Compact Topological Semigroups associated with Oids

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Article history:
Received July 2014
Accepted August 2014
Available online August 2014

Abstract

The known theory for a discrete oid $T$ shows that how to find a subset $T^o$ of $\beta T$ which is a compact right topological semigroup (see section 2 for details). In this paper we try to find an analogue of almost periodic functions for oids. We discover, new compact semigroups by using a certain subspace of functions $U^o(T)$ of $\beta(T)$ for an oid $T$ for which $f^\beta$ is continuous on $T^o \times (T \cup T^{oo} \cup T^{oo})$, where $(T \cup T^{oo} \cup T^{oo})$ is a suitable subspace of $\beta T$ for a wide range.

Mathematical Society Classification: 2010, 54D35.
Keywords: Oid, Jointly continuous function, Compact topological semigroup.

1. Introduction

Let $S$ be a semigroup and topological space. $S$ is called a topological semigroup if the multiplication $(s,t) \to st: S \times S \to S$ is jointly continuous. Cavin and Yood [4] shows that $\beta S$ the Stone-Čech compactification of a discrete semigroup $S$ could be given a semigroup structure, which need not be commutative on $S$ and is continuous in the left-hand variable; (that is for fixed $v \in \beta S$, the map $\mu \to \mu v: \beta S \to \beta S$ is continuous). Indeed the operation on $S$ extends uniquely to $\beta S$, so that $S$ contained in it's topological center [5]. Pym [7] introduced the concept of an oid (see Section 2 for precise definition). Oids are important because nearly all semigroups contain them and all oids are oid-isomorphic [6]. We shall present our theory in a fairly concrete setting, so that our methods and results will be more readily accessible. Through out this paper we will let $T$ be a commutative standard oid with a discrete topology. Then the compact space $\beta T$ produces a compact right topological semigroup, "at
infinity $T^\infty$, so that its topological center is empty and it is not commutative (we refer the reader to [2], for these facts). Our aim of the present paper is to introduce a new compact topological semigroup for an oid $T$, using a certain space of functions on $T$ which have jointly continuous extensions on subspace $T^\infty \times (T \cup T^\infty \cup TT^\infty)$ of $T^\infty \times \beta T$ where $(T \cup T^\infty \cup TT^\infty)$ is a suitable subspace of $\beta T$ which is as large as possible. $C(T)$ is the $C^*$-algebra of all bounded continuous complex valued functions defined on the discrete space $T$ and $C(T)'$ is the dual space of $C(T)$; we indicate the supremum norm on $C(T)$ by $\|\|$.

We define a subset $\mathcal{U}^\infty(T)$ containing all $f \in C(T)$ such that $f^\beta$ is jointly continuous on $T^\infty \times (T \cup T^\infty \cup TT^\infty)$ where $f^\beta$ is a unique continuous extension $f$ to $\beta T$. Then $\mathcal{U}^\infty(T)$ is a $C^*$-subalgebra of $C(T)$ (Lemma 3.3), so that $\mathcal{U}^\infty(T) \subseteq WAP^\infty(T)$ (see [1], for definition). Indeed, $WAP^\infty(T)$ need not be a subset of $\mathcal{U}^\infty(T)$ (Example 4.27). From the functions space $\mathcal{U}^\infty(T)$ we shall able to define an equivalence relation $\mathcal{R}_{\mathcal{U}^\infty(T)}$ on $T^\infty$ by $\mu \sim \mathcal{R}_{\mathcal{U}^\infty(T)} \nu$ if and only if $f^\beta(\mu) = f^\beta(\nu)$ for all $f \in \mathcal{U}^\infty(T)$. This does determine a closed congruence relation on $T^\infty$.

Which makes the quotient $T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)}$ a compact Hausdorff commutative topological semigroup which is a new semigroup to consider. Also, we conclude by establishing some properties of $T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)}$, for example $(T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)})^2$ is not dense in $T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)}$ (Proposition 4.14), it contains $2^c$ idempotents (Theorem 5.4) and $K(T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)})$, the minimal ideal of $T^\infty / \mathcal{R}_{\mathcal{U}^\infty(T)}$ contains a free abelian group on $2^c$ generators (Theorem 6.2).

2. Definitions and preliminaries

Let $x = (x(n))_{n \in \mathbb{N}}$ be any sequence consisting of $1$'s and $\infty$'s. Write

$1.1 = 1, \infty = \infty, 1 = 1$. We define

$$supp(x(n))_{n \in \mathbb{N}} = \{n \in \mathbb{N} : x(n) = \infty\},$$

and write

$$T = \{(x(n))_{n \in \mathbb{N}} : supp(x(n))_{n \in \mathbb{N}} \text{ is finite and non-empty} \}.$$ 

A commutative standard oid is the set $T$ together with the product $xy$ defined in $T$ if and only if $(supp x) \cap (supp y) = \emptyset$ to be $(x(n)y(n))$. Thus the product $x(n)y(n)$ is required to be defined if and only if either $x(n)ory(n)$ is 1. Obviously, the product in $T$ is associative where defined and $supp(xy) = (supp x) \cup (supp y)$ whenever $xy$ is defined in $T$ (oids are discussed in [7]). Any commutative standard oid $T$ can be considered as $\bigoplus_{n=1}^{\infty} \{1, \infty\} \setminus \{(1, 1, \ldots, 1)\}$. We use epithet “standard” to indicate that the index set is $\mathbb{N}$ (in [7], oids could have any index set). For $x, y \in T, supp x < supp y$ means that $n < m$ if $n \in supp x$ and $m \in supp y$, and $supp x_a \to \infty$ for some net $(x_a)$ in $T$ will mean that for arbitrary $k \in \mathbb{N}$ eventually $\min(supp x_a) > k$. Then for
a fixed $t \in T$, eventually $\text{supp} t < \text{supp} x_\alpha$ and so eventually $tx_\alpha$ is defined in $T$. Write $u_n = (1,1,\ldots,\infty,1,1,\ldots)$ (with $\infty$ in the $n$th place). Put $\{u_n: n \in \mathbb{N}\}$.

Then $U$ is a countable subset of $T$. Moreover, any $x \in T$ can be written uniquely as a finite product $x = u_{i_1}u_{i_2} \ldots u_{i_k}$ with $i_1 < i_2 < \ldots < i_k; \text{supp} x = \{i_1, \ldots, i_k\}$. The compact space $\beta T$ is the Stone-$\check{C}$ech compactification of the discrete space $T$ and if $f$ maps $T$ to some compact space, $f^\beta$ is the unique continuous extension of $f$ to $\beta T$. We define

$$T^\omega = \{\mu \in \beta T: \mu = \lim_\alpha x_\alpha \text{ with } \text{supp} x_\alpha \to \infty\}.$$

Obviously, $T \cap T^\omega = \emptyset$. For $\mu \in \beta T, \nu \in T^\omega$ the product $\mu \nu \in C(T)^*$ is defined by $\mu \nu = \nu L_\nu$, where $L_\nu f(t) = \lim_\beta f(ty_\beta)$, if $t \in T, f \in C(T)$ and $y_\beta \to \nu$ with $\text{supp} y_\beta \to \infty$. Then $L_\nu f \in C(T), L_\nu f(t) = (L_t f)^\beta(\nu)$. Further, $L_\nu$ is a bounded linear operator on $C(T)$. Of course $\mu \in \beta T$ is a bounded linear functional on $C(T)$, with $\|\mu\| \leq 1$, if $f(\mu) = f^\beta(\mu)$. In fact, the product $(\mu, \nu) \to \mu \nu: \beta T \times T^\omega \to T^\omega$ is defined and is right continuous and left continuity holds when $\mu = t \in T[1]$. Also $\mu \nu = \lim_\alpha \lim_\beta x_\alpha y_\beta$ where $(x_\alpha)$ is a net in $T$ with $x_\alpha \to \mu$. If $\mu \in T^\omega$, then $L_{\mu \nu} = L_{\nu \circ L_\nu}$, so that $(\mu, \nu) \to \mu \nu: T^\omega \times T^\omega \to T^\omega$ is a binary operation on $T^\omega$ relative to which that $T^\omega$ is a compact right topological semigroup. If $\subseteq T$, then $1_A$ denotes the indicator function of $A$, that is, the function whose value is 1 on $A$ and 0 on $T \setminus A$.

**Remark 2.1.** For $\nu \in T^\omega$ and $\mu \in \beta T$, $\nu \mu$ can not always be defined in a standard odi $T$, if we require that multiplication is right continuous. This is true even if $\mu \in T$. If $z_n = u_1u_2 \ldots u_n$, $n \in \mathbb{N}$, $u_n \in U$ and $z_n \to \lambda \in \beta T$ for some subnet $(z_{n_i})$ of $(z_n)$, then for any $t \in T$, $\lim_i tz_{n_i}$ is not defined. But we can define $\nu \mu$ for standard odis only on a subset of $T^\omega \times \beta T$. This subset includes $T^\omega \times (T \cup T^\omega)$. Now let $x_\alpha \to \mu$ in $\beta T$ with $\text{supp} x_\alpha \to \infty$ and let $\lambda = \lambda'$, where $t \in T$, $\lambda' \in T^\omega$ such that $y_\beta \to \lambda'$ with $\text{supp} y_\beta \to \infty$. Then eventually $t x_\alpha y_\beta$ and for such $\alpha$, eventually $\text{supp} x_\alpha < \text{supp} y_\beta$, so that eventually $tx_\alpha y_\beta$ is defined in $T$ and hence $\lim_\alpha \lim_\beta (tx_\alpha y_\beta) = t\mu \lambda'$ ($= \mu \lambda$) (see[1], Definition 3.5). Therefore, we can defined $\mu \lambda$ on $T^\omega \times (T \cup T^\omega \cup TT^\omega)$, whenever $(T \cup T^\omega \cup TT^\omega)$ is a suitable subspace of $\beta T$ for a wide range.

**Definition 2.2.** (i) The **cardinal function** is the map $c: T \to \mathbb{N}$ given by $c(x) = \text{card}(\text{supp} x)$ (that is, the number of elements of the support of $x$). Then $c$ extends to a unique continuous extension $c^\beta$ from $\beta T$ into the one-point compactification $\mathbb{N} \cup \{\infty\}$. If $(\text{supp} x) \cap (\text{supp} y) = \emptyset$ so that $xy$ is defined in $(xy) = c(x) + c(y)$, and so for $\mu \in \beta T, \nu \in T^\omega$ then $c^\beta(\mu \nu) = c^\beta(\mu) + c^\beta(\nu)$. Thus $c^\beta$ is a homomorphism on $T^\omega$. We denote $1/c(x)$ by $h(x)$ for $x \in T$.

(ii) The **length function** is the map $l: T \to \mathbb{N}$ by letting $l(x)$ (The length Of support of $x$) be the integer $i_k - i_1 + 1$ where $\text{supp} x = \{i_1, \ldots, i_k\}$. 221
Then $l$ extends to a unique continuous extension $l^β$ from $βT$ into the one-point compactification $\mathbb{N} \cup \{∞\}$. We denote $1/l(x)$ by $r(x)$ for $x \in T$.

(iii) The $z$-function is the map $z:T → \mathbb{Z}^+$ by letting $z(x)$ be the largest number of consecutive 1’s between $\min(\text{supp } x)$ and $\max(\text{supp } x)$. Then $z$ extends to a unique continuous extension $z^β$ from $βT$ into the one-point compactification $\mathbb{Z}^+ \cup \{∞\}$. We denote $1/z(x) + 1$ by $k(x)$ for $x \in T$.

We next have some useful results which we will need later.

**Proposition 2.3.** For $μ ∈ T^∞, ν ∈ (T ∪ T^∞ ∪ TT^∞)$ then $l^β(μν) = ∞$.

**Proof.** Let $ν = t ∈ T$, and let $x_α → μ$ for some net $(x_α)$ in $T$ with $\text{supp } x_α → ∞$. Then eventually $\text{supp } t < \text{supp } x_α$, so that eventually $l(tx_α) = ∞$. Since $tx_α → tμ$ in $βT$ and $l^β$ is continuous on $βT$, from which it follows that $l^β(μt) = l^β(tμ) = ∞$. If $ν ∈ T^∞$, and $γ_β → ν$ for some net $(γ_β)$ in $T$ with $\text{supp } γ_β → ∞$, then $l^β(μν) = \lim_β \lim_γ l(x_αγ_β) = ∞$, by a similar reason. Suppose that $ν = tλ, λ ∈ T^∞$. Then $tμλ ∈ T^∞$, since $T^∞$ is a semigroup, hence $l^β(μν) = l^β(μtλ) = l^β(tμλ) = ∞$ and the result follows.

The next result is an immediate consequence of Definition 2.2(ii), Proposition 2.3.

**Corollary 2.4.** For $μ ∈ T^∞, ν ∈ (T ∪ T^∞ ∪ TT^∞)$. Then $r^β(μν) = 0$.

**Proposition 2.5.** Let $μ ∈ T^∞, ν ∈ (T ∪ T^∞ ∪ TT^∞)$. Then $z^β(μν) = ∞$.

**Proof.** This uses Definition 2.2(iii), the proof is parallel to that of Proposition 2.3.

**Corollary 2.6.** Let $μ ∈ T^∞, ν ∈ (T ∪ T^∞ ∪ TT^∞)$. Then $k^β(μν) = 0$.

Proof is straightforward.

3. Space of jointly continuous functions

Our aim of the present section is to introduce a new kind of $C^*$-subalgebra of the $C^*$-algebra $C(T)$. In this section we try to find an analogue of almost periodic functions for oids.

**Definition 3.1.** Let $T$ be a commutative standard oid. We use $U^∞(T)$ to denote the set of all bounded complex valued functions on $T$ for which $(μ, ν) → f^β(μν): T^∞ \times (T ∪ T^∞ ∪ TT^∞) → \mathbb{C}$ is jointly continuous. Clearly $U^∞(T)$ is conjugate closed and contains all constant functions.

**Example 3.2.** (i) Let $h = 1/c$ be as in Definition 2.2(i). Then by a routine
argument we see that for $\mu \in T^\infty, \nu \in (T \cup T^\infty \cup TT^\infty)$, $c^\beta (\mu \nu) = c^\beta (\mu) + c^\beta (\nu)$, and so $(\mu, \nu) \to h^\beta (\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is jointly continuous. Therefore $h \in \mathcal{U}^\infty(T)$.

(ii) Let $r = 1/l$ be as in Definition 2.2(ii). Then by Corollary 2.4, $r^\beta (\mu \nu) = 0$ for $\mu \in T^\infty$ and $\nu \in (T \cup T^\infty \cup TT^\infty)$, and so $(\mu, \nu) \to r^\beta (\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is jointly continuous. Thus $r \in \mathcal{U}^\infty(T)$.

(iii) Let $k = 1/z + 1$ be as in Definition 2.2(iii). Then by Corollary 2.6, $k^\beta (\mu \nu) = 0$ for $\mu \in T^\infty$, $\nu \in (T \cup T^\infty \cup TT^\infty)$, and so $(\mu, \nu) \to k^\beta (\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is jointly continuous. Hence $k \in \mathcal{U}^\infty(T)$.

**Lemma 3.3.** $\mathcal{U}^\infty(T)$ is a $C^*$-subalgebra of the $C^*$-algebra $C(T)$.

**Proof.** It is easily seen that $\mathcal{U}^\infty(T)$ is a subalgebra of the algebra $C(T)$. To prove that $\mathcal{U}^\infty(T)$ is a $C^*$-subalgebra it is enough to prove that $\mathcal{U}^\infty(T)$ is a closed subalgebra of $C(T)$ because the other conditions are satisfied easily. For this purpose, let $(f_n)_{n \in \mathbb{N}}$ be any sequence in $\mathcal{U}^\infty(T)$, $f \in C(T)$ with $\|f_n - f\| \to 0$, as $n \to \infty$. Suppose that $\mu_\alpha \to \mu$ in $T^\infty$, $\nu_\alpha \to \nu$ in $(T \cup T^\infty \cup TT^\infty)$. Then given $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $\|f_n - f\| < \varepsilon/3$ for all $n \geq k$. Fix $n_0 > k$. Then choose $\alpha_0$ such that $\alpha > \alpha_0$.

$$\left| f^\beta_n (\mu_\alpha \nu_\alpha) - f^\beta (\mu \nu) \right| < \varepsilon/3.$$ For such $\alpha$, then

$$\left| f^\beta (\mu_\alpha \nu_\alpha) - f^\beta (\nu_\alpha) \right| \leq \left| f^\beta (\mu_\alpha \nu_\alpha) - f^\beta_n (\mu_\alpha \nu_\alpha) \right| + \left| f^\beta_n (\mu_\alpha \nu_\alpha) - f^\beta_n (\mu \nu) \right| + \left| f^\beta_n (\mu \nu) - f^\beta (\mu \nu) \right|$$

$$\leq \left\| f^\beta - f^\beta_n \right\| + \left| f^\beta_n (\mu_\alpha \nu_\alpha) - f^\beta_n (\mu \nu) \right| + \left| f^\beta_n - f^\beta \right|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

Hence $\lim_n f^\beta_n (\mu_\alpha \nu_\alpha) = f^\beta (\mu \nu)$ and so $(\mu, \nu) \to f^\beta (\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty)$ is jointly continuous, as desired. $\square$

Our next result will be useful in later.

**Theorem 3.4.** Let $f \in \mathcal{U}^\infty(T), \eta \in T^\infty$. Then $L_\eta f \in \mathcal{U}^\infty(T)$.

**Proof.** It is easy to check that $\nu \eta \in (T \cup T^\infty \cup TT^\infty)$ whenever $\nu \in (T \cup T^\infty \cup TT^\infty)$. From this and that the product $(\mu, \nu) \to \mu \nu$ is right continuous, it follows that the map $\nu \to \nu \eta$ is continuous of $(T \cup T^\infty \cup TT^\infty)$ into itself. Therefore the composite map $(\mu, \nu) \to (\mu, \nu \eta) \to f^\beta (\mu \nu \eta)$ is continuous from $T^\infty \times (T \cup T^\infty \cup TT^\infty)$ to $\mathbb{C}$ for each $f \in \mathcal{U}^\infty(T)$. Thus

$$f^\beta (\mu \nu \eta) = (\mu \nu \eta)(f) = \mu \nu \alpha L_\eta (f) = \nu \alpha (L_\eta f) = (L_\eta f)^\beta (\mu \nu).$$

It follows that $(\mu, \nu) \to (L_\eta f)^\beta (\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is continuous, and therefore by Definition 3.1, $L_\eta f \in \mathcal{U}^\infty(T)$, as desired. $\square$

**Definition 3.5.** For $f \in \mathcal{U}^\infty(T)$ and $\nu \in (T \cup T^\infty \cup TT^\infty)$, we define $L_\nu f^\beta (\mu) = f^\beta (\mu \nu)$, where $\mu \in T^\infty$.

**Remark 3.6.** It is easy to check that $L_\nu f^\beta$ is continuous on compact space.
Since \((\mu, \nu) \rightarrow \beta T \times T^\infty \rightarrow \beta T\) is right continuous and left continuity holds when \(\mu = t \in T\). Moreover, \(L_v\) is a bounded linear operator with \(\|L_v\| \leq 1\).

**Theorem 3.7.** Let \(f \in C(T)\). Then \((\mu, \nu) \rightarrow f^\beta(\mu \nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \rightarrow \mathbb{C}\) is jointly continuous if and only if \(\nu \rightarrow L_v f^\beta: (T \cup T^\infty \cup TT^\infty) \rightarrow C(T^\infty)\) is norm continuous.

**Proof.** Define \(\psi: T^\infty \times (T \cup T^\infty \cup TT^\infty) \rightarrow \mathbb{C}\) by \(\psi(\mu, \nu) = f^\beta(\mu \nu)\). Then \(\psi\) is a bounded function, since \(f^\beta\) is continuous on \(\beta T\). It follows readily that \(\mu \rightarrow \psi(\mu, \nu): T^\infty \rightarrow \mathbb{C}\) is a continuous function for each \(\nu \in (T \cup T^\infty \cup TT^\infty)\).

Let \(C(T^\infty)\) have the uniform norm. Since \(T^\infty\) is a compact space and \((T \cup T^\infty \cup TT^\infty)\) is a subspace of \(\beta T\), \(\psi\) is jointly continuous if and only if the mapping \(\nu \rightarrow \psi(0, \nu):(T \cup T^\infty \cup TT^\infty) \rightarrow C(T^\infty)\) is continuous (see [10], Chapter 1, Lemma 1.8(a)).

**Lemma 3.8.** If \(\nu \rightarrow L_v f^\beta: T^\infty \rightarrow C(T^\infty)\) is norm-continuous, then \(\{L_v f^\beta: \nu \in T^\infty\}\) is relatively norm compact in \(C(T^\infty)\).

**Proof.** Is straightforward. \(\square\)

The next result is an immediate consequence of Definition 3.1, Theorem 3.7 and Lemma 3.8.

**Corollary 3.9.** Let \(f \in \mathcal{U}^\infty(T)\). Then \(\{L_v f^\beta: \nu \in T^\infty\}\) is a norm relatively compact in \(C(T^\infty)\).

### 4. Compact topological semigroups

In this section by starting with \(\mathcal{U}^\infty(T)\) we will produce a new compact commutative topological semigroup, and make an investigation of its properties.

Assume that \(\tau\) is the topology induced on a compact right topological semigroup \(T^\infty\) by \(\beta T\) and \(\tau_{\mathcal{U}^\infty(T)}\) is the weak topology induced on \(T^\infty\) by the family \(\{f^\beta: f \in \mathcal{U}^\infty(T)\}\). Then the identity map from \((T^\infty, \tau)\) onto \((T^\infty, \tau_{\mathcal{U}^\infty(T)})\) is continuous, thus \((T^\infty, \tau_{\mathcal{U}^\infty(T)})\) is compact [9].

**Definition 4.1.** For \(\mu, \nu \in T^\infty\), define \(\mu \mathcal{R}_{\mathcal{U}^\infty(T)} \nu\) if and only if \(f^\beta(\mu) = f^\beta(\nu)\) for all \(f \in \mathcal{U}^\infty(T)\). Clearly \(\mathcal{R}_{\mathcal{U}^\infty(T)}\) is a closed relation on \((T^\infty, \tau_{\mathcal{U}^\infty(T)})\).

**Remark 4.2.** It should be noted from Definition 3.1, it follows that if \(f \in \mathcal{U}^\infty(T)\), \((\mu, \nu) \rightarrow f^\beta(\mu \nu): T^\infty \times (T \cup T^\infty) \rightarrow \mathbb{C}\) is separately continuous, so that \(f^\beta(\mu \nu) = f^\beta(\nu \mu)\) for all \(\mu, \nu \in T^\infty\). Therefore \(f \in \mathcal{W}_1^\infty(T)\)(see [1], for details).
Proposition 4.3. $\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ is a congruence relation on $T^{\omega}$.

Proof. To prove that $\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ is congruence, we use Remark 4.2 and Theorem 3.4. Let $\mu, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \mu'$ and $\nu, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \nu'$ where $\mu, \nu, \mu', \nu' \in T^{\omega}$.

Pick $f \in \mathcal{U}(T)$. Then

$$f^\beta(\mu \nu) = (L_\nu f)^\beta(\mu) = (L_\nu f)^\beta(\mu') = f^\beta(\nu \mu') = (L_{\nu'} f)^\beta(\nu) = (L_{\nu'} f)^\beta(\nu') = f^\beta(\nu' \mu') = f^\beta(\mu' \nu')$$

Thus $\mu \nu, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \mu' \nu'$, as claimed. □

Proposition 4.4. $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$ is a compact topological semigroup.

Proof. We know that $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$ is a compact space. Now let $(\mu_\alpha)$ be a net in $T^{\omega}$ converging to $\mu$ in $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$. Then $\mu_{\alpha_\beta} \to \mu_0$ in $(T^{\omega}, \tau_{\mathcal{U}^{\omega}}(T))$ for some subnet $(\mu_{\alpha_\beta})$ of $(\mu_\alpha)$. Since identity map from $(T^{\omega}, \tau)$ onto $(T^{\omega}, \tau_{\mathcal{U}^{\omega}}(T))$ is continuous, it follows that $\mu_{\alpha_\beta} \to \mu_0$ in $(T^{\omega}, \tau_{\mathcal{U}^{\omega}}(T))$. Hence for each $f \in \mathcal{U}^{\omega}(T)$, $f^\beta(\mu_0) = \lim_{\beta} f^\beta(\mu_{\alpha_\beta}) = f^\beta(\mu)$. So $\mu_0, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \mu$. Similarly if $\nu_\alpha \to \nu$ and $\nu_{\alpha_\beta} \to \nu_0$ in $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$ then $\nu_0, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \nu$. Hence, as $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$ is a congruence relation on $T^{\omega}$ by Proposition 4.3, then $\mu_0 \nu_0, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \mu \nu$, so that $f^\beta(\mu_0 \nu_0) = f^\beta(\mu \nu)$ for all $f \in \mathcal{U}^{\omega}(T)$. Now let $\mu \to \mu, \nu_\alpha \to \nu$ in $(T^{\omega}, \mathcal{R}^{\omega}_{\mathcal{U}}(T))$, let $(\mu_{\alpha_\beta} \nu_{\alpha_\beta})$ be a subnet of $(\mu_\alpha \nu_\alpha)$. Using compactness of $(T^{\omega}, \tau)$ we find subsequents (as $\mu_{\alpha_{\beta_\gamma}}, \nu_{\alpha_{\beta_\gamma}} \to \mu_0, \nu_{\alpha_{\beta_\gamma}} \to \nu_0$ in $(T^{\omega}, \tau)$. Then for each $f \in \mathcal{U}^{\omega}(T)$, we have $\lim_{\gamma} f^\beta(\mu_{\alpha_{\beta_\gamma}} \nu_{\alpha_{\beta_\gamma}}) = f^\beta(\mu_0 \nu_0) = f^\beta(\mu \nu)$, since $(\mu, \nu) \to f^\beta(\mu \nu); T^{\omega} \times (T \cup T^{\omega} \cup TT^{\omega}) \to \mathbb{C}$ is jointly continuous, hence $\mu, \nu \to f^\beta(\mu \nu)$ as required. □

Corollary 4.5. Let the quotient semigroup $T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ have the quotient topology. Then $T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ is compact Hausdorff topological semigroup.

Proof. Use Definition 4.1 and Proposition 4.3. □

Corollary 4.6. $T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ is a commutative semigroup.

Proof. Take $\mu, \nu \in T^{\omega}$ and $f \in \mathcal{U}^{\omega}(T)$. Then $f^\beta(\mu \nu) = f^\beta(\nu \mu)$ (Remark 4.2), and thus $\mu \nu, \mathcal{R}^{\omega}_{\mathcal{U}}(T) \nu \mu$, which implies the assertion. □

We conclude this section with some results (both algebraic, topological) on $T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T)$.

Theorem 4.7. $K(T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T))$ the minimal ideal of $T^{\omega}/\mathcal{R}^{\omega}_{\mathcal{U}}(T)$ is compact topological group.
\textit{Proof.} This follows from Corollaries 4.5, 4.6 and Corollary 1.5.3[3]. \hfill \Box

\textbf{Remark 4.8.} For each $n \in \mathbb{N}$, write $H_n = \{ \mu \in T^\infty: \ c^\beta(\mu) = n \}$ and $H_\infty = \{ \mu \in T^\infty: \ c^\beta(\mu) = \infty \}$. Then $T^\infty = H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots \cup H_\infty$. Hence $H_n$ is clopen and each $\mu \in H_n$ is a limit of a net $(x_\alpha)$ in $T$ with $c(x_\alpha) = n$ for each $\alpha$. Further, $H_n H_m \subseteq H_{n+m}$ for all $m, n \in \mathbb{N}$, so $H_1 \cup H_2 \cup \ldots \cup H_n \cup \ldots$ is a subsemigroup of $T^\infty$. Recall that by Definition 2.2(ii), $l$ is the length function and $r(x) = 1/l(x)$, $x \in T$.

\textbf{Lemma 4.9.} Let $\xi \in H_n$ for some $n \in \mathbb{N}$ with $l^\beta(\xi) < \infty$, let $\pi: T^\infty \rightarrow T^\infty/R_{U^\infty(T)}$ be the quotient map. Then $\pi(\xi)$ is not a product.

\textit{Proof.} Let $g_1 \pi = l^\beta$. Then $g_1$ is a continuous function on $T^\infty/R_{U^\infty(T)}$, since $r = 1/l$ and $r \in U^\infty(T)$ (Example 3.2 (ii)) and that $T^\infty/R_{U^\infty(T)}$ have the quotient topology (see [9], Chapter 3, Theorem 9). If $\pi(\xi) = \pi(\xi_1) \pi(\xi_2)$ for some $\xi_1, \xi_2 \in T^\infty$, then $l^\beta(\xi) = l^\beta(\xi_1 \xi_2) = \infty$ (Proposition 2.3), which contradicts $l^\beta(\xi) < \infty$. \hfill \Box

\textbf{Theorem 4.10.} $T^\infty/R_{U^\infty(T)}$ has no identity.

\textit{Proof.} If $\pi(e)$ is an identity element for $T^\infty/R_{U^\infty(T)}$, then $\pi(\xi) = \pi(e) \pi(\xi) = \pi(\xi) \pi(e)$ for all $\xi \in T^\infty$, which is impossible by Lemma 4.9.

\textbf{Remark 4.11.} It is easy to verify that, $l^\beta(e) = \infty$, whenever $\pi(e)$ is an idempotent in $T^\infty/R_{U^\infty(T)}$. We denote the set of all idempotents in $T^\infty/R_{U^\infty(T)}$ by $E(T^\infty/R_{U^\infty(T)})$. Thus we obtain that $E(T^\infty/R_{U^\infty(T)}) \subseteq \{ \pi(\xi): \xi \in T^\infty, l^\beta(\xi) = \infty \}$.

\textbf{Proposition 4.12.} Let $\xi \in H_n$ for some $n \in \mathbb{N}$ with $l^\beta(\xi) < \infty$. Then $\pi(\xi)$ is not a left zero.

\textit{Proof.} Left zeros are idempotents and we saw above that $l^\beta(\xi) = \infty$ if $\xi$ is an idempotent. \hfill \Box

We next have the following theorem.

\textbf{Theorem 4.13.} $T^\infty/R_{U^\infty(T)}$ is not a left zero semigroup.

\textbf{Proposition 4.14.} $(T^\infty/R_{U^\infty(T)})^2$ is not dense in $T^\infty/R_{U^\infty(T)}$.

\textit{Proof.} Let $g, \pi = r^\beta$, where $\pi: T^\infty \rightarrow T^\infty/R_{U^\infty(T)}$ is the quotient map. Then $g_\pi$ is continuous on $T^\infty/R_{U^\infty(T)}$, since $r \in U^\infty(T)$ (Example 3.2 (ii)), and that $T^\infty/R_{U^\infty(T)}$ have the quotient topology. By Corollary 2.4,

$$(T^\infty/R_{U^\infty(T)})^2 \cap g_\pi^{-1}(0,1) = \emptyset$$

But $g_\pi^{-1}(0,1)$ is a non-empty open set in $T^\infty/R_{U^\infty(T)}$ as claimed. \hfill \Box
Remark 4.15. Let $\mu_0 \in T^\infty$ be the cluster point of the sequence $(u_n u_{n+1})_{n=1}^\infty$ in $\beta T$, where $u_n \in U$ for all $n \in \mathbb{N}$, let $r_\beta = r_\beta(\mu_0)$ (Proposition 4.14). Then $r_\beta(\mu_0) = 1/4$ and since by Corollary 2.4, $r_\beta((T^\infty)^2) = 0$ which implies that $\pi(\mu_0)$ is not the limit of a net of elements $(T^\infty/\mathcal{U}^\infty)_T^2$. Thus we obtain an alternative proof of the Proposition 4.14 which will be required in the next result.

Theorem 4.16. $T^\infty/\mathcal{U}^\infty(T)$ is not a left (resp, right) simple semigroup.

Proof. Indeed, $T^\infty/\mathcal{U}^\infty(T) \pi(\mu_0) \subseteq (T^\infty/\mathcal{U}^\infty(T))^2$, and $T^\infty/\mathcal{U}^\infty(T) \pi(\mu_0)$ is closed in $T^\infty/\mathcal{U}^\infty(T)$. Thus $T^\infty/\mathcal{U}^\infty(T) \pi(\mu_0) \neq T^\infty/\mathcal{U}^\infty(T)$ by Proposition 4.14, so $T^\infty/\mathcal{U}^\infty(T)$ is not a left simple semigroup. Thus $T^\infty/\mathcal{U}^\infty(T)$ is not a group (see [3], for more details).

From Theorem 4.16 and Definition 1.5.6[3], we get the following result.

Corollary 4.17. $T^\infty/\mathcal{U}^\infty(T)$ is not topologically left (resp, right) simple.

Corollary 4.18. $T^\infty/\mathcal{U}^\infty(T)$ is not cancellative (and hence is not group).

Proof. Use Theorem 4.16 and Corollaries 3.13, 3.14[3].

Remark 4.19. If for each $k \in \mathbb{N}$, let $x_m^{(k)} = u_m u_{m+1} \ldots u_{m+k-1}, m \in \mathbb{N}$ and $y_n = u_n u_{n+1} \ldots u_{n+2}, n \in \mathbb{N}$, where $u_n \in U$ for all $n$, then supp $x_m^{(k)} \rightarrow \infty$, supp $y_n \rightarrow \infty$. Let $\mu(k), \nu \in T^\infty$ be the cluster points of $(x_m^{(k)})_{m=1}^\infty, (y_n)_{n=1}^\infty$ in $\beta T$ respectively. Then $l_\beta(\mu(k)) = k$, $l_\beta(\nu) = \infty$. Now, suppose that $g_k \pi = l_\beta(\nu)$ (Lemma 4.9). Then $g_k \pi(\mu(k)) = l_\beta(\mu(k)) = k, g_k \pi(\nu) = l_\beta(\nu) = \infty$, which implies that $g_k$ and $l_\beta$ map $T^\infty/\mathcal{U}^\infty(T)$ and $T^\infty$ onto the one-point compactification $\mathbb{N} \cup \{\infty\}$ respectively.

Next we shall prove the following result.

Proposition 4.20. $(g_k^{-1}(\infty))^2$ is not dense in $g_k^{-1}(\infty)$ (and hence is not dense in $T^\infty/\mathcal{U}^\infty(T)$).

Proof. Let $k$ be as in Definition 2.2 (iii). Define $g_k \pi = k_\beta$, where $\pi: T^\infty \rightarrow T^\infty/\mathcal{U}^\infty(T)$ is the quotient map. Then $g_k$ is a continuous function on $T^\infty/\mathcal{U}^\infty(T)$, since $k \in \mathcal{U}^\infty(T)$ (Example 3.2 (iii)). Suppose now that $\pi(\nu) \neq g_k^{-1}(\infty)$, since $l_\beta(\nu) = \infty$ and $g_k \pi = l_\beta$. On the other hand, $k_\beta(\nu) = 1$, since $x(\nu) = 0$ (see Definition 2.2 (iii)). But $(g_k^{-1}(\infty))^2 \subseteq (T^\infty/\mathcal{U}^\infty(T))^2$, an application of Corollary 2.6 then shows that $g_k(g_k^{-1}(\infty))^2 =$
0. Hence, $\pi(\nu) \notin c\ell(g^{-1}_T(\infty))^2$, which implies the desired conclusion.

Recall that by Definition 2.2(ii), $c^\beta$ is a continuous homomorphism on $T^\infty$, and $h = 1/c$, $h \in U^\infty(T)$ (Example 3.2 (i)). Let $\pi: T^\infty \to T^\infty/RU^\infty(T)$ be the quotient map, and let $g c = c^\beta$. Then $g_c$ is a continuous homomorphism on $T^\infty/RU^\infty(T)$. Let $\mu^{(k)}, \nu$ be as in Remark 4.19. Then $c^{\beta}(\mu^{(k)}) = k, c^{\beta}(\nu) = \infty$, and so we obtain that $g_c$ maps $T^\infty/RU^\infty(T)$ onto the one-point compactification $\mathbb{N} \cup \{\infty\}$. □

The proof of the following proposition is essentially the same as that of Proposition 4.20.

**Proposition 4.21.** $(g^{-1}_c(\infty))^2$ is not dense in $g^{-1}_c(\infty)$ (and hence is not dense in $T^\infty/RU^\infty(T)$).

**Theorem 4.22.** Let $\pi: T^\infty \to T^\infty/RU^\infty(T)$ be the quotient map. Then the set $\{\pi(\xi): \xi \in T^\infty, l^\beta(\xi) < \infty\}$ is not dense in $T^\infty/RU^\infty(T)$.

**Proof.** Let $\left(\chi_n\right)_{n=1}^\infty, \nu \in T^\infty$ be as in Remark 4.19. Put $Y = \{\chi_n: n \in \mathbb{N}\}$ and let $1_Y$ be the indicator function of $Y$. Then $1_Y(\nu) = 1$. We complete the proof by showing that $1_Y \in U^\infty(T)$. To see this, suppose that $\mu \in T^\infty, \eta \in (T \cup T^\infty \cup TT^\infty)$ and let $x_\alpha \to \mu$ for some net $(x_\alpha)$ in $T$ with $\text{supp } x_\alpha \to \infty$. If $\eta = t \in T$, then eventually $t < \text{supp } x_\alpha$ and for such $\alpha$, eventually $\text{supp } x_\alpha < \text{supp } \chi_n$, so that eventually $tx_\alpha \notin Y$. Hence $1_Y(t \mu) = 0$. If $\eta \in T^\infty$ and $y_\beta \to \eta$ for some net $(y_\beta) \in T$ with $\text{supp } y_\beta \to \infty$, then $1_Y(\mu \nu) = \lim_\alpha \lim_\beta 1_Y(x_\alpha y_\beta) = 0$ by a similar reason. Finally, if $\eta = t\lambda, \lambda \in T^\infty$. Then $\mu \lambda \in T^\infty$ since $T^\infty$ is a semigroup and hence $1_Y(\mu \eta) = 1_Y(t \mu \lambda) = 0$. Consequently, $1_Y \in U^\infty(T)$.

Let $g_1, \pi = 1_T$. Then $g_1$ is continuous on $T^\infty/RU^\infty(T)$. Take $\xi \in T^\infty$ with $l^\beta(\xi) = k, k \in \mathbb{N}$. There exists a net $(z_\nu)$ in $T$ such that $z_\nu \to \xi$ with $\text{supp } z_\nu \to \infty$. It follows that, eventually $l(z_\nu) = k$, hence eventually $z_\nu \notin Y$.

Therefore $1_Y(\xi) = 0$, so $g_1, \pi(\xi) = 0$. On the other hand, $g_1, \pi(\nu) = 1_Y(\nu) = 1$ and hence $\pi(\nu) \notin c\ell\{\pi(\xi): \xi \in T^\infty, l^\beta(\xi) < \infty\}$, which implies the desired conclusion. □
For remainder of this section we consider the more general results of describing about inclusion between $\mathcal{U}^\infty(T)$ and $\textit{WAP}^\infty(T)$ (see [1], for more details).

**Theorem 4.23.** For $f \in C(T)$, if $(\mu, \nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \to \mathbb{C}$ is separately continuous, then $(\mu, \nu, \eta) \to f^\beta(\mu\eta\nu): T^\infty \times (T \cup T^\infty \cup TT^\infty) \times T^\infty \to \mathbb{C}$ is also separately continuous.

**Proof.** Let $(\mu_a)$ is a net in $T^\infty$ converging to $\mu \in T^\infty$ and let $\nu \in (T \cup T^\infty \cup TT^\infty)$, $\eta \in T^\infty$.

(i) If $\nu \in T^\infty$, then as is readily verified that $f^\beta(\mu_a\nu) \to f^\beta(\mu\nu)$, since $\nu \eta \in T^\infty$ and $(\mu, \nu) \to \mu\nu: \beta T \times T^\infty \to \beta T$ is right continuous.

(ii) If $\nu = t\lambda$, $t \in T$ and $\lambda \in T^\infty$. Then $\lambda\eta \in T^\infty$ since $T^\infty$ is semigroup and $t\mu_a \to t\mu$ in $T$. Using Definition 3.5[1] and that $(\mu, \nu) \to \mu\nu: \beta T \times T^\infty \to \beta T$ is right continuous, it follows that $f^\beta(\mu_a\nu) \to f^\beta(\mu\nu)$.

(iii) If $\nu = t \in T$, then by a similar argument, $f^\beta(\mu_a\nu) \to f^\beta(\mu\nu)$.

Let $(\eta_a)$ is a net in $T^\infty$ converging to $\eta \in T^\infty$. Take $\mu \in T^\infty$, $\nu \in (T \cup T^\infty \cup TT^\infty)$.

(i) Let $\nu \in T^\infty$. Then $\mu\nu \in T^\infty$. In fact, $\eta_a \to \eta$ in $(T \cup T^\infty \cup TT^\infty)$ and by hypothesis, $(\mu, \nu) \to f^\beta(\mu\nu)$ is separately continuous on $T^\infty \times (T \cup T^\infty \cup TT^\infty)$, from which it follows that $f^\beta(\mu\eta_a) \to f^\beta(\mu\eta)$.

(ii) Let $\nu = t\lambda$, $t \in T$ and $\lambda \in T^\infty$. Then $\mu\lambda \in T^\infty$. Indeed, $t\eta_a \to t\eta$ in $(T \cup T^\infty \cup TT^\infty) \subseteq \beta T$ and $(\mu, \nu) \to f^\beta(\mu\nu)$ is separately continuous on $T^\infty \times (T \cup T^\infty \cup TT^\infty)$, hence $f^\beta(\mu\eta_a) \to f^\beta(\mu\eta)$.

(iii) Let $\nu = t \in T$. The proof is similar.

Finally, suppose that $(\nu_a)$ is a net in $(T \cup T^\infty \cup TT^\infty)$ converging to $\nu \in (T \cup T^\infty \cup TT^\infty)$, and let $\mu, \eta \in T^\infty$. Then $\nu_a \eta \to \nu\eta$ in $(T \cup T^\infty \cup TT^\infty)$. But $(\mu, \nu) \to f^\beta(\mu\nu)$ is separately continuous on $T^\infty \times (T \cup T^\infty \cup TT^\infty)$, hence $f^\beta(\mu\nu_a\eta) \to f^\beta(\mu\nu\eta)$. This proves our assertion. $\square$

**Theorem 4.24.** For $f \in C(T)$, let both $(\mu, \nu) \to f^\beta(\mu\nu): T^\infty \times (T \cup T^\infty) \to \mathbb{C}$ and $(\mu, \nu, \eta) \to f^\beta(\mu\eta\nu): T^\infty \times (T \cup T^\infty) \times T^\infty \to \mathbb{C}$ are separately continuous. Then $f \in W^\infty_2(T)$.

**Proof.** We use Theorem 3.11[1]. Suppose $\mu, \nu, \eta \in T^\infty$, and $(x_a)$ is a net converging to $\mu$ with $\text{supp}\ x_a \to \infty$. Then $f^\beta(x_a\eta\nu) \to f^\beta(\mu\eta\nu)$, since $(\mu, \nu) \to \mu\nu: \beta T \times T^\infty \to \beta T$ is right continuous, and $\eta\nu \in T^\infty$. By hypothesis, $(\mu, \nu) \to f^\beta(\mu\nu)$ is separately continuous on $T^\infty \times (T \cup T^\infty)$, hence from Remark 4.2, and that $T^\infty$ is a semigroup we see that
\[ f^\beta (\mu \eta \nu) = f^\beta (\nu \mu \eta) = f^\beta (\eta \mu \nu) = f^\beta (\eta \nu \mu) = f^\beta (\mu \nu \eta). \]

On the other hand, \( \lim_{n} f^\beta (x_n \eta \nu) = \lim_{n} f^\beta (\eta x_n \nu) = f^\beta (\eta \mu \nu), \) since \( (\mu, \nu, \eta) \to f^\beta (\mu \nu \eta) \) is separately continuous on \( T^\infty \times (T \cup T^\infty) \times T^\infty \) and \( x_n \to \mu \min (T \cup T^\infty). \) Thus \( f^\beta (\mu \eta \nu) = f^\beta (\eta \nu \mu), \) and we have that \( f \in W^\infty_2 (T), \) as desired. \[ \Box \]

Definition 3.1 and the preceding theorems now imply the following result.

**Corollary 4.25.** Let \( T \) be a commutative standard oid. Then \( \mathcal{U}^\infty (T) \subseteq W^\infty_2 (T). \)

**Corollary 4.26.** Let \( T \) be a commutative standard oid. Then \( \mathcal{U}^\infty (T) \subseteq W^\infty_{\text{AP}} (T). \)

The proof now follow by the preceding Corollary, Remark 4.2 and Definition 4.1[1]. \[ \Box \]

**Example 4.27.** We have already seen that \( \mathcal{U}^\infty (T) \subseteq W^\infty_{\text{AP}} (T). \) Now we show that the converse is not true. For this purpose, consider two sets \( E = \left\{ (x(n))_{n \in \mathbb{N}} \in T : x(1) = 1 \right\} \) and \( F = \left\{ (x(n))_{n \in \mathbb{N}} \in T : x(1) = \infty \right\}. \) Then \( T = E \cup F. \) Now let \( (u_{2m})_{m \in \mathbb{N}} \) and \( (u_{2n+1})_{n \in \mathbb{N}} \) be two sequences in \( T \) such that \( u_{2m}, u_{2n+1} \in U \) for all \( m, n. \) Take \( u_1 \in U. \) Put \( D = \{ u_1 u_{2m} u_{2n+1} : m > n \} \subseteq F. \)

Define \( f : T \to \mathbb{C} \) by

\[
    f(x) = \begin{cases} 
        1/c(x), & x \in E \\
        1, & x \in D \\
        0, & \text{otherwise}
    \end{cases} \quad (x \in T)
\]

Then \( f \in W^\infty_1 (T) \) (Example 3.7[1]). We show that \( f \in W^\infty_2 (T) \) (Definition 3.9[1]). Assume that \( (x_\alpha), (y_\beta) \) and \( (z_\gamma) \) be nets in \( T \) with \( \text{supp} x_\alpha \to \infty, \) \( \text{supp} y_\beta \to \infty \) and \( \text{supp} z_\gamma \to \infty. \) Then (for all sufficiently large) \( x, y_\beta, z_\gamma \) \( x_\alpha (1) = 1, y_\beta (1), z_\gamma (1) = 1 \), so that \( x_\alpha, y_\beta, z_\gamma \in E. \) It is therefore easy to verify that \( f \in W^\infty_2 (T), \) since \( h = 1/c \in W^\infty_2 (T) \) (Example 3.10[1]), hence \( f \in W^\infty_{\text{AP}} (T). \) To prove that \( f \notin \mathcal{U}^\infty (T), \) we remind the reader that for any \( f \in \mathcal{U}^\infty (T), f^\beta (t \mu \nu) = f^\beta (t \nu \mu), \) where \( t \in T \) and \( \mu, \nu \in T^\infty. \) We assume that \( \mu, \nu \) be the cluster points of the sequences \( (u_{2m})_{m \in \mathbb{N}}, (u_{2n+1})_{n \in \mathbb{N}}. \)
respectively. Then \( \mu, \nu \in T^{\omega}, \) since \( \text{supp} \ u_{2m} \to \infty \) and \( \text{supp} \ u_{2n+1} \to \infty. \)

Hence, \( \lim_m \lim_n f(u_1u_{2m}u_{2n+1}) = 0 \)

\( \lim_m \lim_n f(u_1u_{2m}u_{2n+1}) \) = \( 1, \) so that

Both iterated limits off \( (u_1u_{2m}u_{2n+1}) \) exist. But \( \lim_m \lim_n f(u_1u_{2m}u_{2n+1}) = f^\beta (u_1 \mu \nu), \) and \( \lim_n \lim_m f(u_1u_{2m}u_{2n+1}) = f^\beta (u_1 \nu \mu), \) which is clearly impossible. Thus \( f \notin \mathcal{U}^{\omega}(T), \) as claimed.

\[\Box\]

5. Idempotents

Recall that \( T^{\omega} \) is a compact right topological semigroup and \( R^{\omega}(T) \) is a relation on \( T^{\omega} \) (Definition 4.1). Then \( R^{\omega}(T) \) is congruence and under certain topology on \( T^{\omega}, \) is also a closed relation, so that the quotient space \( T^{\omega}/R^{\omega}(T) \) is a compact topological semigroup (Corollary 4.5), which is commutative (Corollary 4.6). In this section we are concerned with obtaining of idempotents \( T^{\omega}/R^{\omega}(T). \) In connection with the present section special suboids of an oid \( T \) play an important role. Each special suboid corresponds to a strictly increasing sub sequence of \( \mathbb{N}. \) A more details analysis of special suboids can be found in [1].

For an infinite subset \( \Delta \subseteq \mathbb{N} \) the special suboid of an oid \( T \) corresponding to

The strictly increasing sequence of \( \Delta \) is denoted by \( S(\Delta). \) Then \( S(\Delta) \) produces a compact right topological semigroup \( S^{\omega}(\Delta) \) which is a sub semigroup of \( T^{\omega}. \)

Indeed, \( \pi(S^{\omega}(\Delta)) \) is a compact commutative sub semigroup of \( T^{\omega}/R^{\omega}(T), \) where \( \pi: T^{\omega} \to T^{\omega}/R^{\omega}(T) \) is the quotient map.

**Proposition 5.1.** Let \( \Delta \subseteq \mathbb{N} \) be an in finite set and let \( 1_{S(\Delta)} \) be the indicator function of \( S(\Delta). \) Then \( 1_{S(\Delta)}^\beta (\mu \nu) = 1_{S(\Delta)}^\beta (\mu) 1_{S(\Delta)}^\beta (\nu) \) for all \( \mu \in T^{\omega} \) and \( \nu \in (T \cup T^{\omega}) \cap T^{\omega} \cup TT^{\omega}. \)

**Proof.** It is a straightforward argument that \( 1_{S(\Delta)} (xy) = 1_{S(\Delta)}(x) 1_{S(\Delta)}(y). \) For \( x, y \in T \) such that \( \text{(supp} \ x) \cap \text{(supp} \ y) = \emptyset. \) Suppose that \( t \in T, \mu, \nu \in T^{\omega}, \) such that \( t x_a \to \mu \) for some net \( (x_a) \) in \( T \) with \( \text{supp} x_a \to \infty. \) Then eventually \( \text{supp} t < \text{supp} x_a, \) so that eventually \( 1_{S(\Delta)} (tx_a) = 1_{S(\Delta)}(t) 1_{S(\Delta)}(x_a). \)

Since \( [0,1] \) is a compact commutative semigroup with the usual multiplication, and \( t \mu = \lim_a tx_a, \) it follows that \( 1_{S(\Delta)}^\beta (t \mu) = 1_{S(\Delta)}^\beta (t) 1_{S(\Delta)}^\beta (\mu). \) Now let \( \nu \in T^{\omega} \) with \( y_\nu \to \nu \) for some net \( (y_\nu) \) in \( T \) such that \( \text{supp} y_\nu \to \infty. \) Then by a similar reason, we see that

\[ 1_{S(\Delta)}^\beta (\mu \nu) = \lim_a \lim_\beta 1_{S(\Delta)}^\beta (x_a y_\nu) = \lim_a 1_{S(\Delta)}^\beta (x_a) 1_{S(\Delta)}^\beta (\nu) = 1_{S(\Delta)}^\beta (\mu) 1_{S(\Delta)}^\beta (\nu). \]
Finally if \( v = t\lambda, t \in T, \lambda \in T^\infty \) then \( \mu\lambda \in T^\infty \) and \( \mu v = \mu t\lambda \). Thus, by above paragraph, we obtain that \( 1_{S(A)}^\beta(\mu v) = 1_{S(A)}^\beta(\mu)1_{S(A)}^\beta(v) \), as desired. \( \Box \)

**Corollary 5.2.** For an infinite set \( A \subseteq \mathbb{N} \), the indicator function \( 1_{S(A)} \) of \( S(A) \) is in \( \mathcal{U}^\infty(T) \).

**Proof.** To show that \( 1_{S(A)} \in \mathcal{U}^\infty(T) \), it is adequate by Theorem 3.7, to show that \( v \to L_v f^\beta : (T \cup T^\infty \cup TT^\infty) \to C(T^\infty) \) is norm-continuous. Assume that \( (v_\alpha) \) be any net in \( (T \cup T^\infty \cup TT^\infty) \) such that \( v_\alpha \to v \) in \( (T \cup T^\infty \cup TT^\infty) \). Then \( 1_{S(A)}^\beta(v_\alpha) \to 1_{S(A)}^\beta(v) \). Thus for \( \varepsilon > 0 \), there exists \( \alpha_0 \) such that for \( \alpha \geq \alpha_0 \),
\[
\left| 1_{S(A)}^\beta(\mu)1_{S(A)}^\beta(v_\alpha) - 1_{S(A)}^\beta(\mu)1_{S(A)}^\beta(v) \right| < \varepsilon/2.
\]
However, by definition of \( 1_{S(A)}^\beta \), we have \( 1_{S(A)}^\beta(\mu) \in \{0,1\} \) for all \( \mu \in T^\infty \), which is clearly that from Proposition 5.1, and for \( \alpha \geq \alpha_0 \),
\[
\left| 1_{S(A)}^\beta(\mu)1_{S(A)}^\beta(v_\alpha) - 1_{S(A)}^\beta(\mu)1_{S(A)}^\beta(v) \right| < \varepsilon/2.
\]
Hence, for \( \alpha \geq \alpha_0 \),
\[
1_{S(A)}^\beta(v_\alpha) - 1_{S(A)}^\beta(v) \to v \text{ in } \mathcal{U}(T^\infty).
\]
We finish the present section by giving the following main result. In connection with this result we will use non-principal ultrafilters on \( \mathbb{N} \). We remind the reader that, if \( \mathcal{F} \) is a non-principal ultrafilter on \( \mathbb{N} \) and \( A \in \mathcal{F} \), then \( A \) is an infinite set. Moreover, the number of non-principal ultrafilters on \( \mathbb{N} \) is \( 2^\mathbb{C} \) [8].

**Theorem 5.4.** \( T^\infty/\mathcal{R}_{\mathcal{U}^\infty}(T) \) contains at least \( 2^\mathbb{C} \) idempotents.

**Proof.** This uses Corollary 5.2, the proof is essentially the same as that of Proposition 7.1[1]. \( \Box \)

6. Free abelian groups

*In connection with Theorem 4.7, of the Section 4, it should be mentioned that, it is possible for a compact commutative topological semigroup \( T^\infty/\mathcal{R}_{\mathcal{U}^\infty}(T) \) have a minimal idempotent with a unique minimal left ideal and a unique minimal right ideal, so that \( K(T^\infty/\mathcal{R}_{\mathcal{U}^\infty}(T)) \), the minimal ideal of \( T^\infty/\mathcal{R}_{\mathcal{U}^\infty}(T) \) is a maximal group (see [3], for more details). The aim of this Section is the search for existence in \( K(T^\infty/\mathcal{R}_{\mathcal{U}^\infty}(T)) \) a free abelian group on \( 2^\mathbb{C} \) generator. Let us first give the following result.*
Lemma 6.1. Let $\psi$ be an arbitrary function from $\mathbb{N}$ to $\mathbb{R}$, let $f: T \to \mathbb{C}$ be defined by $f(t) = \exp\sum[\psi(n) : n \in \text{supp} t]$. Then $f \in \mathcal{U}^\infty(T)$.

Proof. It is easily seen that $f^\beta(t\mu) = f(t)f^\beta(\mu)$ for $t \in T, \mu \in T^\infty$, since $f$ is an old-map (that is, $f(st) = f(s)f(t)$ whenever $s, t \in T$ with $(\text{supp} s) \cap (\text{supp} t) = \emptyset$). Moreover, $f^\beta$ is a homomorphism of $T^\infty$ to the circle group $\mathbb{I}$.

Thus $f^\beta(\mu v) = f^\beta(\mu)f^\beta(v)$ for $\mu \in T^\infty, v \in (T \cup T^\infty \cup TT^\infty)$. To prove that $f \in \mathcal{U}^\infty(T)$, we show that $v \to L_vf^\beta:(T \cup T^\infty \cup TT^\infty) \to C(T^\infty)$ is norm-continuous (Theorem 3.7). Now suppose that $(v_\alpha)$ be any net in $(T \cup T^\infty \cup TT^\infty)$ with $v_\alpha \to v$ in $(T \cup T^\infty \cup TT^\infty)$.

Then $f^\beta(v_\alpha) \to f^\beta(v)$, hence for $\varepsilon > 0$, there exists $\alpha_0$ such that for $\alpha \geq \alpha_0, |f^\beta(v_\alpha) - f^\beta(v)| < \varepsilon$. But since $|f^\beta(\mu)| = 1$ for all $\mu \in \beta T$, it follows directly that for $\alpha \geq \alpha_0$ and for all $\mu \in T^\infty$,

$$|f^\beta(\mu f^\beta(v_\alpha)) - f^\beta(\mu f^\beta(v))| = |f^\beta(\mu)||f^\beta(v_\alpha) - f^\beta(v)| < \varepsilon/2.$$  

Thus, for $\alpha \geq \alpha_0, \|L_v f^\beta - L_v f^\beta\| < \varepsilon$, that is $f \in \mathcal{U}^\infty(T)$. \hfill $\Box$

We remind the reader that, if $M = \{\mu \in T^\infty: c^\beta(\mu) = 1\}$, then it is easy to verify that $M = \{\mu \in T^\infty: \mu \in \text{cl}\{u_m : m \in \mathbb{N}\}\}$, and $\text{card}(M) = 2^c$ (see [2], Remark 5.8). Recall that, $\pi: T^\infty \to T^\infty/\mathcal{R}_{\mathcal{U}}(T)$ is the quotient map which is a Continuous epimorphism. We now give the following result which is the last major result of this section.

Theorem 6.2. $K(T^\infty/\mathcal{R}_{\mathcal{U}}(T))$ contains a free abelian group on $2^c$ generators.

Proof. This use Lemma 6.1 and Corollary 4.26, the proof is similar to that of Theorem 8.1[1]. \hfill $\Box$

References


