On the inclusion graphs of S-acts

Abdolhossein Delfan, Hamid Rasouli*, Abolfazl Tehranian

Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran.

Abstract

In this paper, we define the inclusion graph \( \text{Inc}(A) \) of an S-act \( A \) which is a graph whose vertices are non-trivial subacts of \( A \) and two distinct vertices \( B_1, B_2 \) are adjacent if \( B_1 \subset B_2 \) or \( B_2 \subset B_1 \). We investigate the relationship between the algebraic properties of an S-act \( A \) and the properties of the graph \( \text{Inc}(A) \). Some properties of \( \text{Inc}(A) \) including girth, diameter and connectivity are studied. We characterize some classes of graphs which are the inclusion graphs of S-acts. Finally, some results concerning the domination number of such graphs are given.

Keywords: S-Act, inclusion graph, diameter, girth, domination number.

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1. Introduction and preliminaries

The notion of an S-act over a monoid \( S \) is a fundamental concept in algebra, theoretical computer science and a variety of applications like automata theory and mathematical linguistics. Assigning graphs to algebraic structures is an approach to study algebraic properties via graph-theoretic properties. In this direction, many authors, e.g. [2, 3, 4, 7, 11, 12, 14], have been performed in connecting graph structures to various algebraic objects. Recently, inclusion graphs attached to rings, vector spaces and groups have been studied in [1, 8, 5]. Moreover, some works associating graphs to S-acts can be found in [6, 9, 13].

In this paper, we associate a graph \( \text{Inc}(A) \) to an S-act \( A \), called the inclusion graph of \( A \), whose vertices are non-trivial subacts of \( A \) in such a way that two distinct vertices \( B_1, B_2 \) are adjacent if \( B_1 \subset B_2 \) or \( B_2 \subset B_1 \). We investigate the relationship between the algebraic properties of an S-act \( A \) and the properties of the graph \( \text{Inc}(A) \). First we determine the girth and diameter of \( \text{Inc}(A) \). Then some classes of graphs which are the inclusion graphs of S-acts are characterized. Finally, we present some results dealing with the domination number of such graphs.

The following is a brief account of some basic definitions about S-acts and graphs.

*Corresponding author

Email addresses: a.delfan@khoiau.ac.ir (Abdolhossein Delfan), hrasouli@srbiau.ac.ir (Hamid Rasouli), tehranian@srbiau.ac.ir (Abolfazl Tehranian)
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Throughout this paper, unless otherwise stated, $S$ denotes a monoid with the identity $1$. By a (left) $S$-act, we mean a non-empty set $A$ on which $S$ acts unitarily, that is, $(st)a = s(ta)$ and $1a = a$ for all $s, t \in S$ and $a \in A$. A (non-empty proper) subset $B$ of $A$ is called a (non-trivial) subact of $A$ if $sb \in B$ for every $s \in S, b \in B$. The set of all non-trivial subacts of $A$ is denoted by $\text{Sub}(A)$. A non-empty subset $I$ of $S$ is said to be a left ideal of $S$ if $st \in I$ for any $s \in S, t \in I$. Considering $S$ as an $S$-act, any left ideal of $S$ is a subact of $S$. An element $\theta \in A$ is said to be a zero element, if $s\theta = \theta$ for all $s \in S$. A simple $S$-act is the one with no non-trivial subact. A completely reducible $S$-act is one which is a disjoint union of simple subacts. For more information about $S$-acts and related notions, the reader is referred to [10].

Let $G$ be a (simple) graph with a vertex set $V(G)$. By order of $G$, we mean the cardinality of $V(G)$ which is simply denoted by $|G|$. For any $u, v \in V(G)$, a $u,v$-path (or $u - v$) is a path with starting vertex $u$ and ending vertex $v$. The distance between two vertices $u, v$, denoted by $d(u, v)$, is defined as the length of the shortest path joining $u$ and $v$ if it exists, and otherwise, $d(u, v) = \infty$. The diameter of $G$, denoted by $\text{diam}(G)$, is the largest distance between pairs of vertices of $G$. The number of vertices adjacent to a vertex $v$ is called the degree of $v$ and denoted by $\deg(v)$. The girth of a graph is the length of its shortest cycle, and a graph with no cycle has infinite girth. A null graph is a graph with no edges. A graph is connected if there is a path between every two distinct vertices. A complete graph is a graph in which every pair of distinct vertices are adjacent. We denote the complete graph with $n$ vertices by $K_n, n \in \mathbb{N}$. A path and a cycle of length $n$ are denoted by $P_n$ and $C_n$, respectively. Two graphs $G_1, G_2$ are isomorphic if and only if there exists a bijection from $V(G_1)$ to $V(G_2)$ preserving the adjacency and non-adjacency. For undefined terms and concepts about graphs, one may consult [15].

2. Main results

In this section we first determine the girth of the graph $\text{Inc}(A)$ for an $S$-act $A$. Then we characterize those cycles which are inclusion graphs of some $S$-acts. Moreover, we study connectivity and diameter for the inclusion graphs. Finally, the domination number of such graphs is briefly studied.

Note that the inclusion graph for a simple $S$-act is undefined because it has no vertex. So we consider non-simple $S$-acts when dealing with their inclusion graphs throughout the paper.

Remark 2.1. It is clear that if $A$ and $B$ are isomorphic $S$-acts, then their graphs $\text{Inc}(A)$ and $\text{Inc}(B)$ are equivalent. The converse is not true in general. To see this, take the monoid $S = \{1, s\}$ where $s^2 = 1$. Consider two $S$-acts $A = \{a, b, c\}$ with trivial action and $B = \{a, b, c, d\}$ presented by the following action table:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>s</td>
<td>a</td>
<td>b</td>
<td>d</td>
<td>c</td>
</tr>
</tbody>
</table>

The non-trivial subacts of $A$ and $B$ are

$A_1 = \{a\}, \ A_2 = \{a, b\}, \ A_3 = \{b\}, \ A_4 = \{b, c\}, \ A_5 = \{c\}, \ A_6 = \{a, c\},$

and

$B_1 = \{a\}, \ B_2 = \{a, b\}, \ B_3 = \{b\}, \ B_4 = \{b, c, d\}, \ B_5 = \{c, d\}, \ B_6 = \{a, c, d\},$

respectively. Then $\text{Inc}(A) \cong \text{Inc}(B) \cong C_6$ whereas $A$ and $B$ are not isomorphic $S$-acts.
It is natural to ask whether a graph is isomorphic to the inclusion graph of an $S$-act. Here we consider complete graphs and cycles and characterize those ones satisfying this property.

We say that an $S$-act $A$ is uniserial, if all of its subacts are totally ordered by inclusion, or equivalently, for any two (cyclic) subacts $B$ and $C$ of $A$, either $B \subseteq C$ or $C \subseteq B$. This generalizes the well-known notion of a uniserial module extensively studied in the literature.

Clearly, for each $S$-act $A$, the graph $\mathcal{Ine}(A)$ is complete if and only if $A$ is a uniserial $S$-act. The following example shows that every complete graph is the inclusion graph of a (uniserial) $S$-act. As we shall see, this is not the case for cycles in general.

**Example 2.2.**

(i) Consider the monogenic semigroup $S = \{s, s^2, s^3, \ldots, s^{n+1}\}$, $s^{n+2} = s^{n+1}$, $n \in \mathbb{N}$. Then all distinct non-trivial left ideals of $S$ form the chain

$$\langle s^n \rangle \subset \langle s^{n-1} \rangle \subset \langle s^{n-2} \rangle \subset \cdots \subset \langle s \rangle,$$

where $\langle s^k \rangle = \{s^i \mid k + 1 \leq i \leq n + 1\}$, for every $1 \leq k \leq n$. So $S$ is a uniserial $S$-act and clearly the graph $\mathcal{Ine}(S)$ is isomorphic to the complete graph $K_n$. In particular, the inclusion graph of the monogenic semigroup $S = \{s, s^2, s^3, s^4\}$, $s^5 = s^4$, is isomorphic to the cycle $C_3$ with the vertices $I_1 = \{s^4\}$, $I_2 = \{s^3, s^4\}$, $I_3 = \{s^2, s^3, s^4\}$:

![Diagram](image)

(ii) The non-trivial left ideals of the semigroup $S = (\mathbb{N}, +)$ are exactly the sets $n + \mathbb{N} = \{n + k \mid k \in \mathbb{N}\}$ where $n \in \mathbb{N}$. Further, $m + \mathbb{N} \subset n + \mathbb{N}$ if and only if $m > n$, for every $m, n \in \mathbb{N}$. Then $S$ is a uniserial $S$-act and the graph $\mathcal{Ine}(S)$ is complete with countably infinite vertices.

(iii) The cycle $C_4$ is the inclusion graph of no $S$-act. Indeed, suppose that $C_4$ is the inclusion graph of an $S$-act $A$ and $B_1, B_2, B_3$ and $B_4$ are all non-trivial subacts of $A$ as the following:

![Diagram](image)

With no loss of generality, assume that $B_1 \subset B_2$. Then $B_3 \subset B_2$, $B_3 \subset B_4$, $B_1 \subset B_4$. It is easily seen that $B_1 \cup B_3 \neq A, B_i$ for all $i \in \{1, 2, 3, 4\}$ which is a contradiction.

(iv) Let $A = \{a, b\}$ be an $S$-act with trivial action. Then $B_1 = \{a\}$ and $B_2 = \{b\}$ are only non-trivial subacts of $A$ which are not adjacent and so $\text{girth}(\mathcal{Ine}(A)) = \infty$.

**Theorem 2.3.** For each $S$-act $A$, $\text{girth}(\mathcal{Ine}(A)) \in \{3, 6, \infty\}$.

**Proof.** First we show that for each $n > 6$, $\text{girth}(\mathcal{Ine}(A)) \neq n$. On the contrary, let $B_1 - B_2 - \cdots - B_n - B_1$ be the shortest cycle of order $n$. If $B_1 \cup B_4 \neq A$, then $B_1 - B_1 \cup B_4 - B_4$ is a path with shorter length between $B_1$ and $B_4$ which is a contradiction. So $B_1 \cup B_4 = A$, and by the same way, $B_1 \cap B_4 = \emptyset$, $B_1 \cup B_5 = A$ and $B_1 \cap B_5 = \emptyset$. Hence, $B_4 = B_5$ which is a contradiction. It remains to show that $\text{girth}(\mathcal{Ine}(A)) \neq 4, 5$. Let $B_1 - B_2 - B_3 - B_4 - B_1$, where $B_1 \subset B_2$, be the shortest cycle in $\mathcal{Ine}(A)$. Then $B_3 \subset B_2$, $B_3 \subset B_4$ and
Let $B_1 \subset B_4$. It is easily seen that $B_1 \cup B_3 \neq A_i B_4$ for all $i \in \{1, 2, 3, 4\}$. Then $B_1 - B_1 \cup B_3 - B_2 - B_1$ forms a cycle of order 3 which is a contradiction. If $B_1 - B_2 - B_3 - B_4 - B_5 - B_1$, where $B_1 \subset B_2$, is the shortest cycle in $\text{Inc}(A)$, then $B_3 \subset B_2$, $B_3 \subset B_4$, $B_5 \subset B_4$ and $B_5 \subset B_1$. This implies that $B_5$ is adjacent to $B_2$ which is a contradiction.

In light of Remark 2.1, Example 2.2 and Theorem 2.3, those cycles which are the inclusion graphs are fully characterized.

**Corollary 2.4.** The cycle $C_n$ is the inclusion graph of an $S$-act if and only if $n = 3$ or $n = 6$.

In the following, we study the connectivity and diameter of the inclusion graphs.

**Theorem 2.5.** Let $A$ be an $S$-act. Then $\text{Inc}(A)$ is disconnected if and only if it is a null graph with $|\text{Inc}(A)| = 2$. Moreover, if $\text{Inc}(A)$ is connected, then $\text{diam}(\text{Inc}(A)) \leq 3$.

**Proof.** Suppose that $|\text{Inc}(A)| \geq 3$. We show that there exists a path between $B_1, B_2$ for every two distinct non-trivial subacts $B_1, B_2$ of $A$. Let $B_1$ and $B_2$ be non-adjacent. If $B_1 \cap B_2 \neq \emptyset$ or $B_1 \cup B_2 \neq A$, then there exists a $B_1, B_2$-path. Now let $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = A$. Since $|\text{Inc}(A)| \geq 3$, $A$ contains a non-trivial subact $B_3$ with $B_3 \neq B_1, B_2$. If $B_1 \cap B_3 = \emptyset$ and $B_1 \cup B_3 = A$, then $B_2 = B_3$ which is a contradiction. So either $B_1 \cap B_3 \neq \emptyset$ or $B_1 \cup B_3 \neq A$. In the same way, either $B_2 \cap B_3 \neq \emptyset$ or $B_2 \cup B_3 \neq A$. We consider the following cases:

**Case 1.** Let $B_1 \cap B_3 \neq \emptyset$ and $B_2 \cap B_3 \neq \emptyset$. Note that $B_1 \cap B_3 \neq B_2, B_2 \cap B_3 \neq B_1$. Then

$$B_1 - B_1 \cap B_3 - B_3 - B_2 \cap B_3 - B_2,$$

is a $B_1, B_2$-path provided that $B_1 \cap B_3 \neq B_1, B_3$ and $B_2 \cap B_3 \neq B_2, B_3$. Otherwise, we get a path with shorter length between $B_1, B_2$. Hence, $d(B_1, B_2) \leq 4$.

**Case 2.** Let $B_1 \cap B_3 \neq \emptyset$ and $B_2 \cup B_3 \neq A$. We have $B_1 \cap B_3 \neq B_2, B_2 \cup B_3 \neq B_1$. Then

$$B_1 - B_1 \cap B_3 - B_3 - B_2 \cup B_3 - B_2,$$

is a $B_1, B_2$-path provided that $B_1 \cup B_3 \neq B_1, B_3$ and $B_2 \cup B_3 \neq B_2, B_3$. Otherwise, we get a path with shorter length between $B_1, B_2$. Hence, $d(B_1, B_2) \leq 4$.

Other cases have the same proof.

The converse is obvious. For the second part, first note that the above proof implicitly states that if $\text{Inc}(A)$ is connected, then $\text{diam}(\text{Inc}(A)) \leq 4$. We claim that 4 is impossible for the diameter. Assume on the contrary that $\text{Inc}(A)$ is a connected inclusion graph of an $S$-act $A$ with $\text{diam}(\text{Inc}(A)) = 4$. Then there exist two distinct vertices $B_1, B_2$ in $\text{Inc}(A)$ for which $B_1 - B_2 - B_3 - B_4 - B_5$ is the shortest $B_1, B_2$-path. It is clear to see that $B_1 \cup B_4 = A, B_1 \cap B_4 = \emptyset, B_1 \cup B_5 = A$ and $B_1 \cap B_5 = \emptyset$. Thus we get $B_4 = B_5$ which is a contradiction.

**Corollary 2.6.** Let $A$ be an $S$-act with two zero elements and $|A| \geq 3$. Then $\text{Inc}(A)$ is connected.

**Proof.** If $\theta_1$ and $\theta_2$ are two zero elements of $A$, then the sets $\{\theta_1\}, \{\theta_2\}$ and $\{\theta_1, \theta_2\}$ are distinct non-trivial subacts of $A$. Hence, by Theorem 2.5, $\text{Inc}(A)$ is connected.

In what follows, we study the connectivity of the inclusion graphs of cyclic, free and cofree $S$-acts. Let us first recall some definitions from [10].

By a cyclic $S$-act, we mean an $S$-act $A$ generated by an element $a \in A$, that is, $A = Sa$ where $Sa = \{sa \mid s \in S\}$.

An $S$-act $A$ is called free if it has a basis $X$, i.e., each element $a \in A$ is uniquely represented as $a = sx$ for some $s \in S$ and $x \in X$. In this case, $A \cong \bigwedge_{x \in X} S$. The dual categorical notion of free is the cofree $S$-act which is isomorphic to an $S$-act of the form $X^S$, the set of all maps from $S$ to a non-empty set $X$, with the action given by $(sf)(t) = f(ts)$ for $s, t \in S$ and $f \in X^S$. The set $X$ is called a cobasis for $A$. 
**Proposition 2.7.** Let A be an S-act. Then the following assertions hold:

(i) If A is cyclic, then Inc(A) is connected. In particular, Inc(S) is connected.

(ii) If A is a free S-act with a basis X where |X| > 2, then Inc(A) is connected.

(iii) If A is a cofree S-act and |A| ≥ 3, then Inc(A) is connected.

**Proof.**

(i) Consider a cyclic S-act A with disconnected inclusion graph. Using Theorem 2.5, A has only two non-trivial subacts, say B and C, such that B ∪ C = A and B ∩ C = ∅. Clearly, B and C are simple subacts of A so that A is completely reducible. Note that a cyclic S-act is completely reducible if and only if it is simple (see [10, Lemma I.5.32]). This implies that A is simple which is a contradiction.

(ii) It follows from hypothesis that the number of non-trivial subacts of A is greater than 2. Hence, Theorem 2.5 gives the assertion.

(iii) Using the assumption, A can be considered as the S-act X^S for a cobasis X where |X| > 1. Since every constant map in A is a zero element and there exist exactly |X| constant maps in A, A contains at least two zero elements and hence Inc(A) is connected by Corollary 2.6.

A non-trivial subact M of an S-act A is called minimal, if B ⊆ M for some subact B of A implies that B = M. We denote the set of all minimal subacts of A by Min(A).

**Remark 2.8.** Let A be an S-act. If deg(M) < ∞ for a minimal subact M of A, then the number of minimal subacts of A is finite. Indeed, if M_1, M_2, M_3, ⋯ be infinite minimal subacts of A other than M, then the infinite strict ascending chain

\[ M ⊆ M \cup M_1 ⊆ M \cup M_1 \cup M_2 ⊆ \cdots, \]

gives that deg(M) = ∞ which is a contradiction. Further, if Inc(A) is complete, then A contains at most one minimal subact.

**Theorem 2.9.** Let A be an S-act and Inc(A) have no cycle. Then Inc(A) is a null graph (with one or two vertices) or P_i where i ∈ {1, 2, 3, 4}.

**Proof.** It follows from the assumption that A has a minimal subact. If M_1, M_2, M_3 are three distinct minimal subacts of A, then

\[ M_1 - M_1 \cup M_2 - M_2 - M_2 \cup M_3 - M_3 - M_3 \cup M_1 - M_1, \]

is a cycle which is a contradiction. So |Min(A)| ≤ 2. The following cases may occur.

**Case 1.** Let A have only one minimal subact, say M. Then every subact of A contains M. We claim that |Inc(A)| ≤ 3. On the contrary, let B_1, B_2, B_3 be another distinct non-trivial subacts of A. Since Inc(A) has no cycle, B_1 ∪ B_2 = B_2 ∪ B_3 = B_3 ∪ B_1 = M whence B_1 = B_2 which is a contradiction. Thus the graph Inc(A) is one of the graphs: one-vertex graph, or the paths P_1 or P_2.

**Case 2.** Let A have two distinct minimal subacts, say M_1, M_2. If M_1 ∪ M_2 = A, then A has no another non-trivial subact and Inc(A) is a null graph with two distinct vertices. If M_1 ∪ M_2 ≠ A, then Inc(A) contains at least the three vertices M_1, M_2, M_1 ∪ M_2, we claim that |Inc(A)| ≤ 5. Assume contrarily that B_1, B_2, B_3 are another distinct non-trivial subacts of A. We show that each B_i contains only one minimal. Otherwise, M_1 ∪ M_2 ⊆ B_i and then M_1 - M_1 ∪ M_2 - B_i - M_1 is a cycle which is a contradiction. Moreover, if B_1 and B_2 intersect in a minimal subact as M_1, then B_1 ∪ B_1 ≠ A because M_2 ⊆ B_1 ∪ B_3 in this case the cycle M_1 - B_1 - B_1 ∪ B_1 - M_1 yields a contradiction. Therefore, each B_i contains only one minimal subact and each minimal subact is contained in only one B_i. This contradicts the number of B_i’s. So, in addition to M_1, M_2, M_1 ∪ M_2, Inc(A) contains at most two another vertices. It is straightforward to see that Inc(A) is one of the paths P_2 or P_3 or P_4. □
Here we study the domination number of the inclusion graphs and determine them for the graphs of some S-acts.

Let G be a graph. The (open) neighborhood $N(x)$ of a vertex $x \in V(G)$ is the set of vertices which are adjacent to $x$. For a subset $T$ of vertices, we put

$$N(T) = \bigcup_{x \in T} N(x), \quad N[T] = N(T) \cup T.$$ 

A set of vertices $T$ in $G$ is a dominating set, if $N[T] = V(G)$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$ and is denoted as $\gamma(G)$.

An $S$-act $A$ is said to be Artinian, if every descending chain of subacts of $A$ terminates. It can be easily seen that every non-empty subact of an Artinian $S$-act contains a minimal subact.

**Proposition 2.10.** Let $A$ be an $S$-act. Then $\gamma(\mathcal{Inc}(A)) \leq 2$ provided that each of the following assertions hold:

(i) $A$ contains a minimal subact;

(ii) $A$ contains a zero element;

(iii) $|\text{Sub}(A)| < \infty$;

(iv) $|A| < \infty$;

(v) $A$ has trivial action;

(vi) $A$ is Artinian.

**Proof.**

(i) Let $M$ be a minimal subact of $A$ and $W := \{B \in \text{Sub}(A) \mid M \not\subseteq B\}$. If $W = \emptyset$, then for every non-trivial subact $B$ of $A$, $M \subseteq B$ and so $\{|M|\}$ is a dominating set. If $W \neq \emptyset$, then $\{M, \bigcup_{B \in W} B\}$ forms a dominating set in $\mathcal{Inc}(A)$. Hence, $\gamma(\mathcal{Inc}(A)) \leq 2$.

(ii) Using (i), $\{z\}$ is a minimal subact of $A$ where $z$ is a zero element.

The assertions (iii),(iv),(v) and (vi) are consequences of (i). \hfill \Box

**Proposition 2.11.** The following assertions hold:

(i) Let $A$ be the coproduct of a family $\{A_i \mid i \in I\}$ of $S$-acts with $|I| > 1$ and $\gamma(\mathcal{Inc}(A_j)) = 1$ for some $j \in I$. Then $\gamma(\mathcal{Inc}(A)) = 2$.

(ii) If $F$ is a free $S$-act with a non-singleton basis and $\gamma(\mathcal{Inc}(S)) = 1$, then $\gamma(\mathcal{Inc}(F)) = 2$.

**Proof.**

(i) Suppose that $\{T\}$ is a dominating set of $\mathcal{Inc}(A_j)$. Let $B = \bigcup_{i \in I} B_i$ be a non-trivial subact of $A$ where $B_i$’s are (possibly empty) subacts of $A_i$. If $B_j \subseteq T$, then $B \subseteq \bigcup_{i \in I} U_i$ where $U_j = T, U_i = A_i$ for all $i \neq j$ and if $T \subseteq B_j$, then $T \subseteq B$. Thus $\bigcup_{i \in I} U_i \cup T$ is a dominating set of $\mathcal{Inc}(A)$ so that $\gamma(\mathcal{Inc}(A)) \leq 2$. Now we show that $\gamma(\mathcal{Inc}(A)) \neq 1$. On the contrary, let $\{B = \bigcup_{i \in I} B_i\}$ be a dominating set of $\mathcal{Inc}(A)$ and $s \in I$. Then one of the subacts $\bigcup_{i \neq s} A_i$ or $A_s$ is non-adjacent to $B$ in $\mathcal{Inc}(A)$ which is a contradiction.

(ii) follows from (i). \hfill \Box

**Example 2.12.** Consider the monoid $S = \{1, s\}$ where $s$ is an idempotent element and the $S$-act $A = \{a, b, c\}$ with the action defined by $1c = c, sc = a$ and $a, b$ are fixed elements. Then all non-trivial subacts of $A$ are the sets $\{a\}, \{b\}, \{a, b\}$ and $\{a, c\}$. It is clear that $\{\{a\}, \{b\}\}$ is a dominating set in $\mathcal{Inc}(A)$ and $\gamma(\mathcal{Inc}(A)) = 2$.

An independent set in a graph is a set of pairwise non-adjacent vertices. The independence number of $G$, written as $\alpha(G)$, is the maximum size of an independent set.

**Remark 2.13.** In [13], it has shown that the independence number of the intersection graph of an $S$-act $A$ equals the number of minimal subacts of $A$. But this is not the case for the inclusion graphs. For instance, let $A = \{a, b, c, d\}$ be an $S$-act with trivial action. Then $\alpha(\mathcal{Inc}(A)) = 6$ whereas $|\text{Min}(A)| = 4$.
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