**Singular Values of One Parameter Family** \( \frac{b^z - 1}{z} \)

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**Abstract**

The singular values of one parameter family of entire functions \( f_\lambda(z) = \frac{\lambda}{z} \frac{b^z - 1}{z} \) and \( f_\lambda(0) = \lambda \ln b \), \( \lambda \in \mathbb{R} \setminus \{0\}, \ z \in \mathbb{C}, \ b > 0, \ b \neq 1 \) are investigated. It is shown that all the critical values of \( f_\lambda(z) \) belong to the right half plane for \( 0 < b < 1 \) and the left half plane for \( b > 1 \). It is described that the function \( f_\lambda(z) \) has infinitely many singular values. It is also found that all these singular values are bounded and lie inside the open disk centered at origin and having radius \( |\lambda \ln b| \).

**Keywords:** Critical values, Singular values.

**1. Introduction**

It is known that if singular values exist in transcendental functions, then it is very crucial to determine the dynamical behavior of these functions. The dynamics of one parameter family \( \lambda e^z \), that has only one singular value, is studied in detail [1, 2, 3]. This exponential family is simpler than other families of functions which involving exponential maps [4, 5, 6, 7] and having more than one or infinitely many singular values. The singular values and dynamics of family \( \frac{e^z - 1}{z} \) are studied by Kapoor and Prasad [8]. The singular values of one parameter families of functions are investigated in [9, 10, 11].

In this work, the singular values of one parameter family of function \( \frac{b^z - 1}{z} \) for \( b > 0, \ b \neq 1 \) which is a generalized family of function \( \frac{e^z - 1}{z} \), is considered. Suppose, for this purpose, the following one parameter family of functions...
\[ T = \left\{ f_\lambda (z) = \lambda \frac{b^z - 1}{z} \text{ and } f_\lambda (0) = \lambda \ln b : \lambda \in \mathbb{R} \setminus \{0\}, z \in \mathbb{C}, b > 0, b \neq 1 \right\} \]

A point \( z^* \) is said to be a critical point of \( f(z) \) if \( f'(z^*) = 0 \). The value \( f(z^*) \) corresponding to a critical point \( z^* \) is called a critical value of \( f(z) \). A point \( w \in \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) is said to be an asymptotic value for \( f(z) \), if there exists a continuous curve \( \gamma : [0, \infty) \to \hat{\mathbb{C}} \) satisfying \( \lim_{t \to \infty} \gamma(t) = \infty \) and \( \lim_{t \to \infty} f(\gamma(t)) = w \). A singular value of \( f \) is defined to be either a critical value or an asymptotic value of \( f \). A function \( f \) is called critically bounded or it is said to be a function of bounded type if the set of all singular values of \( f \) is bounded, otherwise unbounded type. The importance of singular values in the dynamics of a transcendental functions can be seen in [12, 13, 14].

This paper is organized as follows: It is shown that, in Theorem 2.1, the function \( f_\lambda \in T \) has no zeros in the left half plane and \( f_\lambda (z) \) maps the right half plane inside the open disk centered at origin and having radius \( |\lambda \ln b| \) for \( 0 < b < 1 \). It is also found that, in Theorem 2.2, the function \( f_\lambda \in T \) has no zeros in the right half plane and \( f_\lambda (z) \) maps the left half plane inside the open disk centered at origin and having radius \( |\lambda \ln b| \) for \( b > 1 \). In Theorem 2.3, it is proved that the function \( f_\lambda \in T \) has infinitely many singular values. It is seen that all the singular values are bounded and lie inside the open disk centered at origin and having radius \( |\lambda \ln b| \) in Theorem 2.4.

2. Infinitely Many Bounded Singular Values of \( f_\lambda \in T \)

Let us denote the left half plane and the right half plane by \( H^- = \{z \in \hat{\mathbb{C}} : \text{Re}(z) < 0\} \) and \( H^+ = \{z \in \hat{\mathbb{C}} : \text{Re}(z) > 0\} \) respectively. The function \( f_\lambda \in T \) has no zeros in the left half plane and \( f_\lambda (z) \) maps the right half plane inside the open disk centered at origin and having radius \( |\lambda \ln b| \) for \( 0 < b < 1 \), are found in the following theorem:

**Theorem 2.1.** Let \( f_\lambda \in T \) for \( 0 < b < 1 \). Then,

(a) \( f'_\lambda (z) \) has no zeros in the left half plane \( H^- \).

(b) \( f_\lambda (z) \) maps the right half plane \( H^+ \) inside the open disk centered at origin and having radius \( |\lambda \ln b| \).

**Proof.** (a) Since \( f'_\lambda (z) = \lambda \frac{(z \ln b - 1) b^z + 1}{z^2} = 0 \), which implies \( b^{-z} = 1 - z \ln b \). Then, the real and imaginary parts of this equation are

\[ b^{-z} \cos(y \ln b) = 1 - x \ln b \]  \( \quad (1) \)

\[ b^{-z} \sin(y \ln b) = y \ln b \]  \( \quad (2) \)

Thus, it shows that the function \( f'_\lambda (z) \) has no zeros in \( H^- \) for \( 0 < b < 1 \).
(b) Suppose that the line segment $\gamma$ is defined by $\gamma(t) = tz$, $t \in [0,1]$. Further, let the function $h(z) = b^z$ for an arbitrary fixed $z \in \mathbb{C}$. Then

$$\int_{\gamma} h(z)dz = \int_{0}^{1} h(\gamma(t))\gamma'(t)dt = z\int_{0}^{1} b^t dt = \frac{1}{\ln b} (b^z - 1)$$

Since $M = \max_{t \in [0,1]} |h(\gamma(t))| = \max_{t \in [0,1]} |b^t| < 1$ for $z \in H^+$, then

$$|b^z - 1| = \left| \ln b \int_{\gamma} h(z)dz \right| \leq M |z| \ln b < |z| |\ln b|$$

$$\left| \frac{b^z - 1}{z} \right| < |\ln b| \quad \text{for all} \quad z \in H^+.$$

Hence,

$$|f_\lambda(z)| = \left| \frac{b^z - 1}{z} \right| < |\lambda \ln b| \quad \text{for all} \quad z \in H^+.$$

Therefore, $f_\lambda(z)$ maps $H^+$ inside the open disk centered at origin and having radius $|\lambda \ln b|$. The proof of the theorem is completed for $0 < b < 1$.

In the following theorem, it is shown that the function $f_\lambda \in T$ has no zeros in the right half plane and $f_\lambda(z)$ maps the left half plane inside the open disk for $b > 1$:

**Theorem 2.2.** Let $f_\lambda \in T$ for $b > 1$. Then,

(i) $f_\lambda'(z)$ has no zeros in the right half plane $H^+$.

(ii) $f_\lambda(z)$ maps the left half plane $H^-$ inside the open disk centered at origin and having radius $|\lambda \ln b|$.

**Proof.** For $b > 1$, the proof of the theorem may be obtained similarly as Theorem 2.1.

The following theorem describes that the function $f_\lambda \in T$ has infinitely many singular values:

**Theorem 2.3.** Let $f_\lambda \in T$. Then, the function $f_\lambda(z)$ possesses infinitely many singular values.

**Proof.** For critical points, $f_\lambda'(z) = 0$. It gives the equation $(z \ln b - 1)b^z + 1 = 0$. After simplifying, using the real and imaginary parts of this equation

$$\frac{y \ln b}{\sin(y \ln b)} - b^{y \cot(y \ln b) - \frac{1}{\ln b}} = 0 \quad (3)$$

$$x = \frac{1}{\ln b} - y \cot(y \ln b) \quad (4)$$

It is seen that, from Figure 1(a) for $0 < b < 1$ and Figure 2(b) for $b > 1$, the Eq. (3) has infinitely many solutions since intersections are increasing on horizontal axis for expanding interval.
For $b = 0.4$

![Graph for $b = 0.4$]

(a) For $b = 0.4$

For $b = 4$

![Graph for $b = 4$]

Figure 1: Graphs of $\frac{y \ln b}{\sin(\ln b)} - b^{\frac{1}{\ln b}} \cot(\ln b)$

Let $\{y_k\}_{k=-\infty, k \neq 0}$ be the solutions of Eq. (3). Then, from Eq. (4), $x_k = \frac{1}{\ln b} - y_k \cot(y_k \ln b)$ for $k$ nonzero integer. For $z_k = x_k + iy_k$, the critical values $f_{\lambda}(z_k) = \lambda b^{\frac{1}{z_k}} - 1$ are distinct for different $k$. It shows that the function $f_{\lambda}(z)$ has infinitely many critical values for $0 < b < 1$ and $b > 1$.

Since $f_{\lambda}(z) \to \infty$ as $z \to \infty$ along both positive and negative real axes for $0 < b < 1$ and $b > 1$ respectively, then the point 0 is a finite asymptotic value of $f_{\lambda} \in T$.

Therefore, it conclude that the function $f_{\lambda} \in T$ possesses infinitely many singular values for $0 < b < 1$ and $b > 1$. This proves the theorem.

In the following theorem, it is proved that $f_{\lambda} \in T$ has bounded singular values and lie inside the open disk:

**Theorem 2.3.** Let $f_{\lambda} \in T$. Then, all the singular values of $f_{\lambda}(z)$ are bounded and lie inside the open disk centered at origin and having radius $|\lambda \ln b|$.

**Proof.** For $0 < b < 1$, by Theorem 2.1(a), the function $f'_{\lambda}(z)$ has no zeros in the left half plane $H^-$. Hence, all the critical points lie in the right half plane $H^+$. By using Theorem 2.1(b), the function $f_{\lambda}(z)$ maps $H^+$ inside the open disk centered at origin and having radius $|\lambda \ln b|$. It follows that all the critical values of $f_{\lambda}(z)$ are lying inside the open disk centered at origin and having radius $|\lambda \ln b|$ for $0 < b < 1$. 

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Similarly, for \( b > 1 \), using Theorem 2.2 (i) and (ii), it is easily deduce that all the critical values of \( f_{\lambda}(z) \) are lying inside the open disk centered at origin and having radius \( |\lambda| \ln b \).

Since \( f_{\lambda}(z) \) has only one asymptotic value 0, so all the singular values of \( f_{\lambda} \in T \) are bounded and lie inside the open disk centered at origin and having radius \( |\lambda| \ln b | \).

References


