COMMENT ON AND A CHARACTERIZATION OF THE CONCEPT OF COMPLETE RESIDUATED LATTICE

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Abstract. We prove that some properties of the definition of complete residuated lattice [2,4] can be derived from the other properties. Furthermore we prove that the concept of strictly two-sided commutative quantale [1,3] and the concept of complete residuated lattice are equivalent notions.

1. Introduction

Definition 1. A structure $(L, \lor, \land, *, \rightarrow, \bot, \top)$ is called a complete residuated lattice iff

1. $(L, \lor, \land, \bot, \top)$ is a complete lattice whose greatest and least element are $\top, \bot$ respectively,
2. $(L, *, \top)$ is a commutative monoid, i.e.,
   a) $*$ is a commutative and associative binary operation on $L$, and
   b) $\forall a \in L, a * \top = \top * a = a$,
3. $(a) *$ is isotone,
   b) $\rightarrow$ is a binary operation on $L$ which is antitone in the first and isotone in the second variable,
   c) $\rightarrow$ is couple with $*$ as: $a * b \leq c$ iff $a \leq b \rightarrow c \ \forall a, b, c \in L$.

The following proposition illustrates that the conditions (3)(a) and (3)(b) are consequences from the other conditions. Therefore conditions (3)(a) and (3)(b) should be omit from Definition 1 to be consistent.

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Proposition 1. The conditions (3)(a) and (3)(b) are obtained from the commutativity of * and from (3)(c).

Proof. Let \( a_1, a_2, b \in L \) s.t. \( a_1 \leq a_2 \).

(3)(a) Since \( a_2 \ast b \leq a_2 \ast b \), then \( a_2 \leq b \rightarrow (a_2 \ast b) \) and so \( a_1 \leq b \rightarrow (a_2 \ast b) \). So \( a_1 \ast b \leq a_2 \ast b \). Since * is commutative, then \( b \ast a_1 \leq b \ast a_2 \). Hence * is isotone.

(3)(b) Since \( a_2 \rightarrow b \leq a_2 \rightarrow b \), then \( (a_2 \rightarrow b) \ast a_2 \leq b \). So, \( (a_2 \rightarrow b) \ast a_1 \leq b \) which implies that \( a_2 \rightarrow b \leq a_1 \rightarrow b \), i.e., \( \rightarrow \) is antitone in the first variable. Since \( b \rightarrow a_1 \leq b \rightarrow a_1 \), then \( (b \rightarrow a_1) \ast b \leq a_1 \leq a_2 \). So, \( b \rightarrow a_1 \leq b \rightarrow a_2 \), i.e., \( \rightarrow \) is isotone in the second variable.

For the following definition we refer to [1,3].

Definition 2. A structure \((L, \lor, \land, \ast, \rightarrow, \bot, \top)\) is called a strictly two-sided commutative quantale iff

1. \((L, \lor, \land, \bot, \top)\) is a complete lattice whose greatest and least element are \( \top, \bot \) respectively,
2. \((L, \ast, \top)\) is a commutative monoid,
3. \((a \ast \lor_{j \in J} b_j = \lor_{j \in J}(a \ast b_j) \forall a \in L, \forall \{b_j \mid j \in J\} \subseteq L, (b) \rightarrow \) is a binary operation on \( L \) defined by : \( a \rightarrow b = \lor_{\lambda \ast a \leq b} a \forall a, b \in L \).

Lemma 1. In any strictly two-sided commutative quantale \((L, \lor, \land, \ast, \rightarrow, \bot, \top)\), * is isotone.

Proof. Let \( a_1, a_2, b \in L \) s.t. \( a_1 \leq a_2 \). Now, \( b \ast a_2 = b \ast (a_1 \lor a_2) = (b \ast a_1) \lor (b \ast a_2) \). Then \( b \ast a_1 \leq b \ast a_2 \). Hence * is commutative, then \( a_1 \ast b \leq a_2 \ast b \). Hence * is isotone.

Theorem 1. A structure \((L, \lor, \land, \ast, \rightarrow, \bot, \top)\) is complete residuated lattice iff it is strictly two-sided commutative quantale.

Proof. : First, since for every \( \lambda \in L \) s.t. \( a \ast \lambda \leq b \) we have \( \lambda \leq a \rightarrow b \). Then \( \lor_{\lambda \ast a \leq b} \lambda \leq a \rightarrow b \). Since \( a \rightarrow b \leq a \rightarrow b \), then \( (a \rightarrow b) \ast a \leq b \). So, \( a \rightarrow b \in \{\lambda \in L \mid \lambda \ast a \leq b\} \). Hence \( \lor_{\lambda \ast a \leq b} \lambda = a \rightarrow b \).

Second, since * is isotone, then \( \lor_{j \in J}(a \ast b_j) \leq a \ast \lor_{j \in J} b_j \). Now, \( \forall j \in J, b_j \leq a \rightarrow (a \ast b_j) \) which implies that \( \lor_{j \in J} b_j \leq a \rightarrow \lor_{j \in J}(a \ast b_j) \). Thus \( a \ast \lor_{j \in J} b_j \leq \lor_{j \in J}(a \ast b_j) \). Hence \( a \ast \lor_{j \in J} b_j = \lor_{j \in J}(a \ast b_j) \).

\( \Rightarrow \) : Let \( a \ast b \leq c \). Then \( b \rightarrow c = \lor_{\lambda \ast b \leq c} \lambda \geq a \). Conversely, let \( a \leq b \rightarrow c \). Then, \( a \leq \lor_{\lambda \ast b \leq c} \lambda \). So from Lemma 1, \( a \ast b \leq (\lor_{\lambda \ast b \leq c} \lambda) \ast b = \lor_{\lambda \ast b \leq c} (\lambda \ast b) \leq c \).

Definition 3 [1]. A structure \((L, \lor, \land, \ast, \rightarrow, \bot, \top)\) is called a complete MV-algebra iff the following conditions are satisfied:

1. \((L, \lor, \land, \ast, \rightarrow, \bot, \top)\) is a strictly two-sided commutative quantale;
2. \( \forall a, b \in L, (a \rightarrow b) \rightarrow b = a \lor b \).
**Corollary 1.** \((L, \lor, \land, *, \to, \bot, \top)\) is a complete MV-algebra iff \((L, \lor, \land, *, \to, \bot, \top)\) is a complete residuated lattice satisfies the additional property

\((MV)\ (a \to b) \to b = a \lor b \ \forall a, b \in L.\)

**References**