WHEN IS A QUASI-P-PROJECTIVE MODULE DISCRETE?

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Abstract. It is well-known that every quasi-projective module has $D_2$-condition. In this note it is shown that for a quasi-p-projective module $M$ which is self-generator, duo, then $M$ is discrete.

1. Introduction and preliminaries

Throughout, $R$ is an associative ring with identity and right $R$-modules are unitary. Let $M$ be a right $R$-module. A module $N$ is called $M$-generated if there is an epimorphism $M^{(I)} \rightarrow N$ for some index set $I$. In particular, $N$ is called $M$-cyclic if it is isomorphism to $M/L$ for submodule $L \subseteq M$. Following [3] a module $M$ is called self-generator if it generates all its submodules. For standard notation and terminologies, we refer to [4], [3].

Let $M$ be a right $R$-module. A right $R$-module $N$ is called $M$-$p$-projective if every homomorphism from $N$ to an $M$-cyclic submodule of $M$ can be lifted to an $R$-homomorphism from $N$ to $M$. A right $R$-module $M$ is called quasi-$p$-projective, if it is $M$-$p$-projective. A submodule $A$ of $M$ is said to be a small submodule of $M$ (denoted by $A \ll M$) if for any $B \subseteq M$, $A + B = M$ implies $B = M$. A module $M$ is called hollow if every its submodule is small.

In [2], S.Chotchaisthit showed that a quasi-$p$-injective module $M$ is continuous, if $M$ is duo and semiprefect. Here we study, when a quasi-$p$-projective module is discrete.

Consider the following conditions for a module $M$ which have studied in [3] : $D_1$: For every submodules $N$ of $M$ there exist submodules $K$, $L$ of $M$ such that $M = K \oplus L$ and $K \leq N$ and $N \cap L \ll L$.

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\[ D_2: \text{If } N \text{ is a submodule of } M \text{ such that } M/N \text{ is isomorphism to a direct summand of } M, \text{ then } N \text{ is a direct summand of } M. \]

\[ D_3: \text{For every direct summands } K, L \text{ of } M \text{ with } M = K + L, K \cap L \text{ is a direct summand of } M. \]

If the module \( M \) satisfies \( D_1 \) and \( D_2 \) then it is called a \emph{discrete} module.

It is clear that if \( M \) is hollow, then it has \( D_1 \) and \( D_2 \) conditions, since hollow module is indecomposition.

\section{2. Main results}

Recall that a submodule \( N \) of \( M \) is called a \emph{fully invariant} submodule if \( s(N) \subseteq N \), for any endomorphism \( s \) of \( M \). A right \( R \)-module is called a \emph{duo} module if every submodule is fully invariant. A ring \( R \) is right duo if every right ideal is two sided. The proof of the following Lemma is routine.

\begin{lemma}
Let \( M \) be a duo right \( R \)-module and \( A \) its direct summand. Then:
\begin{enumerate}
\item \( A \) is itself a duo module;
\item If \( M \) is a self-generator, then \( A \) is also a self-generator.
\end{enumerate}
\end{lemma}

\begin{proof}
(1) Let \( f \in \text{End}(A) \), \( \pi : M \to A \), \( i : A \to M \) be the projection and inclusion maps. Then \( g = i \pi \in \text{End}(M) \). It follows that for any submodule \( X \) of \( A \), \( f(X) = g(X) \subseteq X \), proving our Lemma.

(2) Let \( M = A \oplus B \). Then \( f(M) = f(A) + f(B) \) for any \( f \in \text{End}(M) \). Let \( X \) be a submodule of \( A \). Since \( M \) is a self-generator, we can write \( X = \sum_{f \in I} f(M) = \sum_{f \in I} (f(A) + f(B)) \), for some subset \( I \) of \( \text{End}(M) \). Since \( f(B) \subseteq B \), it follows that \( f(B) = 0 \) for all \( f \in I \). Hence \( X = \sum_{f \in I} f(A) \). Moreover, \( f \) can be considered as an endomorphism of \( A \), since \( f(A) \subseteq A \). This shows that \( A \) is a self-generator.
\end{proof}

\begin{lemma}
Let \( M \) be a quasi-p-projective. If \( S = \text{End}(M_R) \) is local, then for any non-trivial fully invariant \( M \)-cyclic submodules \( A \) and \( B \) of \( M \), \( A + B \neq M \).
\end{lemma}

\begin{proof}
Let \( 0 \neq s(M) = A \) and \( 0 \neq t(M) = B \), \( s, t \in S \) and \( A + B = M \). Define the map \( f : M = (s+t)(M) \to M/(A \cap B) \) such that \( f((s+t)(m)) = s(m) + (A \cap B) \). For any \( m, m' \in M \), \( (s+t)(m) = (s+t)(m') \) implies \( s(m-m') \in t(m-m') \subseteq A \cap B \). So \( s(m) = s(m') \in (A \cap B) \). Clearly \( f \) is an \( R \)-homomorphism. By quasi-p-projective, there exist \( g \in S \) such that \( \pi \circ g = f \) and \( \pi : M \to M/(A \cap B) \) is natural epimorphism. It follows \( \pi \circ g(s+t)(m) = \pi(s(m)) \). Then \( ((1-g) \circ s - g \circ t)(M) \subseteq (A \cap B) \). Since \( S \) is local, \( g \) or \( 1-g \) is invertible. If \( 1-g \) is invertible, we have \( (s-(1-g)^{-1} \circ g \circ t)(M) \subseteq (A \cap B) \). \( A \subseteq (s-(1-g)^{-1} \circ g \circ t)(M) \subseteq (1-g)^{-1}(A \cap B) \subseteq (A \cap B) \). Then \( A \subseteq (A \cap B) \), that is contradiction. If \( g \) is invertible we have \( B \subseteq (g^{-1} \circ (1-g) \circ s - t) \subseteq g^{-1}(A \cap B) \subseteq (A \cap B) \). Then \( B \subseteq (A \cap B) \), that is contradiction.
\end{proof}

\begin{corollary}
If \( M \) is quasi-p-projective duo module which is a self-generator with local endomorphism ring, then \( M \) is hollow, hence it is discrete.
\end{corollary}

\begin{proof}
It is clear by Lemma 2.2.
\end{proof}

\begin{lemma}
Let \( M = \bigoplus_{i \in I} B_i \) be duo module. Then for any submodule \( A \) of \( M \) we have \( A = \bigoplus_{i \in I}(A \cap B_i) \).
\end{lemma}
Proof. See [1].

**Corollary 2.5.** Let $M$ be a duo module. If $A$ and $B$ are direct summands of $M$, then so $A \cap B$.

**Proof.** Let $M = A \oplus A_1 = B \oplus B_1$, then by lemma 2.4 $B = B \cap (A \oplus A_1 = (A \cap B) \oplus (B \cap A_1)$. hence $M = (A \cap B) \oplus (B \cap A_1) \oplus B_1$. So $A \cap B$ is a direct summand of $M$. □

**Theorem 2.6.** Let $M = \oplus_{i \in I} M_i$ be quasi-p-projective module where each $M_i$ is hollow. If $M$ is duo module, $\text{Rad}(M) \ll M$ then $M$ is discrete.

**Proof.** By Lemma 2.4 every submodule $A$ of $M$ can be written in the form $A = \oplus_{j \in J} (A \cap M_j)$ where $J \subseteq I$ and $A \cap M_j \neq 0$. Since $A \cap M_j$ is small in $M_j$ we see that $A$ is small in $M$. Thus we have proved. □

**Theorem 2.7.** Suppose that $M$ is semisimple quasi-p-projective duo module and $\text{Rad}(M) \ll M$. If $M$ is self-generator, then $M$ is discrete.

**Proof.** We have $M = \oplus_{i \in I} M_i$ such that $M_i$ is simple, then $\text{End}(M_i)$ is local. By Lemma 2.1 each $M_i$ is duo and self-generator. Since any direct summand of a quasi-p-projective is again quasi-p-projective, it follows from Corollary 2.3 that each $M_i$ is discrete. From Theorem 2.6 that $M$ is discrete, proving our Theorem. □

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