ON $\Phi$-FIXED POINT FOR MAPS ON UNIFORM SPACES

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Abstract. The concept of fixed point is extended to $\Phi$-fixed point for those maps on uniform spaces. Two results are presented, first for single-valued maps and second for set-valued maps.

1. Introduction and preliminaries

The fixed point theorem has applications in almost all branches of mathematics. The considering of the existence of fixed point for a mapping, is expressed in metric spaces, and some authors have extended this result in some other versions [1], [2], [3] and [4]. M.A. Khamsi and W.A. Kirk [6] have collected many results in fixed point theory which is a good source in this branch. Here, we would improve their results for single-valued and set-valued maps in uniform spaces, which is a generalization for metric space.

Definition 1.1. Let $X$ be a nonempty set and $\Phi \subset 2^{X \times X}$ satisfies in the following:

1) For any $u \in \Phi$, $\Delta = \{<x,x>: x \in X\} \subseteq u$.

2) If $u \in \Phi$ and $u \subset \upsilon$, then $\upsilon \in \Phi$.

3) If $u, \upsilon \in \Phi$, then $u \cap \upsilon \in \Phi$.

4) For any $u \in \Phi$, there exists $\upsilon \in \Phi$ such that, $\upsilon \upsilon \upsilon \subseteq u$,

where, $\upsilon \upsilon = \{(x,z): \exists y \in X; (x,y) \in \upsilon$ and $(y,z) \in \upsilon\}$.

5) $u \in \Phi$ imply that, $u^{-1} \in \Phi$,

where, $u^{-1} = \{(x,y): (y,x) \in u\}$.

Then, $\Phi$ is said to be a uniform structure for $X$ and $(X, \Phi)$ a uniform space.

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Definition 1.2. Let \((X, \Phi)\) be a uniform space and \(T : X \to X\) be a single-valued mapping, \(x_0 \in X\) is said to be \(\Phi\)-fixed point for \(T\), if \((x_0, Tx_0) \in \bigcap_{u \in \Phi} u\).

Definition 1.3. Let \((X, \Phi)\) be a uniform space and \(T : X \to 2^X\) a set-valued mapping, then \(x_0 \in X\) is said to be a \(\Phi\)-fixed point for \(T\) if there exists \(z \in Tx_0\) such that \((x_0, z) \in \bigcap_{u \in \Phi} u\).

We set \(u[x] = \{y \in X; <x, y> \in u\}\) for any \(x \in X, u \in \Phi\).

2. Main results

Theorem 2.1. Suppose that \((X, \Phi)\) is a uniform space and \(T : X \to X\) a single-valued map. If there is \(z \in X\) such that for any \(\nu \in \Phi\), \(\nu[z] \cap \nu[Tz] \neq \emptyset\), then \(T\) has at least one \(\Phi\)-fixed point in \(X\).

Proof. To show that \(T\) has at least one \(\Phi\)-fixed point in \(X\), we must prove there exists at least one \(x_0\) of \(X\), such that \((x_0, Tx_0) \in \bigcap_{u \in \Phi} u\). Suppose on the contrary, assume that for any \(x_0 \in X\) there exists \(u_0 \in \Phi\), such that \((x_0, Tx_0) \notin u_0\). According to the property of uniform space, there exists \(\nu \in \Phi\), such that \(\nu v \subseteq u_0\). Therefore, \((x_0, Tx_0) \notin \nu v\). Hence, for any \(y \in X\), \((x_0, y) \notin \nu\) or \((y, Tx_0) \notin \nu\). Then, for any \(y \in X\), \(y \notin \nu[x_0]\) or \(y \notin \nu[Tx_0]\). Therefore, we obtain \(\nu[x_0] \cap \nu[Tx_0] = \emptyset\), which is a contradiction by assumption. Hence, there exists \(x_0 \in X\) such that for any \(u \in \Phi\), \((x_0, Tx_0) \in \bigcap_{u \in \Phi} u\), i.e., \(x_0\) is \(\Phi\)-fixed point for \(T\) in \(X\).

The following result is a direct consequence Following Theorem 2.1

Corollary 2.2. It should be noticed in Theorem 2.1, if \((X, \Phi)\) is a Hausdorff uniform space, then \(T\) has at least one fixed point in \(X\).

Theorem 2.3. Suppose that \((X, \Phi)\) is a uniform space and \(T : X \to 2^X\) is a set-valued mapping. If there exists at least one \(x_0 \in X\) such that for any \(u \in \Phi\) and for any \(z \in Tx_0\), \(u[x_0] \cap u[z] \neq \emptyset\), then \(x_0\) is a \(\Phi\)-fixed point for \(T\).

Proof. We will prove that, there is at least one \(x_0 \in X\) and there exists \(z \in Tx_0\) such that \((x_0, z) \in \bigcap_{u \in \Phi} u\). On the contrary, for any \(x_0 \in X\), and for any \(z \in Tx_0\), there exists \(u_0 \in \Phi\) such that \((x_0, z) \notin u_0\). Hence, there is \(\nu \in \Phi\) such that \(\nu v \subseteq u_0\), we have \((x_0, z) \notin \nu v\). Therefore, for any \(y \in X\), \((x_0, y) \notin \nu\) or \((y, z) \notin \nu\). Then for any \(y \in X\), \(y \notin \nu[x_0]\) \cap \(\nu[z]\), i.e., \(\nu[x_0] \cap \nu[z] = \emptyset\) which is a contradiction. There is at least one \(x_0 \in X\) which is a \(\Phi\)-fixed point for \(T\).

Following is a direct result of Theorem 2.3

Corollary 2.4. In Theorem 2.3, if \((X, \Phi)\) be a Hausdorff uniform space, then in fact \(x_0\) is a fixed point for \(T\).
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