On Generalized Ricci-Recurrent LP-Sasakian Manifolds

Rajesh Kumar1, Jay Prakash Singh2*, Jagannath Chowdhury2+

1 Department of Mathematics, Pachhunga University College, Aizawl, Mizoram, India.
rajesh_mzu@yahoo.com
2 Department of Mathematics and Computer Science, Mizoram University, Aizawl, Mizoram, India.
jpsmaths@gmail.com
*jagai_76@yahoo.com

Article history:
Received November 2014
Accepted December 2014
Available online December 2014

Abstract

The object of the present paper is to study a generalized Ricci-recurrent LP-Sasakian manifold. Here we show that the generalized Ricci-recurrent LP-Sasakian manifold admitting cyclic Ricci tensor is an Einstein manifold.

Keywords: Generalized Ricci-recurrent manifold, Ricci-recurrent manifold, LP-Sasakian manifold, Einstein manifold


1. Introduction

In 1950, A.G. Walker [1] introduced the idea of recurrent manifolds. In 2012, De and Mallick [17] defined almost pseudo concircularly symmetric manifolds. In the same year Taleshian and N. Asghari [18] defined Lorentzian $\alpha$-Sasakian manifolds. On the otherhand, De and Guha [2] introduced generalized recurrent manifold with the non-zero 1-form $A$ and another non-zero associated 1-form $B$. Such a manifold has been denoted by $GK_n$. If the associated 1-form $B$ becomes zero, then the manifold $GK_n$ reduces to a recurrent manifold introduced by Ruse [3] which is denoted by $K_n$.

The idea of Ricci-recurrent manifold was introduced by Patterson [4]. He denoted such a manifold by $R_n$. Ricci-recurrent manifolds have been studied by many authors [5], [6], [7].

In 1989, K. Matsumoto [8] introduced the notion of LP-Sasakian manifold. Then I. Mihai and R. Rosca [9] introduced the same notion independently and they obtained several results on this manifold. LP-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [10], U.C. De and et.al., [11].
In 1995, De, Guha and Kamila [12] introduced and studied a type of Riemannian manifold \((M_n, g)\) \((n > 2)\) whose Ricci tensor \(S\) of type \((0,2)\) satisfies the condition

\[
(V_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),
\]

where \(A\) and \(B\) are two 1-forms, \(B\) is non zero, \(P, Q\) are two vector fields such that

\[
g(X, P) = A(X),
\]

\[
g(X, Q) = B(X).
\]

for every vector field \(X\). Such a manifold was called a generalized Ricci-recurrent manifold and an \(n\) dimensional manifold of this kind was denoted by \(GR_n\). If the 1-form \(B\) vanishes identically, then the manifold reduces to a Ricci recurrent manifold introduced by Patterson (1952).

In this paper it is proved that in a generalized Ricci-recurrent LP-Sasakian manifold the vector fields \(P\) and \(Q\) defined by (2) and (3) are in opposite direction. In the last section it is proved that if a generalized Ricci-recurrent LP-Sasakian manifold admits a cyclic Ricci-tensor then the manifold becomes an Einstein manifold.

2. Preliminaries

An \(n\)-dimensional differentiable manifold \(M_n\) is called an LP-Sasakian manifolds if it admits a \((1,1)\) tensor field \(\varphi\), a vector field \(\xi\), a 1-form \(\eta\) and a Lorentzian metric \(g\) which satisfy

\[
(V_X \varphi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,
\]

where \(V\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\).

Let \(S\) and \(r\) denote respectively the Ricci tensor of type \((0, 2)\) and the scalar curvature of \(M_n\). It is known that in an LP-Sasakian manifold \(M_n\), the following relations hold

\[
\varphi(X) = 0, \quad \eta(\varphi X) = 0,
\]

\[
\eta(\xi) = -1,
\]

\[
\varphi^2(X) = X + \eta(X)\xi,
\]

\[
g(X, \xi) = \eta(X),
\]

\[
\nabla_X \xi = \varphi(X),
\]

\[
(V_X \eta)(Y) = g(X, \varphi Y) = g(\varphi X, Y),
\]

\[
R(X, Y)\xi = \eta(Y)X - \eta(X)Y,
\]

\[
S(X, \xi) = (n - 1)\eta(X).
\]

for any vector fields \(X, Y\).

The above results will be used in the next sections.
3. Generalized Ricci-recurrent LP-Sasakian manifolds

In this section we suppose that a generalized Ricci-recurrent manifold is an LP-Sasakian manifold. Then Ricci tensor $S$ of a generalized Ricci-recurrent manifold will satisfy the condition (1).

We have,

$$(\nabla_X S)(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$  \hfill (13)

Therefore from (1) and (13), we get

$$A(X)S(Y, Z) + B(X)g(Y, Z) = XS(Y, Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z).$$  \hfill (14)

Putting $Z = \xi$ in relation (14), we get

$$A(X)S(Y, \xi) + B(X)g(Y, \xi) = XS(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi).$$

Using (8), (9) and (12), we get

$$(n - 1)A(X)\eta(Y) + B(X)\eta((Y) = (n - 1)[(\nabla_X \eta)(Y) + \eta(\nabla_X Y)] - (n - 1)\eta(\nabla_X Y) - S(Y, \varphi(X))$$

$$= (n - 1)(\nabla_X \eta)(Y) - S(Y, \varphi(X)).$$  \hfill (15)

Now,

$$(\nabla_X g)(Z, \xi) = Xg(Z, \xi) - g(\nabla_X Z, \xi) - g(Z, \nabla_X \xi)$$

using (8) and (9) in the above equation, we have

$$(\nabla_X g)(Z, \xi) = X\eta(Z) - \eta(\nabla_X Z) - g(Z, \varphi(X)).$$

Since,

$$g(X, \varphi(Z)) + g(\varphi(Z), X) = 0,$$

Therefore, from the above equation, we get

$$(\nabla_X g)(Z, \xi) = (\nabla_X \eta)(Z) + g(\varphi(Z), X).$$  \hfill (16)

Again, since $\nabla g = 0$.

Therefore, from equation (16), we have

$$(\nabla_X \eta)(Z) = -g(\varphi(Z), X).$$  \hfill (17)

Hence from (15) and (17), we get

$$(n - 1)A(X)\eta(Y) + B(X)\eta((Y) = -(n - 1)g(\varphi(Y), X) - S(Y, \varphi(X)).$$

Putting $Y = \xi$ in the above relation, we get

$$[(n - 1)A(X) + B(X)]\eta(\xi) = -(n - 1)g(\varphi(\xi), X) - S(\xi, \varphi(X))$$  \hfill (18)

Since for every vector field

$$g(\varphi(X), \xi) = \eta(\varphi(X)) = 0$$
In virtue of (5), (6) and (12), we get from (18)

\[(n - 1)A(X) + B(X)\eta(\xi) = 0,
\]

since \(\eta(\xi) = -1\).

Therefore, we get

\[(n - 1)A(X) + B(X) = 0,
\]

which leads to state the following theorem:

**Theorem 3.1.** If a generalized Ricci-recurrent manifold is an LP-Sasakian manifold, then the associated vector fields of the 1-form A and B are in opposite direction.

**4. Generalized Ricci-recurrent LP-Sasakian manifold admitting cyclic Ricci-tensor**

In this section we suppose that a generalized Ricci-recurrent LP-Sasakian manifold admits a cyclic Ricci tensor \(S\), that is

\[
(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0
\]

(20)

then by the virtue of (1) it follows from (20) that

\[
A(X)S(Y, Z) + B(X)g(Y, Z) + A(Y)S(Z, X) + B(Y)g(Z, X)
+ A(Z)S(X, Y) + B(Z)g(X, Y) = 0,
\]

(21)

Putting \(Z = \xi\) in (21) and using (8) and (12), we get

\[
(n - 1)A(X)\eta(Y) + B(X)\eta(Y) + (n - 1)A(Y)\eta(X)
+ B(Y)\eta(X) + A(\xi)S(X, Y) + B(\xi)g(X, Y) = 0.
\]

which implies that

\[
[(n - 1)A(X) + B(X)]\eta(Y) + [(n - 1)A(Y) + B(Y)]\eta(X)
+ A(\xi)S(X, Y) + B(\xi)g(X, Y) = 0.
\]

(22)

By the virtue of (19), we have from (22)

\[
A(\xi)S(X, Y) + B(\xi)g(X, Y) = 0
\]

which gives

\[
S(X, Y) = \mu g(X, Y)
\]

where \(\mu = -\frac{B(\xi)}{A(\xi)}\)

Hence we can state the following theorem:

**Theorem 4.1.** If a generalized Ricci-recurrent LP-Sasakian manifold admits a cyclic Ricci tensor, then it becomes an Einstein manifold.
Corollary 4.1. For n-dimensional generalized Ricci-recurrent manifold $M_n$ with cyclic Ricci tensor, we have the following results:

1. If $M_n$ is a Lorentzian $\beta$-Kenmotsu manifold [13] and [14], then
   $$ A(\xi)S(X,Y) = -2n\beta^2 A(\xi)g(X,Y). $$

2. If $M_n$ is a Lorentzian $\alpha$-Sasakian manifold [15], then
   $$ A(\xi)S(X,Y) = 2n\alpha^2 A(\xi)g(X,Y). $$

3. If $M_n$ is a $(LCS)_n$-manifold [16], then
   $$ A(\xi)S(X,Y) = (n-1)(\alpha^2 - \rho)A(\xi)g(X,Y). $$

Therefore, in view of corollary (4.1) we can state the following theorem:

Theorem 4.2: Let $M_n$ be a generalized Ricci-recurrent manifold with cyclic Ricci tensor. If $M_n$ is one of Lorentzian $\beta$-Kenmotsu manifold, Lorentzian $\alpha$-Sasakian manifold and $(LCS)_n$-manifold with non-zero $A(\xi)$ everywhere, then $M_n$ is Einstein manifold.

References