Fixed Point Theorems for Multi-valued Weakly C -contractive Mappings in Quasi-ordered Metric Spaces

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Abstract
The goal of this paper is to present some common fixed point theorems for multivalued weakly C-contractive mappings in quasi-ordered complete metric space. These results generalizes the existing fixed point results in the literature.

Keywords: Multivalued mapping, Hausdorff distance, Weakly C-contractive mapping, Common fixed point.

1. Introduction

Fixed point theory for contractive mapping first studied by Banach [1]. He proved that every contraction defined on a complete metric space has a unique fixed point. Since then the fixed point theory for single valued and multivalued mappings in metric space has been rapidly developed. In 1972, Chatterjea [2] introduce the concept of C-contraction as follows.

Definition1.1. A mapping $T : X \rightarrow X$ where $(X, d)$ is a metric space is said to be a C-contraction if there exists $k \in (0,0.5)$ such that for all $x, y \in X$ the following inequality holds:

$$d(Tx, Ty) \leq k((d(x, Ty) + d(y, Tx)).$$
Chatterjea [2] proved the following theorem:

**Theorem 1.1.** Every C-contraction in a complete metric space has a unique fixed point.

Choudhury [3] introduce the concept of weakly C-contractive mapping as a generalization of C-contractive mapping and prove that every weakly C-contractive mapping in a complete metric space has a unique fixed point.

**Definition 1.2.** Let \((X, d)\) be a metric space. A mapping \(T : X \to X\), is said to be weakly C-contractive if for all \(x, y \in X\),

\[
d(Tx, Ty) \leq \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \varphi(d(x, Ty), d(y, Tx)),
\]

Where \(\varphi : [0, \infty)^2 \to [0, \infty)\) is a continuous function such that \(\varphi(x, y) = 0\) if and only \(x = y = 0\).

Harjani et al. [5] announced some fixed point results for weakly C-contractive mappings in a complete metric space endowed with a partial order. Meanwhile, Shatanawi [9] proved some fixed point theorems for a nonlinear weakly C-contraction type mapping in metric and ordered metric spaces. In this paper, we introduce the concept of multivalued weakly C-contractive mappings in quasi-ordered partial metric spaces and we prove some existence theorems of common fixed point for such mappings in the context of complete quasi-partial metric spaces under certain conditions.

### 2. Preliminaries

Let \((X, d, \leq)\) be a quasi-ordered metric space, with an order \(\leq\) as a quasi-order (that is, a reflexive and transitive relation) and a metric \(d\). Assume that \(X\) having the following properties which appears in [8]:

**\(\text{(H1)}\):** if \(\{x_n\}\) is a non-decreasing (resp. non-increasing) sequence in \(X\) such that \(x_n \to x\), then \(x_n \leq x\) (resp. \(x_n \geq x\)) for all \(n \in \mathbb{N}\).

Let \(2^X\) denote the family consisting of all nonempty subsets of \(X\) we define the Hausdorff-Pseudo metric in \(H_d : 2^X \times 2^X \to \mathbb{R}_+ \cup \{\infty\}\) given by

\[
H_d(C, D) = \max\{\sup_{a \in C} d(a, D), \sup_{b \in D} d(C, b)\},
\]

where \(d(a, D) = \inf_{b \in D} d(a, b)\), \(d(C, b) = \inf_{a \in C} d(a, b)\).

**Definition 2.1.** Let \((X, d, \leq)\) be a quasi-ordered metric space. We say that \(X\) is sequentially complete if every Cauchy sequence whose consecutive terms are comparable in \(X\) converges.

**Definition 2.2.** [6,7] Let \(X\) be a quasi-ordered metric space. A subset \(D \subseteq X\) is said to be approximative if the multivalued mapping
\[ P_n(x) = \{ y \in D : d(x, y) = d(D, x) \}, \quad \forall x \in X \]

has nonempty values.

The multivalued mapping \( T : X \to 2^X \) is said to have approximative values, AV for short, if \( T x \) is approximative for each \( x \in X \).

The multivalued mapping \( T : X \to 2^X \) is said to have comparable approximative values, CAV for short, if \( T \) has approximative values and, for each \( z \in X \), there exists \( y \in P_{Tz}(x) \) such that \( y \) is comparable to \( z \).

The multivalued mapping \( T : X \to 2^X \) is said to have upper comparable approximative values, UCAV, for short (resp: lower comparable approximative values, LCAV for short) if \( T \) has approximative values and, for each \( z \in X \), there exists \( y \in P_{Tz}(x) \) such that \( y \geq z \) (resp: \( y \leq z \)). It is clear that \( T \) has approximative values if it has compact values. In addition, if \( T \) is single-valued, Then UCAV (LCAV) means that \( T x \geq x \) (\( T x \leq x \) for \( x \in X \).

**Definition 2.3.** The multivalued mappings \( T, S \) are said to have a common fixed point if there is \( x \in X \) such that \( x \in Tx \) and \( x \in Sx \).

In what follows, we give an analogy of the contraction which called multivalued C-weakly contraction mapping will play an important role in this sequel. To this end, we first introduce the following function.

Let \( a \in (0, \infty], R^+_a = [0, a) \). let \( f : R^+_a \to R \) satisfy,

(i) \( f(0) = 0 \) and \( f(t) > 0 \) for each \( t \in (0, a) \)

(ii) \( f \) is non-decreasing on \( R^+_a \)

(iii) \( f \) is continuous

(iv) \( f(t+s) \leq f(t) + f(s) \) for \( s, t \in R^+_a \).

For examples of such function \( f \) we refer to (6).

Define

\[ \mathfrak{B}(0, a) = \{ f \mid f \text{ satisfies (i)-(iv) above} \}. \]

Let \( a \in (0, \infty] \), \( \phi : R^+_a \times R^+_a \to R^+ \) satisfy

(i) \( \phi(t, s) = 0 \) if and only if \( s = t = 0 \).

(ii) \( \phi \) is continuous.

(iii) For any sequence \( \{r_n\} \) with \( \lim r_n = 0 \), there exist \( a \in (0, \frac{1}{2}) \) and \( n_0 \in \mathbb{N} \) such that \( \phi(r_n, 0) \geq (1-a)r_n^a \) (or \( \phi(0, r_n) \geq (1-a)r_n^a \)) for each \( n \geq n_0 \). Define

\[ \Phi([0, a] \times [0, a]) = \{ \phi : \phi \text{ satisfies (i)-(iii) above} \}. \]
Definition 2.3. Let $X$ be a metric space and $d = \sup\{d(x, y) : x, y \in X\}$. Set $a = d$ if $d = \infty$ and $a > d$ if $d < \infty$. Suppose the multivalued mappings $T, S : X \to 2^X$, $f \in \mathcal{F}(0, a)$ and $\varphi \in \Phi([0, f(a-0)) \times [0, f(a-0))]$ satisfy
\[
  f(H_d(Tx, Sy)) \leq f\left(\frac{1}{2}(d(x, Sy) + d(y, Tx))\right) - \varphi(f(d(x, Sy)), f(d(y, Tx)))
\]
for all $x, y \in X$ with $x$ and $y$ comparable. Then we say $T$ and $S$ satisfy weakly $C$-contraction with respect to $f$ and $\varphi$.

Definition 2.4. For two subsets $A$, $B$ of $X$, we say that $r_{AB}$ if, for each $a \in A$, there exists $b \in B$ such that $ab \leq r$ and each $a \in A$ and each $b \in B$ imply that $ab \leq r$. A multi-valued mapping $T : X \to 2^X$ is said to be $r$-non-decreasing ($r$-non-increasing) if $xy \leq r$ implies that $Tx \leq Ty$ ($Ty \leq Tx$) for all $x, y \in X$. $T$ is said to be $r$-monotone if $T$ is $r$-non-decreasing or $r$-non-increasing. The notion of non-decreasing (non-increasing) is similarly defined by writing $\leq$ instead of the notation $\leq$.

3. Main Result

In this section we established common fixed point theorems for multivalued mappings on quasi-ordered complete metric spaces. The idea of the present theorem originate from the study of analogous problem for single-valued mappings in [4] and [9], and multivalued mappings in [6], [7] and [10].

Theorem 3.1. Let $X$ be a quasi-ordered sequentially complete metric space and satisfy (H1). Suppose that the multivalued mappings $T$ and $S$ have UCAV and satisfy the weakly $C$-contraction with respect to $f$ and $\varphi$, then $T$ and $S$ have a common fixed point. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Sx_{2n+1}$ converges to the common fixed point of $T$ and $S$.

Proof: First we show that, if $T$ or $S$ has a fixed point it is a common fixed point of $T$ and $S$. Indeed, let $x$ be a fixed point of $T$ then we have,
\[
  f(d(x, Sx)) \leq f(H_d(Tx, Sx)) \\
  \leq f(0.5(d(x, Sx) + d(x, Tx))) - \varphi(f(d(x, Sx)), f(d(x, Tx))) \\
  = f(0.5d(x, Sx)) - \varphi(f(d(x, Sx), 0) \\
  \leq f(0.5d(x, Sx)) - \varphi(f(d(x, Sx), 0)
\]
This implies that, $\varphi(f(d(x, Sx)), 0) = 0$ and hence $f(d(x, Sx)) = 0$ therefore $d(x, Sx) = 0$. Since $x$ is AV, therefore there exist $y \in P_{x_0}(x)$ such that $d(y, x) = 0$ i.e, $y = x$. Thus
$x \in Sx$. Let $x_0 \in X$, if $x_0 \in Tx_0$ the proof is finished. Otherwise, from the fact that $Tx_0$ has UCAV it follows there exists $x_i \in Tx_0$ with $x_i \neq x_0$ and $x_i \geq x_0$ such that

$$d(x_0, x_i) = \inf_{x \in Tx_0} d(x, x_0) = d(Tx_0, x_0).$$

Again since $Sx_i$ has UCAV it follows there exist $x_2 \in Sx_i$ with $x_2 \neq x_i$ and $x_2 \geq x_i$ such that

$$d(x_i, x_2) = \inf_{x \in Sx_i} d(x, x_i) = d(Sx_i, x_i).$$

By induction and using UCAV, we can find in this way a sequence $\{x_n\}$ in $X$ with $x_{n+1} \geq x_n$ such that $x_{2n+1} \in Tx_{2n}$ and

$$d(x_{2n+1}, x_{2n}) = d(Tx_{2n}, x_{2n}).$$

and $x_{2n+2} \in Sx_{2n+1}$ with

$$d(x_{2n+2}, x_{2n+1}) = d(Sx_{2n+1}, x_{2n+1}).$$

On the other hand

$$d(Tx_{2n}, x_{2n}) \leq \sup_{x \in Sx_{2n-1}} d(Tx_{2n}, x) \leq H_d(Tx_{2n}, Sx_{2n-1}).$$

Therefore

$$d(x_{2n+1}, x_{2n}) \leq H_d(Tx_{2n}, Sx_{2n-1}). \quad (1)$$

Similarly we can show that

$$d(x_{2n+2}, x_{2n+1}) \leq H_d(Sx_{2n+1}, Tx_{2n}). \quad (2)$$

Now we show that $\lim_{n \to \infty} d(x_{n+1}, x_n) = 0$. By using (2) and since $f$ is non-decreasing, we have

$$f(0.5(d(x_{2n+1}, x_{2n+2})) \leq f(H_d(Tx_{2n}, Sx_{2n+1})) \leq f(0.5(f(d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n}))) - \phi(f(d(x_{2n}, Sx_{2n+1})), f(d(x_{2n+1}, Tx_{2n})))) \leq f(0.5(d(x_{2n}, x_{2n+2}))) - \phi(f(d(x_{2n}, x_{2n+2})), 0) \leq f(0.5(d(x_{2n}, x_{2n+2}))). \quad (3)$$

As $f$ is a non-decreasing function, we get

$$d(x_{2n+1}, x_{2n+2}) \leq 0.5d(x_{2n}, x_{2n+2}). \quad (3)$$

Since
\[ d(x_{2n}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}). \]

We have

\[ d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, x_{2n+1}). \quad (4) \]

Similarly, by using (1) one can show that

\[ d(x_{2n}, x_{2n+1}) \leq 0.5d(x_{2n-1}, x_{2n+1}). \quad (5) \]

Thus

\[ d(x_{2n}, x_{2n+1}) \leq d(x_{2n-1}, x_{2n}). \quad (6) \]

From (4) and (6), we have

\[ d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n), \quad \forall n \in N. \quad (7) \]

So, by (7) we get that \( \{d(x_n, x_{n+1}): n \in N\} \) is a non-increasing sequence. Hence there is \( r \geq 0 \) such that

\[ \lim_{n \to \infty} d(x_n, x_{n+1}) = r. \]

By (3) and (5) we have

\[ d(x_n, x_{n+1}) \leq 0.5d(x_{n-1}, x_{n+1}) \]
\[ \leq 0.5\left(d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right). \quad (8) \]

Letting \( n \to \infty \) and using (8), we get that

\[ r \leq \lim_{n \to \infty} 0.5d(x_{n-1}, x_{n+1}) \leq 0.5(r + r). \]

Hence

\[ \lim_{n \to \infty} d(x_{n-1}, x_{n+1}) = 2r. \]

Using the continuity \( f \), \( \varphi \) and (3), we get that

\[ f(r) \leq f(0.5(2r)) - \varphi(f(2r), 0), \]

which implies that \( \varphi(f(2r), 0) = 0 \) and hence \( r = 0 \).

Next we show that \( (x_n) \) is a Cauchy sequence in \( X \). Since \( \lim_{n \to \infty} f(d(x_{n-1}, x_{n+1})) = 0 \), from assumption (iii) of \( \varphi \) there exists \( 0 < a < \frac{1}{2} \) and \( n_0 \in N \) such that

\[ \varphi(f((d(x_{n-1}, x_{n+1})), 0) \geq af(d(x_{n-1}, x_{n+1})) \quad \text{for all} \quad n \geq n_0. \]

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On the other hand, for any given \( \epsilon > 0 \), we choose \( \delta > 0 \) to be small enough such that 
\[
    f(\delta) < \frac{a}{1 - 2a} f(\epsilon).
\]
Moreover, there exists \( n_1 \) such that \( d(x_{n+1}, x_n) \leq \delta \), for each \( n \geq n_1 \).

Now for any numbers \( m > n \geq \max\{n_0, n_1\} \), from the inequality (1) and (2) we have
\[
    f(d(x_{n+1}, x_n)) \leq f(H_d(Tx_n, Sx_{n-1})) \quad \text{or} \quad f(H_d(Tx_{n-1}, Sx_n))
\]
\[
    \leq f\left(0.5(d(x_n, Sx_{n-1}) + d(x_{n-1}, Tx_n))\right)
\]
\[
    - \phi\left(f\left(d(x_n, Sx_{n-1})\right), f\left(d(x_{n-1}, Tx_n)\right)\right)
\]
\[
    \leq f\left(0.5\left(d(x_{n-1}, x_{n+1})\right)\right) - \phi(0, f\left(d(x_{n-1}, x_{n+1})\right))
\]
\[
    \leq f\left(d(x_{n-1}, x_{n+1})\right) - (1 - a)f\left(d(x_{n-1}, x_{n+1})\right)
\]
\[
    \leq af\left(d(x_{n-1}, x_{n+1})\right)
\]
\[
    \leq a\left(f\left(d(x_{n-1}, x_n)\right) + f\left(d(x_n, x_{n+1})\right)\right).
\]

Therefore
\[
    f(d(x_n, x_{n+1})) \leq (a / (1 - a)) f(d(x_{n-1}, x_n)).
\]

Set \( \alpha = \frac{a}{1 - a} < 1 \). By repeating this procedure, for any \( k > n \) we obtain
\[
    f(d(x_k, x_{k-1})) \leq \alpha f(d(x_{k-1}, x_{k-2})) \leq \ldots \leq a^{k-n} f(d(x_n, x_{n-1})).
\]

Therefore, from the assumption of \( f \) we have,
\[
    f(d(x_n, x_m)) \leq f(d(x_m, x_{m-1})) + f(d(x_{m-1}, x_{m-2})) + \ldots + f(d(x_n, x_{n-1}))
\]
\[
    \leq \alpha^{m-n} f(d(x_n, x_{n-1})) + \alpha^{m-n-1} f(d(x_n, x_{n-1})) + \ldots
\]
\[
    + \alpha f\left(d(x_n, x_{n-1})\right)
\]
\[
    = (\alpha - \alpha^{m-n+1} / (1 - \alpha)) f(d(x_n, x_{n-1}))
\]
\[
    < (\alpha / (1 - \alpha)) f(d(x_n, x_{n-1})) < (\alpha / (1 - \alpha)) f(\delta)
\]
\[
    = (a / (1 - 2a)) f(\delta) < f(\epsilon).
\]

This shows that \( d(x_n, x_n) < \epsilon \), so \( \{x_n\} \) is a \( \leq - \) non-decreasing Cauchy sequence. Since \( X \) is a sequentially complete, there exists \( x^* \in X \) such that \( \lim_{n \to \infty} x_n = x^* \). Finally, we prove that
\( x^* \) is a common fixed point of \( T \) and \( S \). For every \( n \in N \), \( (H1) \) guarantees that \( x_n \) is comparable to \( x^* \), so for \( n \in N \) we have,
\[ f(d(x_{2n+2}, Sx^*)) \leq f(\sup_{x \in Tx_{2n+1}} d(x, Sx^*)) \leq f(H_d(Tx_{2n+1}, Sx^*)) \]
\[ \leq f(0.5(d(x_{2n+1}, Sx^*) + d(x^*, Tx_{2n+1}))) - \varphi(f(d(x_{2n+1}, Sx^*)), f(d(x^*, Tx_{2n+1}))) \]
\[ \leq f(0.5(d(x_{2n+1}, Sx^*) + d(x^*, x_{2n+2}))) - \varphi(f(d(x_{2n+1}, Sx^*)), f(d(x^*, x_{2n+2}))), \]
\[ (9) \]

Since \( \varphi \) is l.s.c, letting \( n \to \infty \) in (9) we get
\[ f(d(x^*, Sx^*)) \leq f(0.5d(x^*, Sx^*)) - \varphi(f(d(x^*, Sx^*)), 0). \]

Which implies \( \varphi(f(d(x^*, Sx^*)), 0) = 0 \) and hence \( d(x^*, Sx^*) = 0 \). Since \( Sx^* \) is AV, there exist \( y \in P_{Sx^*} \) such that \( d(y, x^*) = 0 \) i.e. \( y = x^* \), therefore \( x^* \) is a fixed point of \( S \), and so it is a common fixed point. This completes the proof.

Similar to the proof of Theorem 3.1 we have the following Theorem.

**Theorem 3.2.** Let \( X \) be a sequentially complete quasi-ordered metric space and satisfy (H1). Suppose that \( T, S : X \to 2^X \) be two mappings that satisfy weakly \( C \)-contraction with respect to \( f \) and \( \varphi \), and have LCAV. Then \( T \) and \( S \) have a common fixed point. Further, for each \( x_0 \in X \), the iterated sequence \( \{x_n\} \) with \( x_{2n+1} \in Tx_{2n} \) and \( x_{2n+2} \in Sx_{2n+1} \) converges to the common fixed point of \( T \) and \( S \).

**Theorem 3.3.** Let \( X \) be a totally ordered sequentially complete metric space and satisfy (H1) and the following

(H2) \( x \leq y \leq z \) implies that \( d(z, x) \geq d(y, x) \) for all \( x, y, z \in X \).

Suppose that \( T \) and \( S \) satisfy all conditions given in Theorem 3.1 (resp. in Theorem 3.2), then \( T, S \) have a unique common fixed point \( x \in X \) and the iterated convergence of Theorem 3.1 holds.

**Proof:** Theorem 3.1 (resp. Theorem 3.2) ensures existence of common fixed points. To prove the uniqueness, let both \( x \) and \( y \) be common fixed point of \( T \) and \( S \). Since \( (X, \leq) \) is a totally ordered space, we have either \( x > y \) or \( y > x \). Without loss of generality, we assume that the former is true. If \( T \) has UCAV, we have \( x^* \in Tx \), with \( x \leq x^* \) and \( d(x^*, y) = d(Tx, y) \). From our assumption it follows that \( d(x^*, y) \geq d(x, y) \). On the other
hand, \( x \in Tx \) implies that \( d(x^*, y) \leq d(x, y) \). Hence, \( d(x^*, y) = d(x, y) = d(Tx, y) \). If \( x \neq y \), then \( d(x, y) > 0 \). Thus

\[
d(x, y) = d(Tx, y) \leq H_d(Tx, Sy). \tag{10}
\]

If \( T \) has LCAV, so does \( S \), we have \( y^* \in Sy \) with \( y^* \leq y \) and \( d(y^*, x) = d(Sy, x) \). From (H2) it follows that \( d(y^*, x) \geq d(x, y) \). On the other hand, \( y \in Sy \) implies that \( d(y^*, x) \leq d(x, y) \). Hence, \( d(y^*, x) = d(x, y) = d(x, Sy) \). At all events, (10) holds if \( x \neq y \).

\[
f(d(x, y)) \leq f(H_d(Tx, Sy)) \leq f\left(\frac{1}{2}(d(y, Tx) + d(x, Sy))\right) - \varphi(d(y, Tx), d(x, Sy))
\]

\[
= f(d(x, y)) - \varphi(d(x, y), d(x, y)) < f(d(x, y))
\]

This is a contradiction. Consequently, the inequality \( x < y \) is not true. By the same methods we can verify that \( y < x \) is also not true. Thus \( x = y \).

**Theorem 3.3.** Let \( X \) be a sequentially complete quasi-ordered metric space and satisfy (H1). Suppose that \( T, S : X \to 2^X \) be two mappings have AV, are non-decreasing, and weak \( C \)-contraction with respect to \( f \) and \( \varphi \). If there exists \( x_0 \in X \) such that \( \{x_0\} \leq Sx_0 \leq Tx_0 \). Then \( T \) and \( S \) have a common fixed point. Further, the iterated convergence of Theorem 3.1 holds.

**Proof:** let \( x_0 \in X \), if \( x_0 \in Sx_0 \) then is a common fixed point of \( T \) and \( S \) thus the proof is complete. Otherwise, since \( Sx \) has AV, there exist \( x_i \in Sx_0 \) with \( x_i \geq x_0 \) and \( d(x_0, x_i) = d(Sx_0, x_i) \). Since \( x \geq x_i \) for all \( x \in Tx_i \). If \( x_i \in Tx_i \), the proof is finished, otherwise, by means of \( Tx \) is AV, there exist \( x_2 \in Tx_1 \) with \( x_2 \geq x_i \) and \( d(x_1, x_2) = d(Tx_1, x_i) \). Inductively, we can construct a sequence \( x_n \) in \( X \) as \( x_n \neq x_{n-1} \) and \( x_n \geq x_{n-1} \) such that \( x_{2n+i} \in T_{2n+i} \), \( x_{2n+2} \in Sx_{2n+1} \) and (1), (2) hold. Now the rest of the proof is the same as theorem 3.1.

**References**
