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CARTESIAN PRODUCTS OF PQP-M-SPACES

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Abstract. In this paper we define the concept of finite and countable Cartesian products of PqpM-spaces and give a number of its properties. We also study the properties of topologies of those products.

Introduction

Let \((X, P_1)\) and \((Y, P_2)\) are PM-spaces under triangle function \(\ast\) and a pair \((X \times Y, P_1 \times P_2)\) is a finite product of PM-spaces (see Tardiff [11], Urazov [12]), when the function \(P_1 \times P_2 : (X \times Y)^2 \to \Delta^+\) is given by formula:

\[
P_1 \times P_2 (u, v) = P_1 (x_1, y_1) \ast P_2 (x_2, y_2)
\]

for any \(u = (x_1, y_1)\) and \(v = (x_2, y_2)\) in \(X \times Y\). Convolution of Wald space [13], as well as several types of products of PM-spaces, where first defined by Istrâțescu and Vadura [4]. If \(T\) is a \(t\)-norm and \(\ast = \ast_T\), then \(\ast\)-product is the \(T\)-product on defined independently by Egbert [1] and Xavier [14]. It is immediat that \(\ast\)-product of PM spaces is PM-space (see (Sherwood, Taylor [9]), (Höle [3]). In section 1 we extended this notion and results of \(T\)-product of PqpM-spaces. In section 2 we give definition and some results on countable products of PqpM-spaces of type \(\{k_n\}\).

0. Preliminary notes and results

Definition 0.1 ([8]). A distance distribution function is a nondecreasing function \(F : (\mathbb{R}, +, \infty) \to [0, 1]\) which is left-continuous on \((\mathbb{R}, +, \infty)\) and \(F(0) = 0\).
and \( \lim_{x \to \infty} F(x) = 1 \). We denote by \( \Delta^+ \) the set of all distribution functions and by \( \varepsilon_a \) specific distribution function by
\[
\varepsilon_a(t) = \begin{cases} 
1, & \text{for } t > a, \\
0, & \text{for } t \leq a, \ a \in \mathbb{R}.
\end{cases}
\]
The element of \( \Delta^+ \) are partially ordered by
\[
F \leq G \text{ if and only if } F(x) \leq G(x), \text{ for } x \in \mathbb{R}.
\]
For any \( F, G \in \Delta^+ \) and \( h \in (0, 1] \), let \( (F, G, h) \) denote the condition
\[
G(x) = F(x + h) + h \quad \text{for all } x \in (0, h^{-1})
\]
and
\[
d_L(F, G) = \inf\{h : \text{both } (F, G; h) \text{ and } (G, F; h) \text{ hold}\}.
\]
As shown by Sibley [10] the function \( d_L \) is a metric in \( \Delta^+ \) which is a modified form on the well-known Levy metric for distribution functions and the metric space \( (\Delta^+, d_L) \) is compact and hence complete (see [8, pp. 45-49]).

**Definition 0.2** ([8,15]). A binary operation \( \ast : \Delta^+ \times \Delta^+ \to \Delta^+ \) is a triangle function if \( (\Delta^+, \ast) \) is an Abelian monoid with identity \( \varepsilon_0 \) in \( \Delta^+ \) such that, for any \( F, F', G, G' \in \Delta^+ \),
\[
F \ast G \leq F' \ast G' \quad \text{whenever } F \leq F', \ G \leq G'.
\]

Note that a triangle function \( \ast \) is continuous if it is continuous with respect to the metric topology induced by \( d_L \).

Let \( T(\Delta^+) \) denote the family of all triangle functions \( \ast \) then the relation \( \leq \) defined by
\[
\ast_1 \leq \ast_2 \iff F \ast_1 G \leq F \ast_2 G, \quad \text{for all } F, G \in \Delta^+
\]
is a partial order in the family \( T(\Delta^+) \).

The second relation in the set \( T(\Delta^+) \) \( \gg \) is defined by
\[
\ast_1 \gg \ast_2 \iff ((E \ast_2 G) \ast_1 (F \ast_2 H)) \geq ((E \ast_1 F) \ast_2 (G \ast_1 H)),
\]
for all \( E, F, G, H \in \Delta^+ \).

We can see the connection between the two relation: \( \ast_1 \gg \ast_2 \) implies \( \ast_1 \geq \ast_2 \) and following conditions: \( \text{min} \geq \ast \) and \( \text{min} \gg \ast \), for all \( \ast \).

**Definition 0.3** ([2]). A probabilistic-quasi-pseudo-metric-space (briefly, a Pqp-metric space) is a triple \((X, P, \ast)\), where \( X \) is a nonempty set, \( P \) is a function from \( X \times X \) into \( \Delta^+ \), \( \ast \) is a triangle function and the following conditions are satisfied (the value of \( P \) at \((x, y)\) in \( X^2 \) will be denoted by \( P_{xy} \)):
\[
P_{xx} = u_0, \quad \text{for all } x \in X,
\]
\[
P_{xy} \ast P_{yz} \leq P_{xz}, \quad \text{for all } x, y, z \in X.
\]
If \( P \) satisfies also the additional condition
\[
P_{xy} \neq \varepsilon_0 \quad \text{it } x \neq y,
\]
then \((X, P, \ast)\) is called a probabilistic quasi-metric space.
Moreover, if $P$ satisfies the condition of symmetry:

$$P_{xy} = P_{yx},$$

then $(X, P, \ast)$ is called a probabilistic metric space (PM-space).

If the function $Q : X^2 \to \Delta^+$ be defined by

$$Q_{xy} = P_{yx}, \text{ for all } x, y \in X,$$

then a triple $(X, Q, \ast)$ is also a probabilistic-quasi-pseudo-metric space. We say $P$ and $Q$ are conjugate each other.

**Lemma 0.4.** Let $(X, P, Q, \ast)$ be a structure defined by $P_{xy}$-metric $P$ and $\ast_1 \gg \ast_2$ (0.2.2). Then $(X, F_{\ast_1}, \ast)$ is a probabilistic pseudo-metric space whenever the function $F_{\ast_1} : X^2 \to \Delta^+$ is given by:

$$F_{\ast_1} = P_{xy} \ast_1 Q_{xy}, \text{ for all } x, y \in X_0.$$ (0.4.1)

If additionally, $P$ satisfies the condition $P_{xy} \neq u_0$ or $Q_{xy} \neq u_0$ for $x \neq y$, (0.4.2) then $(X, F_{\ast_1}, \ast)$ is a PM-space.

**Lemma 0.5** ([2, Example 9]). If $(X, p)$ is a quasi-pseudometric-space and the function $P_p : X^2 \to \Delta^+$ is defined by

$$P_p(x, y) = \varepsilon_p(x, y), \text{ for all } x, y \in X$$

and $\ast$ is a triangle function such that

$$\varepsilon_a \ast \varepsilon_b \geq \varepsilon_{a+b} \text{ for all } a, b \in R^+,$$

then $(X, P_p, \ast)$ is a proper $P_{qp}$-metric space.

**Theorem 0.6** ([2, Theorem 6]). Let $(X, P, \ast)$ be a $P_{qp}$-metric space under a uniformly continuous $t$-function $\ast$ and, for any $x \in X$, and $t > 0$, the $P$-neighborhood of $x$ be a set

$$N_x^P(t) = \{y \in X : d_P(P_{xy}, u_0) < t\}.$$ 

Then the collection of all $P$-neighborhood form a base for the topology $\tau_P$ on $X$ the $P_{qp}$-metric $Q$ which is a conjugate of $P$ generate a topology $\tau_Q$ on $X$. Thus natural structure associated with a $P_{qp}$-metric is a bitopological space $(X, \tau_P, \tau_Q)$.

It is worthy of note that in the spaces $(X, P_p, Q_q, \ast)$, the $\tau_{P_p}$-topology is equivalent to the $q$-quasi-pseudometric topology $\tau_{P_q}$ (see [2], [11]).

**Lemma 0.7.** Let $(X, P, \ast)$ be a $P_{qp}$-space. Then the relation $\leq_P$ defined by

$$x \leq_P y \text{ if and only if } P_{xy} = \varepsilon$$

is reflexive and transitive, i.e. it is a quasi-order on $X$.

**Proof.** Reflexivity follows immediately from (0.2.1) and transivity is a consequence of (0.3.2).

**Corollary 0.8.** If $P_{qp}$-metric satisfies the assumption (0.4.2), then the relation $\leq_P$ is a partial ordering on $X$. 

Proof. Assume that \( x \leq_P y \) and \( y \leq_P x \). This means that
\[
P_{xy} = \varepsilon_0 \quad \text{and} \quad P_{yx} = u.
\]
By (0.4.2), it follows that \( P_{xy} = P_{yx} = \varepsilon_0 \) if and only if \( x = y \).

**Corollary 0.9.** If \( x \neq y \) imply \( P_{xy} = \varepsilon_0 \) and \( P_{yx} \neq \varepsilon_0 \) or \( P_{xy} \neq \varepsilon_0 \) and \( P_{yx} = \varepsilon_0 \), then \( \leq_P \) is a linear ordering on \( X \).

**Remark 0.10.** If \( Q \) is a conjugate of a \( Pqp \)-metric \( P \), then the relation \( \leq_Q \) generated by \( Q \) is also a quasi-ordering on \( X \) and is the inverse relation of \( \leq_P \).

## 1. Cartesian products of \( PqpM \)-spaces

In this section, we give some properties of Cartesian products of \( PqpM \)-spaces.

**Definition 1.1.** Let \( (X, P_1, *) \) and \( (Y, P_2, *) \) be \( PqpM \)-spaces. The \( * \)-product of \( (X, P_1) \) and \( (Y, P_2) \) is the pair \( (X \times Y, P_1 \times P_2) \), where \( P_1 \times P_2 \) is the function from \( (X \times Y)^2 \) into \( \Delta^+ \) given by
\[
P_1 \times P_2 (u, v) = P_1(x_1, y_1) * P_2(x_2, y_2)
\]
for any \( u = (x_1, y_1) \) and \( v = (x_2, y_2) \) in \( X \times Y \).

**Theorem 1.2.** Let \( (X, P_1, *) \) and \( (X, P_2, *) \) be \( PqpM \)-spaces. Let a mapping \( P_1 \times P_2 : (X \times Y)^2 \rightarrow \Delta^+ \) be given by
\[
P_1 \times P_2 (u, v) = (P_1(x_1, x_2) * P_2(y_1, y_2)) \quad \text{with} \quad *_1 \succ *
\]
for any \( u = (x_1, y_1), \ v = (x_2, y_2) \in X \times Y \). Then \( (X \times Y, P_1 \times P_2, *) \) is a \( PqpM \)-space.

**Proof.** If \( u = v \), then \( x_1 = x_2 \) and \( y_1 = y_2 \). Thus, by (0.6.1), we have
\[
P_1 \times P_2 (u, u) = P_1(x_1, x_1) * P_2(y_1, y_1) = u_0 *_1 u_0 = u_0.
\]
Now, let \( w = (x_3, y_3) \) \( X \times Y \). Then, by (0.3.2) and (0.2.2), we obtain
\[
P_1 \times P_2 (u, v) = P_1(x_1, x_2) * P_2(y_1, y_2)
\]
\[
\geq (P_1(x_1, x_3) * P_1(x_3, x_2)) *_1 (P_2(y_1, y_3) * P_2(y_3, y_2))
\]
\[
\geq (P_1(x_1, x_3) * P_2(y_1, y_3)) * (P_1(x_3, x_2) * P_2(y_3, y_2))
\]
\[
= P_1 \times P_2 (u, v) * P_1 \times P_2 (w, v).
\]
This completes the proof.

**Definition 1.3.** Let \( (X, P_1, *) \) and \( (X, P_2, *) \) be \( PqpM \)-spaces and let \( *_1 \succ * \). Then \( (X \times Y, P_1 \times P_2, *_1) \) is called a *Cartesian \(*_1\)-product* of \( PqpM \)-spaces provided that \( P_1 \times P_2 \) is given by the formula (1.1).

By Definition 0.2 and (0.2.2), it follows that \( \text{Min} \gg * \) holds true for any \( t_{\Delta^+} \)-norm \( * \). Thus it follows that the function given by
\[
P_1 \times P_2 (u, v) = \text{Min} (P_1(x_1, x_2), P_2(y_1, y_2))
\]
\[
= P_1(x_1, x_2) \times P_2(y_1, y_2)
\]
satisfies the conditions of Theorem 1.2.
Theorem 1.4. Let \((X, G_{p_1, *_M})\) and \((Y, G_{p_2, *_M})\) be a \(Pqpm\)-space defined by formula \(G_p(x, y) = G\left(\frac{1}{p(x,y)}\right)\), where \(G \in \Delta^+\) be distinct from \(\varepsilon_0\) and \(\varepsilon_\infty\), which were generated by quasi-pseudo-metric \(p_1\) and \(p_2\), respectively. Let a function \(p_1 \vee p_2 : (X \times Y)^2 \to \mathbb{R}^+\) be defined by

\[
p_1 \vee p_2 (u, v) = \max(p_1(x_1, x_2), p_2(y_1, y_2)),
\]

where \(u = (x_1, x_2)\) and \(v = (y_1, y_2)\) belong to \(X \times Y\). Then the triple \((X \times Y, G_{p_1 \vee p_2, *_M})\) is a \(Pqpm\)-space generated by a quasi-pseudo-metric \(p_1 \vee p_2\).

As a consequence of Theorem 1.2, we have the following:

Corollary 1.5. Let \((X, P, *)\) be a \(Pqpm\)-space. Let \(*_1\) = \(\text{Min}\). Then there are four \(Pqpm\)-metrics on \(X \times X\) generated by the function \(P\), that is, \(P \times P, P \times Q, Q \times P\) and \(Q \times Q\), where \(Q\) is the \(Pqpm\)-metric conjugate with \(P\).

Remark 1.6. Note that, by Definition 1.1, for all \(u, v \in X \times X\), the following equalities hold:

\[
P \times P(u, v) = Q \times Q(u, v), \quad P \times Q(u, v) = Q \times P(u, v).
\]

Therefore, the pairs \(P \times P\) and \(Q \times Q\) as well as \(P \times Q\) and \(Q \times P\) are the mutually conjugate \(Pqpm\)-metrics defined on \(X \times X\). The function \(M(X \times Y)^2 \to \Delta^+\) given by

\[
M(u, v) = P \times Q(u, v) \land Q \times P(u, v) = P \times P(u, v) \land Q \times Q(u, v)
\]

for all \((u, v) \in X \times X\) is a probabilistic pseudo-metric on \(X \times X\).

Corollary 1.7. Let \((X, P, \text{*})\) be a \(Pqpm\)-space. Let the \(t_{\Delta^+}\)-norm \(*\) be continuous at \((\varepsilon_0, \varepsilon_0)\) and \(*_1 = \text{Min}\). Then the topology \(T_{P \times P}\) generated by the function \(P \times P\) is equivalent to the topology \(T_P \times T_P\). Also, the topologies \(T_P \times T_Q, T_Q \times T_P, T_P \times T_Q, T_Q \times T_Q\) are equivalent.

Proof. For an illustration, we prove the first equivalence. Let \(t_1, t_2 > 0\) and \(x, y \in X\). Then we have

\[
N_{x}^P(t_1) \times N_{y}^P(t_2) \in T_P \times T_P.
\]

Let \(t_3 = \max(t_1, t_2)\) and \(u = (x, y) \in X \times X\). Then a \(P \times P\)-neighbourhood of a point \(u \in X \times X\) is of the form:

\[
N_{u}^{P \times P}(t_3) = \{v = (x_1, x_2) : P \times P(u, v)(t_3) > 1 - t_3\}
\]

\[
= \{v = (x_1, x_2) : P_{xx}(t_3) > 1 - t_3 \text{ and } P_{yy}(t_3) > 1 - t_3\},
\]

\[
N_{x}^{P}(t_3) \times N_{y}^{P}(t_3) \subset N_{x}^{P}(t_1) \times N_{y}^{P}(t_2).
\]

On the other hand, for each \(t > 0\) and \(u = (x, y) \in X \times X\), we have

\[
N_{u}^{P \times P}(t) = N_{x}^{P}(t) \times N_{y}^{P}(t).
\]

The remaining cases can be verified similarly. This completes the proof.

Theorem 1.8. Let \((X, P, \text{*})\) be a \(Pqpm\)-space. Assume that the \(t_{\Delta^+}\)-norm \(*\) is continuous and let \(\leq_P\) be the quasi-order generated by \(P\) (in the sense of Lemma 0.7). Then the set \(G(\leq_P) = \{(x, y) \in X^2 : x \leq_P y\}\) is closed in the topology \(T_{P \times P}\).
Proof. Assume that \((x_1, y_1)\) belongs to the \(P \times Q\)-closure of \(G(\leq_P)\) and does not belong to \(G(\leq_P)\). Then, by Corollary 0.9, \(P_{x_1y_1} \neq \varepsilon_0\) and there exists a sequence \(\{(x_n, y_n)\}\) of \(G(\leq_P)\) which is \(P \times Q\)-convergent to \((x_1, y_1)\). This means that

\[
P_{x_1x_n} \rightarrow \varepsilon_0, \quad Q_{y_1y_n} \rightarrow \varepsilon_0.
\]

Thus, by (0.3.2), we have

\[
\varepsilon_0 \neq P_{x_1y_1} \geq P_{x_1x_n} * P_{x_ny_1} \geq P_{x_1x_n} * P_{x_ny_n} * P_{y_ny_1} = P_{x_1x_n} * P_{x_ny_n} * Q_{y_1y_n} = P_{x_1x_n} * \varepsilon_0 * Q_{y_1y_n} = P_{x_1x_n} * Q_{y_1y_n} \rightarrow \varepsilon_0,
\]

which is a contradiction. This completes the proof.

Lemma 1.9. Let \((X, P, *)\) be a \(PqpM\)-space satisfying the condition (0.3.4), and let the \(t_{\Delta^*}\)-norm \(*\) be continuous. Then the set \((\leftarrow, x) = \{y \in X \mid y \leq_P x\}\), where \(\leq_P\) is the order generated by \(P\), is a subset of \(N^Q_x(t)\) for every \(t > 0\).

Proof. If \(y \in (\leftarrow, x)\), then \(y \leq_P x\) and so, by (0.3.4), we have

\[
P_{yx} = Q_{xy} = \varepsilon_0.
\]

Therefore, we have \(y \in N^Q_x(t)\) for every \(t > 0\).

Corollary 1.10. The set \((\leftarrow, x)\) is \(G_\delta\) in the topology \(T_Q\).

Proof. For \(t > 0\), there is a natural number \(n\) such that \(\frac{1}{n} < t\). Then we have

\[
Q_{xy}(t) \geq Q_{xy}(\frac{1}{n}) > 1 - \frac{1}{n} > 1 - t,
\]

which means that

\[
N^Q_x(\frac{1}{n}) \subset N^Q_x(t).
\]

Therefore, we conclude that the family \(\{N^Q_x(\frac{1}{n})\}_{n \in \mathbb{N}}\) satisfies the assertion. This completes the proof.

Lemma 1.11. The set \((\leftarrow, x)\) is \(P\)-closed.

Proof. Assume that \(y\) belongs to the \(P\)-closure of \((\leftarrow, x)\) and \(y \notin (\leftarrow, x)\). Then \(P_{yx} \neq \varepsilon_0\) and, for each \(n \in \mathbb{N}\), there is \(x_n \in (\leftarrow, x)\) such that

\[
P_{yx_n} \rightarrow \varepsilon_0.
\]

Finally, we have

\[
\varepsilon_0 \neq P_{yx} \geq P_{yx_n} * P_{x_ny} = P_{yx_n} * \varepsilon_0 = P_{yx_n} \rightarrow \varepsilon_0,
\]

which is a contradiction. This completes the proof.

Corollary 1.12. The set \([x, \rightarrow] = \{y \in X \mid x \leq_P y\}\) is a \(Q\)-closed and \(G_\delta\) in the topology \(T_P\).

The following result is an immediate consequence of Lemma 1.2:
Theorem 1.13. Let \((X, P, *)\) be a \(PqpM\)-space satisfying the condition of Corollary 0.9 and let the \(t_{\Delta^+}\)-norm \(*\) be continuous. Then the family \(\{([-, x])_{x \in X}\) forms a \(P\)-closed subbase of a topology, which is denoted by \(T([\leftarrow])\). Similarly, the family \(\{[x, \rightarrow]\}_{x \in X}\) forms a \(Q\)-closed subbase of \(T(\rightarrow)\).

We note that these families form, respectively, a \(P\)-closed and \(Q\)-closed base and that the function \(P\) generates such a partial order \(\leq_P\) in \(X\) which is a lattice order.

Lemma 1.14. Let \((X, P, *)\) be a \(PqpM\)-space satisfying the condition of Corollary 0.9 and let the \(t_{\Delta^+}\)-norm \(*\) be continuous. Then the set \(\langle [-, x]\rangle = \{y \in X y <_P x\}\) is \(Q\)-open and the set \(\langle [x, \rightarrow]\rangle = \{y \in X x <_P y\}\) is \(P\)-open.

Proof. By Corollary 0.9, it follows that \(\leq_P\) orders \(X\) linearly. Hence we have \(\langle [-, x]\rangle \subset N^Q_y(t)\) for all \(t > 0\). On the other hand, for each \(y \in \langle [-, x]\rangle\), we have \(Q_{xy} \neq \varepsilon_0\). This means that there exists \(t > 0\) such that \(Q_{yx}(t) > 1 - t\). We thus have \(N^Q_y(t) \subset \langle [-, x]\rangle\). This completes the proof.

Corollary 1.15. Let \((X, P, *)\) be a \(PqpM\)-space satisfying the condition of Corollary 0.2 and let the \(t_{\Delta^+}\)-norm \(*\) be continuous. The family \(\{\langle [-, x]\rangle\}_{x \in X}\) is a \(Q\)-open base for \(T_Q\). Similarly, the family \(\{\langle [x, \rightarrow]\rangle\}_{x \in X}\) is a \(P\)-open base for the topology \(T_P\).

Theorem 1.16. Let \((X, P, *)\) be a \(PqpM\)-space. Then the family \(\{\langle [-, x]\rangle\}_{x \in X}\) is a complete neighbourhood system in the space \(X\). It thus defines some topology on \(X\). Similarly, \(\{\langle [x, \rightarrow]\rangle\}_{x \in X}\) forms a complete neighbourhood system in \(X\).

Proof. It suffices to observe that, for each \(x \in X\), \(x \in \langle [-, x]\rangle\) and, if \(y \in \langle [-, x]\rangle\), then we have 

\[
\langle [-, y]\rangle \subset \langle [-, x]\rangle.
\]

2. Cartesian product in \(PqpM\)-spaces of the type \(\{k_n\}\)

The following result characterizes countable Cartesian products of \(PqpM\)-spaces.

Definition 2.1. Let \(\{(X_n, P_n)\}\) be a sequence of \(PqpM\)-spaces and let the sequence \(\{k_n\}\) of nonnegative numbers satisfy the condition \(\sum_{n \in \mathbb{N}} k_n = 1\). Then the pair \((X, P)\) is called a Cartesian product of \(PqpM\)-spaces of the type \(\{k_n\}\) if \(X = \prod_{n \in \mathbb{N}} X_n\) and \(P : X^2 \to \Delta^+\) is given by

\[
P_{xy} = \sum_{n \in \mathbb{N}} k_n P_n(x_n, y_n),
\]

where \(x = \{x_n\}\) and \(y = \{y_n\}\).

Definition 2.2 ([5]). A function \(T : I^2 \to I (I = \langle 0, 1 \rangle)\) is called a \(t\)-norm if it satisfies the following conditions
(T1) \( T(a, b) = T(b, a) \)

(T2) \( T(a, b) \leq T(c, d) \) whenever \( a \leq c \) and \( \leq d \)

(T3) \( T(a, 1) = a \)

(T4) \( T(T(a, b), c) = T(a, T(b, c)) \), for all \( a, b, c, d \in I \).

(TA) The \( t \)-norm \( T \) is said to be Archimedean if for any \( x, y \in (0, 1) \), there exists \( n \in N \) such that

\[
x^n < y, \text{ that is } x^n \leq y \text{ and } x^n \neq y,
\]

where \( x^0 = 1, x^1 = x \) and \( x^{n+1} = T(x^n, x) \), for all \( n \geq 1 \). We shall now establish the notation related to a few most important \( t \)-norm:

\[
M(x, y) = \text{Min} (x, y), \quad \text{(TM)}
\]

\[
\Pi(x, y) = x \cdot y, \quad \text{(TII)}
\]

\[
W(x, y) = \text{Max} (x + y - 1, 0). \quad \text{(TW)}
\]

The function \( W \) is continuous and Archimedean and we give the following relations among \( t \)-norms

\[
M \geq \Pi \geq W. \quad \text{(TR)}
\]

**Definition 2.3.** Let \( X \) be a nonempty set, \( P : X^2 \to D \), and \( I \) in \( T_I \)-norm. The triple \( (X, P, T) \) is called a quasi-pseudo-Menger space if it satisfies the axioms:

(M1) \( P_{xx} = \varepsilon_0, \ x \in X \),

(M2) \( P_{xz}(t_1 + t_2) \geq T(P_{xy}(t_1), P_{yz}(t_2)) \), for all \( x, y, z \in X \) and \( t_1, t_2 > 0 \).

If \( P \) satisfies the additional condition:

(M3) \( P_{xy} \neq \varepsilon_0 \) if \( x \neq y \), then \( (X, P, T) \) is quasi-Menger space.

Moreover, if \( P \) satisfies the condition of symmetry \( P_{xy} = P_{yx} \), then \( (X, P, T) \) is called a Menger-space (see [5]).

**Definition 2.4.** Let \( (X, P, T) \) be a probabilistic quasi-Menger space \( (PqM) \) and the function \( Q : X^2 \to D \) be defined by

\[
Q_{xy} = P_{yx}, \text{ for all } x, y \in X.
\]

Then the ordered triple \( (X, Q, T) \) is also \( PqM \)-space. The function \( Q \) is called a conjugate \( Pqp \)-metric of the \( P \). By \( (X, P, Q, T) \) we denote the structure generated by the \( Pqp \)-metric \( P \) on \( X \).

**Lemma 2.5.** Let \( (X_n, P_n) \) be a sequence of proper \( PqpM \)-spaces (Lemma 0.5). Then the Cartesian product \( (X, P) \) of the type \( \{ k_n \} \) is also a proper \( PqpM \)-space. Also, if each \( (X_n, P_n) \) is a quasi-pseudo-Menger space with respect to the \( t_I \)-norm of type \( (TA) \), then so is the Cartesian product of type \( \{ k_n \} \). Moreover, the topology \( T_p \) of a Cartesian product of the type \( \{ k_n \} \) generated by \( P \) is equivalent to the product topology.

**Proof.** For proper \( PqpM \)-spaces, the condition \( F \geq u_a \) is equivalent to the statement that \( F(a+) = 1 \). It thus suffices to observe that, if, for some \( a > 0 \),

\[
P_{x_yn}(a+) = 1
\]
for all \(x_n, y_n \in X_n\), then, by (2.1), we obtain
\[
P_{xy}(a+) = \lim_{t \to a} (\Sigma k_n P_{x_n y_n}(t)) = \Sigma k_n = 1.
\]

To prove the second part of the theorem, let us observe that, by the definition of the \(t\)-norm \(W\) and the Menger condition (M2), the following holds:
\[
W(P_{xz}(t), P_{zy}(s)) = \max \left( \Sigma k_n P_{n}(x_n, z_n)(t) + P_{n}(z_n, y_n)(s) - 1, 0 \right)
\]
\[
\leq \max \left( \Sigma k_n \max (P_{n}(x_n, z_n)(t) + P_{n}(z_n, y_n)(s) - 1, 0), 0 \right)
\]
\[
= \Sigma k_n W(P_{n}(x_n, z_n)(t), P_{n}(z_n, y_n)(s))
\]
\[
\leq \Sigma k_n P_{n}(x_n, y_n)(t + s) = P_{xy}(t + s).
\]

Therefore, we have proved that the Cartesian product of the type \(\{k_n\}\) is a quasi-pseudo-Menger space.

In order to prove the third assertion, let us suppose that the sequence \(\{x^n\}\) is \(P\)-convergent to \(x = \{x_k\}\) in \((X, P)\). Then, for each \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that
\[
x^n \in N^P_x(t)
\]
for all \(n > n_0\). Suppose, further, that, for some \(i_0 \in \mathbb{N}\), the sequence \(\{x^n_{i_0}\}\) is not convergent to \(x_{i_0} \in X_{i_0}\) which is the \(i_0\)-th coordinate of \(x\). This means that, for some \(t_0 > 0\), there exists \(m_n > n\) for all \(n\) such that
\[
P_i(x_{i_0}, x_{i_0}^{m_n})(t_0) > 1 - t_0.
\]
Let \(t = k_{i_0} t_0\). Then, for all \(n > n_0\), we get
\[
1 - t = 1 - k_{i_0} t_0
\]
\[
< P_{xx} m_n(t)
\]
\[
= \sum_{i \in \mathbb{N}} k_i P_i(x_i, x_i^{m_n})(t)
\]
\[
\leq \sum_{i = i_0} k_{i_0} + (1 - t_0)
\]
\[
= 1 - k_{i_0} + k_{i_0} - k_{i_0} t_0
\]
\[
= 1 - t,
\]
which is a contradiction. This means that, if \(\{x^n\}\) is \(P\)-convergent, then each sequence \(\{x_i\}\) is \(P_i\)-convergent to \(x_i\) for all \(i \in \mathbb{N}\). Thus the projections onto the \(i\)-th coordinate are continuous. Therefore, the topology of the Cartesian product of the type \(\{k_n\}\) is stronger than the product topology.

Now, let \(U\) be a \(P\)-open set of \(T_P\). Then, if \(x \in U\), there exists a \(P\)-neighbourhood \(N^P_x(t_0) \subset U\). Let \(F \subset \mathbb{N}\) be a finite subset such that
\[
\sum_{j \in F} k_j - (1 - t_0) > 0.
\]
For every $j \in F$, we select $y_j \in N_{x_j}(t_0)$ and fix $t = 1 - (1 - t_0)(\sum_{j \in F} k_j)^{-1}$. Then, for each $y = \{y_j\}$ such that $y_i = y_j$ for $i = j$ where $j \in F$, we get
\[
P_{xy}(t_0) = \sum_{i \in \mathbb{N}} k_i P_i(x_j, y_j)(t_0) > \sum_{j \in F} k_j P_j(x_j, y_j)(t_0) > \sum_{j \in F} k_j(1 - t) = 1 - t_0.
\]
Thus it follows that $y \in N_{x}(t_0)$. Let $U_i$ be $P_i$-open with $U_i = X_i$ for $i \in \mathbb{N} - F$ and $U_i = N_{x_i}(t)$ for $i \in F$. Then we have
\[
x \in \prod_{i \in \mathbb{N}} U_i \subset U,
\]
which shows that $U$ is open in the product topology. This completes the proof.

**Corollary 2.6.** Each finite or countable Cartesian product of quasi-pseudo-Menger spaces is quasi-pseudo-Menger.

**Proof.** By Lemma 2.5, it follows that each finite or countable cartesian product of quasi-pseudo-Menger spaces is a quasi-pseudo-Menger space with respect to the $t$-norm $W$. Since that $\sup\{W(x, x) : x < 1\} = 1$ the topology of it is quasi-pseudo-metrizable (see [6], [7]). Indeed, let $p$ be a quasi-pseudo-metric that generates the topology. Then $(X, G_p)$ of Theorem 1.4 satisfies the required condition.

**Remark 2.7.** We note that the Cartesian products of $PM$-space were studied by Istratescu and Vadura [4], Egbert [1], Sherwood and Taylor [9] and Radu [6].

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