Some Algebraic Structures of Languages

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Abstract
In this paper, suitable operations are defined on the class of partitions of a language which give rise to certain monoids and semigroups. In particular, certain algebraic structures of a language defined over a string are described.

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1. Introduction

The applications of various algebraic structures abound (see [1, 2, 3, 6, 7, 8] for details and related references). In particular, certain algebraic structures have found applications in formal language theory (see [6] for details). Moreover, a number of algebraic structures of partitions of a set and that of an integer have been developed which have useful applications in computer arithmetic, formal languages and sequential machines (see [5, 7] for details). In this paper, suitable operations on the set of partitions of a language are defined which give rise to certain monoids and semigroups. In addition, certain algebraic structures of a language defined over a string are described.
2. Definitions

Definition 2.1 Union, Intersection and Concatenation of Languages

Let $X$ be an alphabet and $X^*$ denote the set of all strings over $X$. A language $L$ is a subset of $X^*$ i.e., $L \subseteq X^*$. Let $L_1$ and $L_2$ be any two languages over $X$. The union of $L_1$ and $L_2$, denoted $L_1 \cup L_2$, is the language $L_1 \cup L_2 = \{u \in X^* | u \in L_1 \text{ or } u \in L_2\}$. The intersection of $L_1$ and $L_2$, denoted $L_1 \cap L_2$, is the language $L_1 \cap L_2 = \{u \in X^* | u \in L_1 \text{ and } u \in L_2\}$. The concatenation (or simply, catenation) of $L_1$ and $L_2$, denoted $L_1L_2$, is the language $L_1L_2 = \{u = u_1u_2 | u_1 \in L_1 \text{ and } u_2 \in L_2\}$. It is immediate to see that the union, intersection and catenation of languages are each associative because union, intersection and catenation of strings are each associative and hence $X^*$ with catenation is a non-commutative monoid (see [2, 4], for details). In the same vein, let $u^*$ be defined as the set of all strings over $u \in X^*$, then $u^*$ with catenation is a commutative monoid.

Definition 2.2 Cardinality bounded languages

Let $X^n$, henceforth called a cardinality bounded language over $X$, denote the set of all strings of length $\leq n$ over $X$. In other words, $\{X^n\}$ is a strictly monotonic increasing nested sequence, and obviously $X^* = X^0 \cup X^1 \cup \ldots \cup X^n \cup \ldots$. However, it gives an alternative form of representation of the usual one viz., $X^* = X^0 \cup X^1 \cup \ldots \cup X^n \cup \ldots$, where $X^n$ is the set of all strings of length $n$ over $X$. Moreover, $\bigcup_{n=0}^{\infty} X^n = \bigcup_{n=0}^{\infty} X^n$, but $\bigcap_{n=0}^{\infty} X^n = \emptyset$, whereas $\bigcap_{n=0}^{\infty} X^n = \{\varepsilon\}$.

It may also be observed that each of $X^n$ is a well-ordered set with $\subseteq$ (inclusion), and hence a finite ordinal, say $\alpha, \beta, \gamma, \ldots$, satisfying the following properties: (i) $\beta \in \alpha \Rightarrow \beta \subset \alpha$ (ii) each $\alpha$ is well-ordered by $\subseteq$ and (iii) neither $\alpha$ nor its element is an element of itself.

For example, let $X = \{0,1\}$, then $X^0 = \{\varepsilon\}$, $X^1 = \{\varepsilon, 0,1\}$, $X^2 = \{\varepsilon, 0,1,00,01,10,11\}$, $X^3 = \{\varepsilon, 0,1,00,01,10,11,000,001,010,011,100,101,110,111\}$, and so on.

Let us recapitulate that the cardinality of a language $L$, denoted $|L|$, is the number of strings in $L$. Thus $X^*$ is countably infinite over any $X$. Moreover, $|X^n| = |X^0| + |X^1| + |X^2| + \ldots + |X^n|$.

Examples

Let $X = \{0\}$. Then,
$$|X^n| = |X^0| + |X^1| + |X^2| + \ldots + |X^n| = \{|\varepsilon|\} + \{|0|\} + \{|00|\} + \{|000|\} + \ldots + \{|0^n|\} = 1^0 + 1^1 + 1^2 + \ldots + 1^n.$$
Let $X = \{0,1\}$. Then,

$$|X^n| = |X^0| + |X^1| + |X^2| + \cdots + |X^n| = |\{\varepsilon\}| + |\{0,1\}| + |\{00,01,10,11\}| + \cdots + |\{0,1\}^n| = 2^0 + 2^1 + 2^2 + \cdots + 2^n.$$

Let $X = \{0,1,2\}$. Then,

$$|X^n| = |X^0| + |X^1| + |X^2| + \cdots + |X^n| = |\{\varepsilon\}| + |\{0,1,2\}| + |\{00,01,02,10,11,12,20,21,22\}| + \cdots + |\{0,1,2\}^n| = 3^0 + 3^1 + 3^2 + \cdots + 3^n.$$

Let $X = \{0,1,2,3\}$. Then,

$$|X^n| = |X^0| + |X^1| + |X^2| + \cdots + |X^n| = |\{\varepsilon\}| + |\{0,1,2,3\}| + |\{00,01,02,03,10,11,12,13,20,21,22,23,30,31,32,33\}| + \cdots + |\{00,001,002,003,010,011,012,013,020,021,022,023,030,031,032,033,100,101,102,103,110,111,112,113,120,121,122,123,130,131,132,133,200,201,202,203,210,211,212,213,220,221,222,223,230,231,232,233,300,301,302,303,310,311,312,313,320,321,322,323,330,331,332,333\}| = 4^0 + 4^1 + 4^2 + \cdots + 4^n.$$

By induction, if $X$ be a $k$—element set, we have

$$|X^n| = k^0 + k^1 + k^2 + \cdots + k^n.$$

### 3. Some algebraic structures of languages

#### 3.1 Monoids of equivalence classes of a partition of a language

Let $R_a$ be a relation on $X^*$ such that for $s, t \in X^*$, $sR_at$ if and only if $s$ and $t$ are of equal length.

It is easy to see that $R_a$ is an equivalence relation on $X^*$ and hence, it partitions $X^*$ into its equivalence classes. In other words, a partition of $X^*$ can be viewed as a collection of disjoint languages of $X^*$, whose union is $X^*$.

Let the equivalence class generated by $S \in X^*$ be denoted $[S]_{R_a}$ or simply $[S]$, and the quotient set $X^*/R_a$ denote the family of all equivalence classes of $X^*$.

Let us define an operation $\ast$ on $X^*/R_a$ such that $[s][t] = [st]$ where $st$ is the catenation of $s$ and $t$. Then, $(X^*/R_a, \ast, [\varepsilon])$ is a monoid of the partition of $X^*$ induced by $R_a$, where $[\varepsilon]$
is the identity of catenation. The operation $*$ is neither commutative nor idempotent, in general. However, the identity element $[\varepsilon]$ is the only idempotent element. Also, the operation $*$ is commutative if $X$ is a singleton.

Moreover, as described in section 2 above, it is easy to see that $(L/R_a, *, [\varepsilon])$ is a commutative monoid where $L = u^*$, $u \in X^*$.

Similarly, for each of the relations $R_b$, $R_c$, and $R_d$ defined on $X^*$ as

(i) $sR_b t$ iff both $s$ and $t$ have the same number of occurrences of each symbol,
(ii) $sR_c t$ iff $s$ and $t$ agree in their first symbols, and
(iii) $sR_d t$ iff $s$ and $t$ agree in their last symbols;

the respective quotient set is a non-commutative and non-idempotent monoid.

Moreover, each of $R_b$, $R_c$, and $R_d$, similar to $R_a$, defined on a language $u^*$, partitions it, and the respective quotient set is a commutative monoid of the partitions of $u^*$.

### 3.2 Monoids of partitions of a language

We introduce further three operations on the class of all partitions of $X^*$.

Let $\mathcal{F}(X^*)$ denote the collection of all partitions of $X^*$ and $S = \{S_1, S_2, \ldots\}$ and $T = \{T_1, T_2, \ldots\}$ be two partitions of $X^*$. Observe that $S'_i$'s and $T'_i$'s are the blocks of $S$ and $T$, respectively, and each block is a subset of $X^*$.

Let a binary operation $\odot$ be defined on $\mathcal{F}(X^*)$ as follows:

For any $S, T \in \mathcal{F}(X^*)$, $S \odot T$ consists of the set of nonempty intersections of every block of $S$ with every block of $T$. It is clear that the operation $\odot$ is both associative and commutative as intersection on languages is associative and commutative. The partition consisting of a unique single block is the identity of $\odot$. It may be observed that $P \odot P = P$ for all $P \in \mathcal{F}(X^*)$ i.e., $\odot$ is idempotent. Thus, $(\mathcal{F}(X^*), \odot)$ or $(\mathcal{F}(X^*), \odot, \{X^*\})$ is a commutative, idempotent monoid.

Let another binary operation $\oplus$ on $\mathcal{F}(X^*)$ be defined as follows:

Let $S, T \in \mathcal{F}(X^*)$. A subset $P$ of $X^*$ belongs to $S \oplus T$ if

(i) $P$ is the union of one or more elements of $S$;
(ii) $P$ is the union of one or more elements of $T$; and
(iii) No element of $P$ satisfies (i) and (ii) except $P$ itself.

Clearly, $\oplus$ is associative and commutative, and the partition consisting of singleton blocks is the identity of the operation $\oplus$ on $\mathcal{F}(X^*)$. Thus, $(\mathcal{F}(X^*), \oplus)$ or $(\mathcal{F}(X^*), \oplus, \{x_0, x_1, x_2, \ldots\})$, where $x_i$'s are the elements of $X^*$, is a commutative, idempotent monoid.

Finally, let a binary operation $\otimes$ be defined on $\mathcal{F}(X^*)$ as follows:
For any $S, T \in \mathcal{F}(X^*)$, $S \odot T$ is the union of every block of $S$ with every block of $T$ if no element of the block of $S$ and/or $T$ appears previously. In the case, a block has an element that appeared previously, it is not included in the union.

It is immediate to see that $\odot$ is associative but non-commutative and every element $P \in \mathcal{F}(X^*)$ is idempotent as $P \odot P = P$ holds. Thus, $(\mathcal{F}(X^*), \odot)$ is only a semigroup as there is no identity element.

It is immediate to see that all the foregoing constructions, described above, hold good for $X^n$ as well.

**Examples**

Let $X = \{0, 1\}$ be an alphabet and $n = 2$. Then, $X^2 = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$.

Let $S, T \in \mathcal{F}(X^2)$ where $S = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$ and $T = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$. Then, the following hold:

(i) $S \odot T = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \in \mathcal{F}(X^2)$, $S \odot S = \{\varepsilon, 0, 1, 00, 01, 10, 11\} = S$, and $S \odot T = T \odot S$. Similarly, results could be computed to show associativity. Thus, $(\mathcal{F}(X^2), \odot)$ is a commutative, idempotent monoid with $\{\varepsilon, 0, 1, 00, 01, 10, 11\}$ as the identity.

(ii) $S \oplus T = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \in \mathcal{F}(X^2)$, $S \oplus S = \{\varepsilon, 0, 1, 00, 01, 10, 11\} = S$, and $S \oplus T = T \oplus S$. Moreover, $I = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$ is the identity element since $I \oplus T = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \oplus \{\varepsilon, 0, 1, 00, 01, 10, 11\} = \{\varepsilon, 0, 1, 00, 01, 10, 11\} = T$, for any $T$. Results could be computed to show that $\oplus$ is associative. Thus, $(\mathcal{F}(X^2), \oplus)$ is a commutative, idempotent monoid with $\{\varepsilon, 0, 1, 00, 01, 10, 11\}$ as the identity.

(iii) $S \otimes T = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \in \mathcal{F}(X^2)$, $T \otimes S = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \in \mathcal{F}(X^2)$, and $T \otimes S \neq S \otimes T$. Moreover, $T \otimes T = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \otimes \{\varepsilon, 0, 1, 00, 01, 10, 11\} = \{\varepsilon, 0, 1, 00, 01, 10, 11\} = T$. In order to show associativity, let $R = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$. Then, $(S \otimes T) \otimes R = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \otimes \{\varepsilon, 0, 1, 00, 01, 10, 11\} = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$, and $S \otimes (T \otimes R) = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \otimes (\{\varepsilon, 0, 1, 00, 01, 10, 11\} \otimes \{\varepsilon, 0, 1, 00, 01, 10, 11\}) = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \otimes \{\varepsilon, 0, 1, 00, 01, 10, 11\} = \{\varepsilon, 0, 1, 00, 01, 10, 11\}$ i.e., $(S \otimes T) \otimes R = S \otimes (T \otimes R)$. Thus, $(\mathcal{F}(X^2), \otimes)$ is an idempotent semigroup.

### 3.3 Some algebraic structures of a language over $u \in X^*$

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Let \( u^* \) denote the set of all strings over \( u \in X^* \). Then \( u^* \) is a commutative monoid under catenation. Moreover, the monoid \( C = (u^*, \circ) \) is isomorphic to the monoid \( N = (\mathbb{N}, .) \), where \( \circ \) and \( . \) denote catenation and multiplication, respectively.

**Proof**

The first part follows by definition.

For the second part, let \( f: C \rightarrow N \) be a function defined as

\[
f(u) = \begin{cases} 
1, & \text{if } u = \varepsilon, \\
n, & \text{if } u = u^n, \quad \forall u \in C,
\end{cases}
\]

where \( u^n \) is the \( n \) - times catenation of \( u \) itself.

It is easy to see that \( \forall u, v \in C, \) since \( f(uv) = f(u_1u_2 ... u_nv_1v_2 ... v_n) = f(u_1)f(u_2) ... f(u_n)f(v_1)f(v_2) ... f(v_n) = f(u_1u_2 ... u_n)f(v_1v_2 ... v_n) = f(u)f(v) \), the function \( f \) is a monoid homomorphism.

Let \( u, v \in C \) such that \( f(u) = f(v) \) i.e., \( f(u_1)f(u_2) ... f(u_n) = f(v_1)f(v_2) ... f(v_n) \). Then, as strings are ordered, we have \( u_1 = v_1, u_2 = v_2, ..., u_n = v_n \) i.e., \( u = v \), which implies that \( f \) is injective. Moreover, by the definition of \( f \), \( \forall n \in \mathbb{N}, \exists u \in C \) such that \( f(u) = n \) i.e., \( f \) is surjective.

Hence \( f \) is an isomorphism.

**Proposition 3.3.1**

A finite \( C = (u^*, \circ) \) is a cyclic group of order \( n \).

**Proof**

Let \( u^* \) be represented as \( \{u^0, u^1, ..., u^{n-1}, ...\} \). A finite \( C \) can be represented as \( (u_n^*, \circ) \) where \( u_n^* \) is the set of \( n \) elements of \( u^* \). Let \( C \) be finite viz., \( C = \{C^i, \circ \}, i = 0,1, ..., n - 1 \), where

\[
C^i = \begin{cases} 
C^{i+1}, & 0 \leq i < n - 1 \\
C^0, & i = n - 1.
\end{cases}
\]

Let \( C^iC^j = C^{i+j}, \ i + j < n \) and \( C^iC^j = C^{i+j-n}, \ i + j \geq n \). Then, it is easy to see that \( C \) is a cyclic group of order \( n \).

**Example**
Let \( u = bba \), \( u^* = \{ \varepsilon, bba, bbabba, \ldots \} \) and \( n = 7 \). Then, \( \mathcal{C} = \{ C^0, C^1, C^2, C^3, C^4, C^5, C^6 \} = \{ \varepsilon, bba, bbabba, bbabbbabba, bbabbabbbabba, bbabbbabbbabba, bbabbbabbbabba \} \).

Observe that \( C^1C^2 = C^3 \), \( C^4C^5 = C^2 \), \( C^6C^1 = C^0 \), etc. Thus, \( \mathcal{C} \) is a cyclic group of order 7.

**Proposition 3.3.2**

Languages of a finite \( u^* \) form a bounded distributive lattice.

**Proof**

Let a finite \( u^* \) be represented as \( u^{n \odot} = u^0 \cup u^1 \cup \ldots \cup u^{n-1} \), and \( G \) be the set of all possible languages of \( u^{n \odot} \). Let \( H \) be a structure consisting of \( G \) with union and intersection representing the (join) \( \lor \) and (meet) \( \land \) operations, respectively. Let \( L_1, L_2, L_3 \in G \). It is straightforward to see that \( L_1 \lor L_2 = L_2 \lor L_1 \), \( L_1 \lor (L_2 \lor L_3) = (L_1 \lor L_2) \lor L_3 \) and \( L_1 \lor L_1 = L_1 \) hold, as the union of languages is associative, commutative and idempotent. Thus, \( (G, \lor) \) is a commutative, idempotent semigroup. Also, as the intersection of languages is commutative, associative and idempotent, \( (G, \land) \) is a commutative, idempotent semigroup.

Moreover, as the absorption properties hold i.e., \( L_1 \lor (L_1 \land L_2) = L_1 \) and \( L_1 \land (L_1 \lor L_2) = L_1 \), and for all \( L_1, L_2 \in G \), \( L_1 \land L_2 = L_1 \) and \( L_1 \lor L_2 = L_2 \) hold, \( H = (G, \lor, \land) \) is a lattice.

Also, \( \forall L \in G \), as \( L \lor \emptyset = L \), \( \emptyset \) is the identity element of the join operation and, as \( L \land G = L \), \( G \) is the identity of the meet operation. Thus, \( H \) is a bounded lattice.

In addition, as \( L_1 \lor (L_2 \land L_3) = (L_1 \lor L_2) \land (L_1 \lor L_3) \) and \( L_1 \land (L_2 \lor L_3) = (L_1 \land L_2) \lor (L_1 \land L_3) \) hold, \( H \) is a bounded distributive lattice.

**Example**

Let \( u = 01 \in X^* \) over an alphabet \( X = \{0,1\} \), and \( u^{3 \odot} = \{ \varepsilon, 01, 0101 \} \). The set \( G \) of all possible languages of \( u^{3 \odot} \) is \( \{ \emptyset, \{ \varepsilon \}, \{01\}, \{0101\}, \{\varepsilon, 01\}, \{\varepsilon, 0101\}, \{01, 0101\}, \{\varepsilon, 01, 0101\} \} \).

Observe that \( \{01\} \lor (\{01\} \land \{01, 0101\}) = \{01\} \), \( \{01\} \land (\{01\} \lor \{01, 0101\}) = \{01\} \) i.e., absorption properties hold. Also, \( \emptyset \lor \{01, 0101\} = \{01, 0101\} \) and \( \emptyset \land \{\varepsilon, 01, 0101\} = \{\varepsilon, 0101\} \) i.e., \( \emptyset \) is the identity for \( \lor \), and \( \{\varepsilon, 01, 0101\} \) is the identity for\( \land \). Similarly, results for various other combinations could be computed.

Thus, \( (G, \lor, \land) \) is a bounded distributive lattice.

**4. Concluding Remarks**

A number of operations were introduced on the class of partitions of a language which gave rise to certain monoids and semigroups. Moreover, cyclic group, commutative monoid and
bounded distributive lattice of a language over a string were introduced. It may be emphasized at this end that the constructions provided in this paper, specially defined on $X^{n*}$, may be found useful to Network segmentation, analysis of large databases, finite state machines, etc. In particular, an alternative representation of a language, developed in definition 2.2, may be exploited for further research.

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