Arens Regularity of Banach Module Actions and the Strongly Irregular Property

Abotaleb Sheikhalii1, Abdolmotaleb Sheikhalii2, Neda Akhlaghi3

1Department of Mathematics, Kharazmi University, Tehran, Iran
E-mail address: Abotaleb.sheikhali.20@gmail.com

2Department of Mathematics, Damghan University, Damghan, Iran
E-mail address: Abdolmotaleb.math88@gmail.com

3Department of Mathematics, Kharazmi University, Tehran, Iran
E-mail address: Neda.akhlaghi1365@gmail.com

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Abstract

Let $X, Y, Z$ be normed spaces. We show that, if $X$ is reflexive, then some extensions andadjointsof
the bounded bilinear map $f: X \times Y \to Z$ are Arens regular. Also the left strongly irregular propertyis
equivalent to the right strongly irregular property. We show that the right module action $\pi^2_n: A^{(n+1)} \times
A^{(n)} \to A^*$ factors, where A is a Banach algebra.

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1. Introductionand Preliminaries

Arens showed in [1] that a bounded bilinear map $f: X \times Y \to Z$ on normed spaces, has two natural
different extensions $f^{***}$, $f^{****}$ from $X^{**} \times Y^{**}$into $Z^{**}$. When these extensions are equal, $f$
is saidto be Arens regular. Throughout the article, we identify a normed space with its canonical image
in thesecond dual.

Let $X, Y, Z$ be normed spaces and $f: X \times Y \to Z$ be a bounded bilinear mapping. The natural
extensions of $f$ are as follows:
i) $f^*: Z^* \times X \to Y^*$, given by $(f^*(z^*, x), y) = (z^*, f(x, y))$ where $x \in X$, $y \in Y$, $z^* \in Z^*$ ($f^*$ is said the adjoint of $f$).

ii) $f^{**}: Y^{**} \times Z^* \to X^*$, given by $(f^{**}(y^{**}, z^*), x) = (y^{**}, f^*(z^*, x))$ where $x \in X$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.

iii) $f^{***}: X^{**} \times Y^{**} \to Z^{**}$, given by $(f^{***}(x^{**}, y^{**}), z^*) = (x^{**}, f^{**}(y^{**}, z^*))$ where $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$, $z^* \in Z^*$.

Let $f^*: Y \times X \to Z$ be the flip of $f$ defined by $f^*(y, x) = f(x, y)$, for every $x \in X$ and $y \in Y$. Then $f^*$ is a bounded bilinear map and it may extends as above to $f^{****}: Y^{**} \times X^{**} \to Z^{**}$. In general, the mapping $f^{****}: X^{**} \times Y^{**} \to Z^{**}$ is not equal to $f^{***}$. When these extensions are equal, then $f$ is Arens regular. If the multiplication of a Banach algebra $A$ enjoys this property, then $A$ itself is called Arens regular. The first and the second Arens products are denoted by $\Box$, $\Diamond$ respectively.

One may define similarly the mappings $f^{****}: Z^{****} \times X^{**} \to Y^{**}$ and $f^{*****}: Y^{*****} \times Z^{***} \to X^{***}$ and the higher rank adjoints. Consider the nets $(x_\alpha) \subseteq X$ and $(y_\beta) \subseteq Y$ converge to $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ in the $w^*$-topologies, respectively, then

$$f^{***}(x^{**}, y^{**}) = w^* - \lim_{\alpha} w^* - \lim_{\beta} f(x_\alpha, y_\beta)$$

and

$$f^{****}(x^{**}, y^{**}) = w^* - \lim_{\alpha} w^* - \lim_{\beta} f(x_\alpha, y_\beta)$$

so Arens regularity of $f$ is equivalent to the following

$$\lim_{\alpha} \lim_{\beta} \{z^*, f(x_\alpha, y_\beta)\} = \lim_{\beta} \lim_{\alpha} \{z^*, f(x_\alpha, y_\beta)\}$$

if the limits exit for each $z^* \in Z^*$. The map $f^{***}$ is the unique extension of $f$ such that

$x^{**} \to f^{***}(x^{**}, y^{**}): X^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $y^{**} \in Y^{**}$ and

$y^{**} \to f^{***}(x, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $x \in X$.

The left topological center of $f$ is defined by

$$Z_l(f) = \{x^{**} \in X^{**}: y^{**} \to f^{***}(x^{**}, y^{**}): Y^{**} \to Z^{**} \text{ is } w^* - w^* \text{ continuous}\}.$$

Since $f^{****}: X^{**} \times Y^{**} \to Z^{**}$ is the unique extension of $f$ such that the map $y^{**} \to f^{****}(x^{**}, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $x^{**} \in X^{**}$, we can set

$$Z_l(f) = \{x^{**} \in X^{**}: f^{***}(x^{**}, y^{**}) = f^{****}(x^{**}, y^{**}), (y^{**} \in Y^{**})\}.$$

The right topological center of $f$ may therefore be defined as

$$Z_r(f) = \{y^{**} \in Y^{**}: x^{**} \to f^{****}(x^{**}, y^{**}): Y^{**} \to Z^{**} \text{ is } w^* - w^* \text{ continuous}\}.$$

Again since the map

$x^{**} \to f^{***}(x^{**}, y^{**}): Y^{**} \to Z^{**}$ is $w^* - w^*$ continuous for each $y^{**} \in Y^{**}$, we can set

$$Z_r(f) = \{y^{**} \in Y^{**}: f^{***}(x^{**}, y^{**}) = f^{****}(x^{**}, y^{**}), (x^{**} \in X^{**})\}.$$

A bounded bilinear mapping $f$ is Arens regular if and only if $Z_l(f) = X^{**}$, or equivalently $Z_r(f) = Y^{**}$. It is clear that $X \subseteq Z_l(f)$ if $Z_l(f) = X$, then the map $f$ is said to be left strongly irregular. Also $Y \subseteq Z_r(f)$ if $Z_r(f) = Y$, then the map $f$ is said to be right strongly irregular. A bounded bilinear mapping $f: X \times Y \to Z$ is said to factor if it is onto. Let $A$ be a Banach algebra, $X$ be a Banach space and $\pi_1: A \times X \to X$ be a bounded bilinear map ($\pi_1$ is said the left module action of $A$...
on $X$. If $\pi_1(ab, x) = \pi_1(a, \pi_1(b, x))$, for each $a, b \in A$, $x \in X$, then the pair $(\pi_1, X)$ is said to be a left Banach $A$–module. A right Banach $A$–module $(X, \pi_2)$ can be defined similarly. A triple $(\pi_1, X, \pi_2)$ is said to be a Banach $A$–module if $(\pi_1, X)$ and $(X, \pi_2)$ are left and right Banach $A$–modules, respectively, and $\pi_1(a, \pi_2(x, b)) = \pi_2(\pi_1(a, x), b)$ for each $a, b \in A$, $x \in X$. Let $(\pi_1, X, \pi_2)$ be a Banach $A$–module. A bounded linear mapping $D : A \to X^*$ is said to be a derivation if $D(ab) = D(a).b + a. D(b)$, for each $a, b \in A$.

2. Arens regularity of bounded bilinear maps

Remark 2.1. Let $f$ be a bounded bilinear map from $X \times Y$ into $Z$. $f^{* * * n}$ means that the number of stars is $3n$ for every $n \in N$.

Let $f$ be a bounded bilinear map and $x^{**} \in X^{**}, y^{**} \in Y^{**}$. If $f$ is Arens regular then for every $z^{*} \in Z^{*}$,

$$
\langle f^{* * * r}(y^{**}, x^{**}), z^{*} \rangle = \langle f^{* * * r}(x^{**}, y^{**}), z^{*} \rangle = \langle f^{* * * r}(x^{**}, y^{**}), z^{*} \rangle = \langle f^{* * * r}(y^{**}, x^{**}), z^{*} \rangle
$$

Therefore, $f^{* * }$ is Arens regular. Now let $f^{* * }$ be Arens regular, for every $x^{**} \in X^{**}, y^{**} \in Y^{**}, z^{*} \in Z^{*}$,

$$
\langle f^{* * * r}(x^{**}, y^{**}), z^{*} \rangle = \langle f^{* * * r}(y^{**}, x^{**}), z^{*} \rangle = \langle f^{* * * r}(y^{**}, x^{**}), z^{*} \rangle = \langle f^{* * * r}(x^{**}, y^{**}), z^{*} \rangle
$$

Hence $f$ is Arens regular if and only if $f^{* * }$ is Arens regular.

Lemma 2.2. If $f : X \times Y \to Z$ is Arens regular and $X$ is a reflexive space, then $f^{* * * n}$ and $f^{* * * n}$ are Arens regular for every $n \in N$.

Proof. First, we show that $f^{* * }$ is Arens regular for an arbitrary $f$. Then we show that $f^{* * * n} = f^{* * * n}$. By [7, Theorem 2.1], for every $x^{****} \in X^{****}, y^{****} \in Y^{****}, z^{***} \in Z^{***}$, we have

$$
\langle f^{* * * r}(x^{****}, y^{****}), z^{***} \rangle = \langle x^{****}, f^{* * * r}(y^{****}, z^{***}) \rangle
$$

$$
= \langle y^{****}, f^{* * * r}(z^{***}, x^{****}) \rangle
$$

$$
= \langle y^{****}, f^{* * * r}(z^{***}, x^{****}) \rangle
$$

$$
= \langle f^{* * * r}(y^{****}, x^{****}), z^{***} \rangle
$$

It follows that $f^{* * }$ is Arens regular. This completes the proof of Arens regularity of $f^{* * * n}$. Now if $f$ is Arens regular then we show that $f^{* * }$ is Arens regular. We should show

$$
(1) \quad f^{* * * n} = f^{* * * n}
$$

Since $f$ is Arens regular,

$$
(2) \quad (f^{* * })^{* * } = (f^{* * })^{* * }
$$

so it is enough to show that

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\[ (3) \quad f^{****} = f^{****} \]

\( f^{**} \) is Arens regular, therefore \( f^{****} = f^{****} \), so from the Arens regularity of \( f^r \), we have \( f^{****} = f^{****} \). Therefore \( f^{****} = f^{****} \). From the Arens regularity of \( f \), \( f^{***} = f^{***} \). Now by [6, Theorem 2.1], for every \( x^{***} \in X^{***} \), \( y^{***} \in Y^{***} \), \( Z^{***} \in Z^{***} \), we have

\[
\langle f^{***}(x^{***}, y^{***}) \rangle = \langle x^{***}, f^{****}(y^{***}) \rangle
\]

Therefore equation (3) holds and \( f^{****} \) is Arens regular. Hence \( f^{****} \) is Arens regular, for every \( n \in N \).

**Lemma 2.3.** Let \( f: X \times Y \rightarrow Z \) be is a bounded bilinear map. If \( X \) is reflexive, then \( f \) and every adjoint and every flip map of \( f \) such that its domain contains \( X, X^*, X^{**}, \ldots \) is Arens regular.

**Proof.** First we show that if \( Y \) is reflexive, then the result holds. \( f^{****}(Z^{***}, X^{**}) \subseteq Y^{***} \) and \( Y^* \) is reflexive, therefore

\[ f^{****}(Z^{***}, X^{**}) \subseteq f^{****}(Z^{***}, X^{**}) \subseteq Y^*. \]

Now by [7, Theorem 2.1], \( f \) is Arens regular. Therefore \( f^r \) is Arens regular, so the result holds.

**Lemma 2.4.** If \( X \) is reflexive and the bounded bilinear map \( f^{****} \) factors, then \( f \) and every adjoint map and every flip map of it is Arens regular.

**Proof.** If \( X \) is reflexive space then by **Lemma 2.3**, \( f \) and \( f^{rt} \) are Arens regular and by [7, Corollary 2.2] it is equivalent that \( f^{****}(Z^{***}, X^{**}) \subseteq Y^* \) and \( f^{****} \) factors, therefore \( Y^{***} \subseteq Y^* \) and it is equivalent that \( Y \) is reflexive. Now for every adjoint map or every flip map, \( X \) or \( X^* \) or \( Y^* \), is contained in a part of its domain. Since these spaces are all reflexive, therefore by **Lemma 2.3** the result holds.

**Theorem 2.5.** Let \( X \) be is reflexive and let \( f^{****} \) factors. Then \( f \) is left strongly irregular if and only if it is right strongly irregular.

**Proof.** By **Lemma 2.3** \( f \) is Arens regular. From the definition, \( f \) is Arens regular if and only if \( Z(f) = X^{**} \). \( X \) is reflexive therefore \( Z(f) = X \), i.e. \( f \) is left strongly irregular, therefore \( f \) is Arens regular if and only if \( f \) is left strongly irregular. On the other hand by **Lemma 2.4**, \( Y \) is also reflexive, therefore by definition
of topological centers, $f$ is Arens regular if and only if $Z_r(f) = Y^{**}$, since $Y$ is reflexive so $Z_r(f) = Y$, thus $f$ is right strongly irregular. therefore $f$ is Arens regular if and only if right strongly irregular. It follows that $f$ is left strongly irregular if and only if right strongly irregular.

3. Module action

In [5] Eshaghi Gordji and Filali show that left module action of a Banach algebra $A$ on $A^{(n)}$ factors.

Now let $\pi_{2n}$ be the right module action of $A$ on $A^{(n)}$. Thus $\pi_{2n}$ maps $A^{(n)} \times A$ into $A^{(n)}$ and $\pi_{2n}$ maps $A^{(n+1)} \times A^{(n)}$ into $A^*$, for every $n \geq 1$. Also $\pi_{2n+1} = \pi_{2n}^* \pi_{2n-1}$ and $\pi_{1n} = \pi_{1n-1}^*$ such that $A^{(0)} = A$, $\pi = \pi_{10} = \pi_{20}$. In the next theorem we show that the right module action factors.

**Theorem 3.1.** Let $A$ be a Banach algebra.

I) If $A$ has a left bounded approximate identity, then $\pi_{2n}^*$ factors for every positive even integer $n$.

II) If $A$ has a right bounded approximate identity, then $\pi_{2n}^*$ factors for odd positive even integer $n$.

**Proof.** I) We use the induction on $n$. Let $n = 2$ and $(e_\beta)$ be a left bounded approximate identity in $A$ with a cluster point $e^{**} \in A^{**}$. Therefore for every $a^{***} \in A^{**}$ we have $\pi_{2n}^*(a^{***}, e^{**}) = a^{***}$.

Let $(a_\alpha^*)$ be a net in $A^*$ with a cluster point $e^{**} \in A^{**}$, so for every $a \in A$,

$$\langle \pi_{2n}^*(a^{***}, e^{**}), a \rangle = \langle a^{***}, \pi_{2n}^* (e^{**}, a) \rangle = \langle a^{***}, \pi_{1n}^* (e^{**}, a) \rangle$$

$$= \lim_{\alpha} \langle \pi_{2n}^* (e^{**}, a), a_\alpha^* \rangle = \lim_{\alpha} \langle \pi_{2n}^* (e^{**}, a), a_\alpha^* \rangle$$

$$= \lim_{\alpha} \langle e^{**}, \pi_{2n}^* (a, a_\alpha^*) \rangle = \lim_{\alpha} \langle e^{**}, \pi_{2n}^* (a, a_\alpha^*) \rangle$$

$$= \lim_{\alpha} \langle a_\alpha^*, a \rangle = \langle a^{***}, a \rangle.$$  

Therefore for $n = 2, \pi_{2n}^*$ factors. Now suppose that the result holds for $n = 2k - 2$. So,

$$\langle \pi_{2(2k-2)}^* (a^{***}, e^{**}), a \rangle = \langle a^{***}, \pi_{2(2k-2)}^* (e^{**}, a) \rangle = \langle a^{***}, \pi_{1(2k-2)}^* (e^{**}, a) \rangle$$

$$= \langle e^{**}, \pi_{1(2k-2)}^* (a, a^{**}) \rangle = \langle e^{**}, \pi_{1(2k-2)}^* (a, a^{**}) \rangle$$

$$= \langle e^{**}, \pi_{2(2k-2)}^* (a, a^{**}) \rangle = \langle a^{***}, \pi_{2(2k-2)}^* (a, e^{**}) \rangle$$

$$= \langle a^{***}, \pi_{2(2k-2)}^* (a, e^{**}) \rangle = \langle \pi_{2(2k-2)}^* (a^{***}, e^{**}), a \rangle$$

thus $\pi_{2(2k)}^*$ factors.

II) Again by induction. Let $(e_\alpha)$ be a right bounded approximate identity in $A$ with a cluster point $e^{**} \in A^{**}$. for $n = 1$ it is enough to show that $\pi_{2n}^*(e^{**}, a^*) = a^*$ for every $a^* \in A^*$.  

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\[ \langle \pi^*_2 (e^{**}, a), a \rangle = \langle e^{**}, \pi^*_2 (a, a) \rangle = \langle e^{**}, \pi^*_1 (a, a) \rangle = \lim_{\alpha} \langle a^*, \pi^*_1 (a, e^{\alpha}) \rangle = \langle a^*, a \rangle. \]

Now suppose that is true for some \( k \), then
\[ \langle \pi^*_2 (e^{**}, a), a \rangle = \langle e^{**}, \pi^*_2 (a, a) \rangle = \langle e^{**}, \pi^*_1 (a, a) \rangle = \langle e^{**}, \pi^*_2 (a, a) \rangle = \langle a^*, \pi^*_1 (a, e^{**}) \rangle \]
\[ = \langle a^*, \pi^*_2 (a, e^{**}) \rangle = \langle e^{**}, \pi^*_2 (a, a^*) \rangle = \langle e^{**}, \pi^*_2 (a, a^*) \rangle = \langle e^{**}, \pi^*_1 (a, a^*) \rangle. \]

so the result holds.

Here is a new proof for the theorem [4.7.1]

**Theorem 3.2.** If \( X \) is a reflexive space and \( D : A \to X^* \) is a derivation, then \( D^{**} \) is also a derivation.

**Proof.** As \( X \) is reflexive, by lemma 2.3 the following module actions are Arens regular,
\[ \pi_1 : A \times X \to X, \quad \pi_2 : X \times A \to X \]
\[ \pi^*_1 : X^* \times A \to X^*, \quad \pi^*_2 : X^* \times A \to X^* \]

Now the maps below are Arens regular by [7, 4.4],
\[ D^{**} : (A^{**}, \Box) \to X^{***}, \quad D^{**} : (A^{**}, \delta) \to X^{***} \]

**References**


