\}$ converges strongly to $Tx$. The map $T : M \rightarrow X$ is demiclosed at 0 if for every sequence $\{x_n\}$ in $M$ converging weakly to $x$ and $\{Tx_n\}$ convergent strongly to 0, we have $Tx = 0$. The set of best approximations of $u \in X$ from $M$ is defined by $P_M(u) = \{x \in M : d(x, u) = \operatorname{dist}(u, M) = \inf_{y \in M} d(u, y)\}$. The set of fixed points of $T$( resp. $I$) is denoted by $F(T)$(resp. $F(I)$). A point $x \in M$ is a common fixed (coincidence) point of $I$ and $T$ if $x = Ix = Tx$ ($Ix = Tx$). The set of coincidence points of $I$ and $T$ is denoted by $C(I, T)$. Two selfmaps $I$ and $T$ of $M$ are called:

1. commuting if $ITx = TIx$ for all $x \in M$;
2. $R$-weakly commuting if for all $x \in M$ there exists $R > 0$ such that $d(ITx, TIx) \leq Rd(Ix, Tx)$;
3. compatible if $\lim_n d(TIx_n, ITx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim_n Tx_n = \lim_n Ix_n = t$ for some $t \in M$;
4. weakly compatible if they commute at their coincidence points, i.e. $ITx = TIx$ whenever $Ix = Tx$.

The set $M$ is called $q$-starshaped with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \leq k \leq 1\}$ joining $q$ to $x$, is contained in $M$ for all $x \in M$. Suppose $M$ is $q$-starshaped with $q \in F(I)$ and is both $T$- and $I$-invariant in a $p$-normed space $X$. Then $T$ and $I$ are called:

5. $R$-subcommuting with $q \in F(I)$ and is both $T$- and $I$-invariant in a $p$-normed space $X$. Then $T$ and $I$ are called:
6. $R$-subweakly commuting on $M$ if there exists a real number $R > 0$ such that $\|ITx - TIx\|_p \leq R \|kTx + (1-k)q - Ix\|_p$ for all $x \in M$, $k \in (0, 1)$. If $R = 1$, then the maps are called 1-subcommuting;
7. $R$-commuting if $ITx = TIx$ for all $x \in C_q(I, T)$, where $C_q(I, T) = \bigcup\{C(I, T_k) : 0 \leq k \leq 1\}$ and $T_kx = (1-k)q + kTx$.

Clearly, commuting maps are $R$-subweakly commuting, $R$-subweakly commuting maps are $R$-subcommuting and $R$-subcommuting maps are $C_q$-commuting but the converse, in each case, does not hold in general (see [8, 11] and references therein).

Following important extension of the concept of starshapedness was defined by Dotson [4] and has been studied by many authors (see [2]-[7],[9]-[18],[20]).

**Definition 1.1.** (Dotson’s convexity). Let $M$ be subset of a $p$-normed space $X$ and $F = \{f_x\}_{x \in M}$ a family of functions from $[0, 1]$ into $M$ such that $f_x(1) = x$ for each $x \in M$. The family $F$ is said to be contractive [4, 5, 12, 14] if there
exists a function \( \phi : (0, 1) \rightarrow (0, 1) \) such that for all \( x, y \in M \) and all \( t \in (0, 1) \), we have \( \| f_x(t) - f_y(t) \|_p \leq [\phi(t)]^p \| x - y \|_p \). The family \( F \) is said to be jointly (weakly) continuous if \( t \rightarrow t_0 \) in \([0,1] \) and \( x \rightarrow x_0 \) (\( x \rightarrow x_0 \) weakly) in \( M \), then \( f_x(t) \rightarrow f_{x_0}(t_0) \) (\( f_x(t) \rightarrow f_{x_0}(t_0) \) weakly) in \( M \). We observe that if \( M \subset X \) is \( q \)-starshaped and \( f_x(t) = (1-t)q + tx, \) then \( f \in M \) for each \( t \in [0,1] \), then \( F = \{ f_x \}_{x \in M} \) is a contractive jointly continuous and jointly weakly continuous family with \( \phi(t) = t \). Thus the class of subsets of \( X \) with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets (see [3, 4, 6, 12, 14]).

2. Main Results

In the papers [16, 17] under consideration, the author defines the so called \((S)\)-convex structure for a linear space \( X \) which is absurd as starshaped sets and hence linear spaces satisfy the so called \((S)\)-convex structure. Therefore, we always define convex and starshaped structure on a nonempty subset \( M \) of \( X \). Thus Definition 1 in [15], Definition 2.7 in [16] and Definition 2.3 in [17] should be modified in the context of a nonempty subset of a linear space \( X \) (see definition 1.1 above). Condition (iv) of the definition has no meanings and should be deleted and in Condition (v) the function \( \phi \) should be from \((0,1) \rightarrow (0,1)\). Similarly, Definition 2.8 [16] should be modified as follows (see [4, 6, 12, 14]):

Let \( T \) be a selfmap of the set \( M \) having a family of functions \( F = \{ f_x \}_{x \in M} \) as defined above. Then \( T \) is said to satisfy the property \((A)\), if \( T(f_x(t)) = f_{Tx}(t) \) for all \( x \in M \) and \( t \in [0,1] \).

Example 2.1. An affine map \( T \) defined on \( q \)-starshaped set with \( Tq = q \) satisfies the property \((A)\). For this note that each \( q \)-starshaped set \( M \) has a contractive jointly continuous family of functions \( F = \{ f_x \}_{x \in M} \) defined by \( f_x(t) = tx + (1-t)q \), for each \( x \in M \) and \( t \in [0,1] \). Thus \( f_x(1) = x \) for all \( x \in M \). Also, if the selfmap \( T \) of \( M \) is affine and \( Tq = q \), we have \( T(f_x(t)) = T(tx + (1-t)q) = tTx + (1-t)q = f_{Tx}(t) \) for all \( x \in M \) and all \( t \in [0,1] \). Thus \( T \) satisfies the property \((A)\); a property considered first time in 2000, by Khan, the author and Thaheem (see [12], Theorems 3.7, 3.10, 3.12). This signifies that \((S)\)-convex structure should be introduced on a nonempty subset \( M \) of a linear space \( X \).

Here is the main result of Nashine [16].

Theorem 2.2. Let \( X \) be a \( p \)-normed space with a \((S)\)-convex structure. Let \( I, D : X \rightarrow X, C \) a subset of \( X \) such that \( T(\partial C) \subset C \) and \( u \in F(T) \cap F(I) \). Suppose that \( D = \overline{P_M(u)} \) and \( T \) is \( I \)-nonexpansive on \( D \cup u \), \( I \) satisfies property \((A)\), \( I \) is continuous, \( TI = IT \) on \( D \), \( cl(T(D)) \) is compact on \( D \). Also assume, range of \( f_0 \) is contained in \( I(D) \). If \( D \) is nonempty, closed and if \( I(D) \subset D \), then \( D \cap F(I) \cap F(T) \neq \emptyset \).

My comments to Theorem 2.2 are as follows:
(a) The condition “range of $f_\alpha$ is contained in $I(D)$” makes the result trivial. As a matter of fact take $f_\alpha(t) = t\alpha$ for each $\alpha \in X$ and $t \in [0,1]$; now $X$ is a linear space with zero element so $\{f_\alpha\}$ is a $(S)$-convex structure with range of $f_\alpha$ equal to $X$. Thus $X \subseteq I(D) \subseteq D \subseteq X$.

(b) The $(S)$-convex structure is not a hereditary property so the set $D$ here is without any convexity structure and hence the statement in the proof of this theorem “$T_n$ is a well-defined map from $D$ into $D$ for each $n$” makes no sense; it is worth mentioning that the entire proof depends on this important fact. Same concerns the proof of Theorem 2 in \[15\].

(c) The statement in the proof of Theorem 2.2, “Since $\text{cl}(T(D))$ is compact, each $\text{cl}(T_n(D))$ is compact” needs to be verified which is crucial for the application of Theorem 2.9 stated in \[16\]. Actually, when $D$ is $q$-starshaped, it has $(S)$-convex structure $f_x(t) = tx + (1-t)q$, for each $x \in D$ and $t \in [0,1]$. Further, if $T_n x = (1-k_n)q + k_n T x$ for all $x \in D$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1, then $\text{cl}(T_n(M))$ is compact for each $n$ provided $\text{cl}(T(D))$ is compact.

The second and last result in \[16\] is the following:

**Theorem 2.3.** Let $X$ be a complete $p$-normed space whose dual separates the points of $X$ with a $(S)$-convex structure. Let $T$, $I : X \to X$, $C$ a subset of $X$ such that $T(\partial C) \subset C$ and $u \in F(T) \cap F(I)$. Suppose that $T$ is $I$-nonexpansive on $D \cup u$, $I$ satisfies property $(A)$, $I$ is weakly continuous, $TI = IT$ on $D$. Also assume that range of $f_\alpha$ is contained in $I(D)$. If $D$ is nonempty, weakly compact and if $I(D) \subset D$, then $D \cap F(I) \cap F(T) \neq \emptyset$.

The above comments (a) and (b) apply to Theorem 2.3 as well.

(d) The author has utilized Theorem 3.2 (stated in \[16\]) in the proof of Theorem 2.3 (see p.56, line 15) which holds for a compact metric space whereas the underlying set $D$ here is assumed to be weakly compact and $I$ is not continuous as well.

(e) The author seems to claim in equality (3.1) that $y_m \to 0$ which can not be true unless $Tx_m \to Ty$ which is impossible under the assumed hypotheses. If we assume that $T$ is completely continuous to assure $Tx_m \to Ty$, then the condition “$I - T$ is demiclosed” becomes superfluous and we directly get the conclusion(see \[5,6,10,12,14\]). Thus the proof of Theorem 2.3 is incomplete and incorrect. Consequently, Remark 3.5–Remark 3.9 in \[16\] are invalid.

(f) For more general and comprehensive results for noncommuting maps namely, $R$-subweakly commuting, $R$-subcommuting and $C_q$-commuting maps defined on the set $M$ satisfying the Dotson's convexity condition (or the so called $(S)$-convex structure), we refer the reader to \[5,6,10,11\].
Comments on the results in [17]

(g) The author defines in the proofs of Theorems 3.1 and 3.2 in [17]: $T_n : \Omega \times P_M(x_0) \rightarrow P_M(x_0)$ by $T_n(\omega, x) = f_{T(\omega,x)}(k_n)$ and claims that each $T_n$ is a random operator without proving the measurability of $T_n$. The measurability of $T_n$ is still an open problem (see [2, 13] and references therein). Thus all the results, Theorems 3.1-3.3 in [17], are deterministic in nature and hence are simple corollaries to more general results in [5, 6, 10, 11].

(h) The author has utilized Lemma 2.5 (stated in [17]) in the proof of his Theorem 3.2 (see p.67, line 29) which holds for a compact metric space whereas the underlying set $P_M(x_0)$ here is assumed to be weakly compact and $g$ is not continuous as well.

(i) The author seems to claim in lines 7 to 12 on page 68, that $y_m \rightarrow 0$ strongly which can not be true unless $T(\omega, \xi_m(\omega)) \rightarrow T(\omega, \xi(\omega))$. This is impossible as $T$ is not assumed to have any type of continuity. Thus the proof of Theorem 3.2 is incomplete and incorrect. Consequently, Remark 3.5–Remark 3.7 in [17] are invalid.

Comments on the results in [18]

The proofs of all the results in [18] depends on the following statement:

If the maps $I$ and $T$ are compatible, then $I$ and $T_n$ are also compatible for each $n \geq 1$ where $T_n(x) = (1 - k_n)q + k_nTx$ for fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1.

Here we give an example to show that the above statement is not correct.

Example 2.4. Let $X = \mathbb{R}$ with usual norm and $M = [1, \infty)$. Let $I(x) = 2x - 1$ and $T(x) = x^2$, for all $x \in M$. Let $q = 1$. Then $M$ is $q$-starshaped with $IQ = q$. Note that $I$ and $T$ are compatible. Further $C(I, T_\frac{1}{3}) = \{1, 2\}$ and $IT_\frac{1}{3}(2) \neq T_\frac{1}{3}I(2)$, which implies that $I$ and $T_\frac{1}{3}$ are not weakly compatible. Thus $I$ and $T_\frac{1}{3}$ are not compatible maps. Consequently, all the results proved in [18] are incorrect.

The results in [18] can be corrected if the compatibility of $I$ and $T$ is replaced by the condition of subcompatibility (see [1]).

References


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