R-robustly measure expansive homoclinic classes are hyperbolic

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Abstract

Let $f: M \to M$ be a diffeomorphism on a closed smooth $n(n \geq 2)$-dimensional manifold $M$ and let $p$ be a hyperbolic periodic point of $f$. We show that if the homoclinic class $H_f(p)$ is $R$-robustly measure expansive then it is hyperbolic.

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1. Introduction

Roughly speaking, definition of expansiveness is, if two orbits are closed then they are one orbit which was introduced by Utz [22]. A main research is to study structure of the orbits in differentiable dynamical systems, and so a goal of differentiable dynamical system is to study stability properties (Anosov, Axiom A, hyperbolic, structurally stable, etc.). Therefore, expansiveness is an important notion to study stability properties. For instance, Mañé [11] proved that if a diffeomorphism is $C^1$ robustly expansive then it is quasi-Anosov. Arbieto [1] proved that for $C^1$ generic an expansive diffeomorphism is Axiom A without cycles. For expansivity, we can find various generalization notations, that is, continuum-wise expansive [5], $n$-expansive [13], and measure expansive [14]. Among that, we study measure expansiveness in the paper. Let $M$ be a closed smooth $n(n \geq 2)$-dimensional Riemmanian manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$-topology. Denote by $d$ the distance on $M$ induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle $TM$. Let $\Lambda$ be a closed $f$-invariant set. We say that $\Lambda$ is hyperbolic if the tangent bundle $T\Lambda M$ has a $Df$-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|D_x f^n|_{E^s}\| \leq C \lambda^n \text{ and } \|D_x f^{-n}|_{E^u}\| \leq C \lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$. If $\Lambda = M$, then we say that $f$ is Anosov.

For any closed $f$-invariant set $\Lambda \subset M$, we say that $\Lambda$ is expansive for $f$, if there is $e > 0$ such that for any $x, y \in \Lambda$ if $d(f^n(x), f^n(y)) \leq e$ then $x = y$. Equivalently, $\Lambda$ is expansive for $f$ if there is $e > 0$ such that
\[ \Gamma_{f}^{\pm}(x) = \{ x \} \] for all \( x \in \Lambda \), where \( \Gamma_{f}^{\pm}(x) = \{ y \in \Lambda : d(f^{i}(x), f^{i}(y)) \leq e \text{ for all } i \in \mathbb{Z} \} \). Let \( M(M) \) be the set of all Borel probability measures on \( M \) endowed with the weak* topology, and let \( M^{*}(M) \) be the set of nonatomic measures \( \mu \in M(M) \). For any \( \mu \in M^{*}(M) \), we say that \( \Lambda \) is \( \mu \)-expansive for \( f \) if \( \mu(\Gamma_{f}^{\pm}(x)) = 0 \). \( \Lambda \) is said to be \textit{measure expansive} for \( f \) if \( \Lambda \) is \( \mu \)-expansive for all \( \mu \in M^{*}(M) \); that is, there is a constant \( e > 0 \) such that for any \( \mu \in M^{*}(M) \) and \( x \in \Lambda \), \( \mu(\Gamma_{f}^{\pm}(x)) = 0 \). Here \( e \) is called a \textit{measure expansive constant} of \( f \mid_{\Lambda} \). Clearly, the expansiveness implies the measure expansiveness, but the converse does not hold in general (see [14, Theorem 1.35]). We say that \( f \) is \textit{quasi-Anosov} if for any \( \nu \in TM \setminus \{ 0 \} \), the set \( \{ ||Df^{n}(v)|| : n \in \mathbb{Z} \} \) is unbounded. Sakai et al. [19] proved that if a diffeomorphism \( f \) is \( C^{1} \) robustly measure expansive then it is quasi-Anosov. Lee [7] proved that for \( C^{1} \) generic \( f \), if \( f \) is measure expansive then it is Axiom A without cycles. It is well known that if \( p \) is a hyperbolic periodic point of \( f \) with period \( \pi(p) \) then the sets

\[ W^{s}(p) = \{ x \in M : f^{n}(p) \rightarrow p \text{ as } n \rightarrow \infty \} \]

and

\[ W^{u}(p) = \{ x \in M : f^{-n}(p) \rightarrow p \text{ as } n \rightarrow \infty \} \]

are \( C^{1} \)-injectively immersed submanifolds of \( M \). A point \( x \in W^{s}(p) \setminus W^{u}(p) \) is called a \textit{homoclinic point} of \( f \) associated to \( p \). The closure of the homoclinic points of \( f \) associated to \( p \) is called the \textit{homoclinic class} of \( f \) associated to \( p \), and it is denoted by \( H_{f}(p) \). It is clear that \( H_{f}(p) \) is compact, transitive, and invariant.

Denote by \( P(f) \) the set of all periodic points of \( f \). Let \( q \) be a hyperbolic periodic point of \( f \). We say that \( p \) and \( q \) are \textit{homoclinically related}, and write \( p \sim q \) if

\[ W^{s}(p) \cap W^{u}(q) \neq \emptyset \text{ and } W^{u}(p) \cap W^{s}(q) \neq \emptyset. \]

It is clear that if \( p \sim q \) then index\((p) = \text{index}(q) \), that is, \( \dim W^{s}(p) = \dim W^{u}(q) \). By the Smale’s transverse homoclinic point theorem, \( H_{f}(p) = \{ q \in P_{f}(f) : q \sim p \} \), where \( \Lambda \) is the closure of the set \( \Lambda \) and \( P_{f}(f) \) is the set of all hyperbolic periodic points. Note that if \( p \) is a hyperbolic periodic point of \( f \) then there is a neighborhood \( U \) of \( p \) and a \( C^{1} \)-neighborhood \( \mathcal{U}(f) \) of \( f \) such that for any \( g \in \mathcal{U}(f) \) there exists a unique hyperbolic periodic point \( p_{g} \) of \( g \) in \( U \) with the same period as \( p \) and \( \text{index}(p_{g}) = \text{index}(p) \). Such a point \( p_{g} \) is called the \textit{continuation} of \( p = p_{f} \). We say that \( \Lambda \) is \textit{locally maximal} if there is a neighborhood \( U \) of \( \Lambda \) such that \( \Lambda = \bigcap_{n \in \mathbb{Z}} f^{n}(U) \).

In differentiable dynamical systems, a main research topic is to study that for a given system, if the system has a property then we consider that a system which is \( C^{1} \)-nearby system has the same property. Then, we consider various type of \( C^{1} \)-perturbation property on a closed invariant set which are the following.

(a) We say that \( H_{f}(p) \) is \( C^{1} \) \textit{robustly \( \mathcal{P} \) property} if there is a \( C^{1} \)-neighborhood \( \mathcal{U}(f) \) of \( f \) such that for any \( g \in \mathcal{U}(f) \), \( H_{g}(p_{g}) \) is \( \mathcal{P} \) property. If \( \mathcal{P} \) is expansive then the expansive constant is uniform, which means that the constant depends on \( f \) (see [15, 16]).

(b) We say that \( H_{f}(p) \) is \( C^{1} \) \textit{persistently \( \mathcal{P} \) property} if there is a \( C^{1} \)-neighborhood \( \mathcal{U}(f) \) of \( f \) such that for any \( g \in \mathcal{U}(f) \), \( H_{g}(p_{g}) \) is \( \mathcal{P} \) property. If \( \mathcal{P} \) is expansive then the expansive constant is not uniform which means that the constant depends on \( g \in \mathcal{U}(f) \) (see [20]).

(c) We say that \( H_{f}(p) \) is \( C^{1} \) \textit{stably \( \mathcal{P} \) property} if there are a \( C^{1} \)-neighborhood \( \mathcal{U}(f) \) of \( f \) and a neighborhood \( U \) of \( H_{f}(p) \) such that for any \( g \in \mathcal{U}(f) \), \( \Lambda_{g}(U) \) is \( \mathcal{P} \) property, where \( \Lambda_{g}(U) = \bigcap_{n \in \mathbb{Z}} g^{n}(U) \) is the continuation of \( H_{f}(p) \). If \( \mathcal{P} \) is expansive, then the expansive constant is not uniform which means that the constant depends on \( g \in \mathcal{U}(f) \) (see [8]).

In the item (c), we can also consider a closed invariant set. We say that a subset \( \mathcal{S} \subset \text{Diff}(M) \) is \textit{residual} if \( \mathcal{S} \) contains the intersection of a countable family of open and dense subsets of \( \text{Diff}(M) \); in this case \( \mathcal{S} \) is dense in \( \text{Diff}(M) \). A property \( \mathcal{P} \) is said to be \( C^{1} \)-\textit{generic} if \( \mathcal{P} \) holds for all diffeomorphisms which belong to some residual subset of \( \text{Diff}(M) \).

Li [10] introduced another \( C^{1} \) robust property which is called \( R \)-\textit{robustly \( \mathcal{P} \) property}. Using to the notion, we consider the following.
Definition 1.1. Let the homoclinic class \( H_f(p) \) associated to a hyperbolic periodic point \( p \). We say that \( H_f(p) \) is \( R \)-robustly measure expansive if there are a \( C^1 \)-neighborhood \( U(f) \) of \( f \) and a residual set \( \mathcal{G} \) of \( U(f) \) such that for any \( g \in \mathcal{G} \), \( H_g(p_g) \) is measure expansive, where \( p_g \) is the continuation of \( p \).

Recently, Pacifico and Vieites [17] proved that a diffeomorphism \( f \) in a residual subset far from homoclinic tangencies are measure expansive. Lee and Lee [9] proved that if the homoclinic class \( H_f(p) \) is \( C^1 \) stably measure expansive then it is hyperbolic. Koo et al. [6] proved that for \( C^1 \) generic \( f \), if a locally maximal homoclinic class \( H_f(p) \) is measure expansive, then it is hyperbolic. Owing to the result, we have the following which is a main theorem of the paper.

**Theorem 1.2.** Let the homoclinic class \( H_f(p) \) associated to a hyperbolic periodic point \( p \). If \( H_f(p) \) is \( R \)-robustly measure expansive then it is hyperbolic.

2. Dominated splitting and Hyperbolic periodic points in \( H_f(p) \)

Let \( M \) be as before, and let \( f \in \text{Diff}(M) \). A periodic point for \( f \) is a point \( p \in M \) such that \( f^n(p) = p \), where \( \pi(p) \) is the minimum period of \( p \). Denote by \( P(f) \) the set of all periodic points of \( f \). For given \( x, y \in M \), we write \( x \to y \) if for any \( \delta > 0 \), there is a \( \delta \)-pseudo orbit \( \{x_i\}_{i=0}^{n} (n > 1) \) of \( f \) such that \( x_0 = x \) and \( x_n = y \). We write \( x \leftrightarrow y \) if \( x \to y \) and \( y \to x \). The set of points \( \{x \in M : x \leftrightarrow x\} \) is called the \textit{chain recurrent set} of \( f \) and is denoted by \( \mathcal{R}(f) \). It is clear that \( P(f) \subset \Omega(f) \subset \mathcal{R}(f) \). Here \( \Omega(f) \) is the non-wandering set of \( f \). Let \( p \) be a hyperbolic periodic point of \( f \). We say that the \textit{chain component} if for any \( x \in M \), \( x \rightarrow p \) and \( p \rightarrow x \) and denote it by \( C_f(p) \). Note that the chain component \( C_f(p) \) of \( f \) is a equivalent class, it is a closed set and \( f \)-invariant set. The following was proved by Bonatti and Crovisier [2].

**Remark 2.1.** There is a residual set \( \mathcal{G}_1 \subset \text{Diff}(M) \) such that for any \( f \in \mathcal{G}_1 \), \( H_g(p) = C_f(p) \) for some hyperbolic periodic point \( p \).

**Proposition 2.2.** Let the homoclinic class \( H_f(p) \) be \( R \)-robustly measure expansive. If \( x \in W^s(p) \cap W^u(p) \), then \( x \in W^s(p) \cap W^u(p) \).

**Proof.** Since \( H_f(p) \) is \( R \)-robustly measure expansive, there exists a \( C^1 \)-neighborhood \( U(f) \) and a residual set \( \mathcal{G} \subset U(f) \) such that for any \( g \in \mathcal{G} \), \( H_g(p_g) \) is measure expansive. Let \( \mathcal{G} = \mathcal{G}_1 \). Since \( x \in W^s(p) \cap W^u(p) \), by [17, Proposition 2.6], there is \( g \in U(f) \cap \mathcal{G} \) such that we can make a small arc \( J \subset W^s(p_g) \cap W^u(p_g) \). Since \( H_g(p_g) = C_g(p_g) \), we know \( J \subset C_g(p_g) \). Let \( \text{diam}(J) = 1 \). We define a measure \( \mu \in M^*(M) \) by \( \mu(C) = \nu(C \cap J) \) for any Borel set \( C \subset M \), where \( \nu \) is a normalized Lebesgue measure on \( J \). Let \( e = 1/4 \) be a measure expansive constant. Since \( J \subset W^s(p_g) \cap W^u(p_g) \), there is \( N > 0 \) such that \( \text{diam}(g^i(J)) \leq 1/4 \) for \( -N \leq i \leq N \), and \( g^i(J) \subset W^s_{e/4}(p_g) \cap W^u_{e/4}(p_g) \) for \( |i| > N \). Thus for all \( i \in \mathbb{Z} \), we know that \( \text{diam}(g^i(J)) \leq e \). Recall that

\[
\Gamma_e(x) = \{ y \in H_g(p_g) : d(g^i(x), g^i(y)) \leq e \text{ for } i \in \mathbb{Z} \}.
\]

We can construct the set

\[
A_e(x) = \{ y \in J : d(g^i(x), g^i(y)) \leq e \text{ for } i \in \mathbb{Z} \}.
\]

Then we know \( A_e(x) \subset \Gamma_e(x) \). Thus we have

\[
0 < \mu(A_e(x)) \leq \mu(\Gamma_e(x)),
\]

which is a contradiction to the measure expansivity of \( H_g(p_g) \). \( \square \)

For \( f \in \text{Diff}(M) \), we say that a compact \( f \)-invariant set \( \Lambda \) admits a \textit{dominated splitting} if the tangent bundle \( T \Lambda \) has a continuous \( Df \)-invariant splitting \( E \oplus F \) and there exist \( C > 0 \), \( 0 < \lambda < 1 \) such that for all \( x \in \Lambda \) and \( n \geq 0 \), we have

\[
\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C \lambda^n.
\]
Theorem 2.3. Let \( H_f(p) \) be the homoclinic class containing a hyperbolic periodic point \( p \). Suppose that \( H_f(p) \) is R-robustly measure expansive. Then there exist a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) of \( f \) and a residual set \( \mathcal{J} \subseteq \mathcal{U}(f) \) such that for any \( g \in \mathcal{J} \), \( H_g(p_g) \) admits a dominated splitting \( T_{H_g(p_g)} \mathcal{M} = E(g) \oplus F(g) \) with index \( (p_g) = \dim E(g) \).

Proof. Suppose that \( H_f(p) \) is R-robustly measure expansive. Then as in the proof of [20, Theorem 1], there is \( m > 0 \) such that for every \( x \in W^s(p) \cap W^u(p) \) there exists \( m_1 \in [90, m] \) such that \( \|Df^{m_1}|_{E(x)}\| \cdot \|Df^{-m_1}|_{F(f^{-m_1}(x))}\| \leq 1/2 \). Since the dominated splitting can be extended by continuity to the

\[
W^s(p) \cap W^u(p) = H_f(p),
\]

we have that \( H_f(p) \) has a dominated splitting \( E \oplus F \).

\[
\text{Theorem 2.4. Let the homoclinic class } H_f(p) \text{ be } R\text{-robustly measure expansive. Then there exist } C > 0, 0 < \lambda < 1 \text{ and } m > 0 \text{ such that for any } q \in H_f(p) \text{ is a hyperbolic periodic point of period } \pi(q) \text{ and } q \sim p \text{, then}
\]

\[
\prod_{i=0}^{k-1} \|Df^{m_i}|_{E_x(f^{-m_i}(q))}\| < C \lambda^k \text{ and } \prod_{i=0}^{k-1} \|Df^{-m_i}|_{E_u(f^{-m_i}(q))}\| < C \lambda^k,
\]

where \( k = \lfloor \pi(q)/m \rfloor \) (\( \lfloor \cdot \rfloor \) represents the integer part).

Proof. Since \( H_f(p) \) is R-robustly measure expansive, there are a \( C^1 \)-neighborhood \( \mathcal{U}(f) \) and a residual set \( \mathcal{J} \subseteq \mathcal{U}(f) \) such that for any \( g \in \mathcal{J} \), \( H_g(p_g) \) is measure expansive. Let \( \mathcal{J} = \mathcal{J}_1 \). Since \( q \in H_f(p) \) and \( p \sim q \), as in the proof of [20], it is enough to show that the family of periodic sequences of linear isomorphisms of \( \mathbb{R}^n \) generated by \( Df \) along the hyperbolic periodic points \( q \in H_f(p) \), \( q \sim p \) and index \( (p_g) = \dim E(g) \) is uniformly hyperbolic. Suppose, by contradiction, that the assume does not hold. Then as in the proof of [18, Theorem B], we may assume that a hyperbolic periodic point \( q \in H_f(p) \) such that the weakest normalized eigenvalue \( \lambda \) is close to 1. Then by Franks lemma, there is \( g \in \mathcal{J} \) such that for any small \( \gamma > 0 \) we can construct a closed small curve \( \mathcal{J}_q \) containing \( q \) or a closed small circle \( \mathcal{C}_q \) centered at \( q \) such that \( \mathcal{J}_q \subset C_g(p_g) \) and two endpoints are related to \( p_g \) and \( \mathcal{C}_q \subset C_g(p_g) \). Note that \( \mathcal{J}_q \) and \( \mathcal{C}_q \) are \( g^{n(q)} \)-invariant, normally hyperbolic, and \( g^{l(q)} \) is the identity map for some \( l > 0 \) (see [18]). For \( \mathcal{J}_q \), we define a measure \( \mu \in \mathcal{M}^s(M) \) by

\[
\mu(C) = \frac{1}{\ell(q)} \sum_{l=0}^{\pi(q)-1} \nu(g^{-i}(C \cap g^l(\mathcal{J}_q)))
\]

for any Borel set \( C \) of \( M \), where \( \nu \) is a normalized Lebesgue measure on \( \mathcal{J}_q \). Let \( \gamma \leq \epsilon \) be a measure expansive constant of \( g|_{H_g(p_g)} \). By [14, Proposition], \( g \) is measure expansive if and only if \( g^n \) is measure expansive for \( n \in \mathbb{Z} \setminus \{0\} \). Let \( \Gamma_g^\delta(x) = (y \in H_g(p_g) : d(g^{l(q)}(x), g^{l(q)}(y)) \leq \epsilon \) for all \( i \in \mathbb{Z} \). Then we have

\[
\left\{ y \in \mathcal{J}_q : d(g^{l(q)}(x), g^{l(q)}(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z} \right\} = \left\{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z} \right\}.
\]

Thus we know

\[
0 < \mu(\{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z} \}) \leq \mu(\Gamma_g^\delta(x)).
\]

Since \( H_g(p_g) \) is measure expansive for \( g \), we know \( \mu(\Gamma_g^\delta(x)) = 0 \). Thus we have

\[
\mu(\{ y \in \mathcal{J}_q : d(g^i(x), g^i(y)) \leq \epsilon \text{ for all } i \in \mathbb{Z} \}) = 0.
\]

This is a contradiction.

For \( \mathcal{C}_q \), case, if \( \mathcal{C}_q \) is irrational rotation then using the Franks’ lemma, there is \( h \in \mathcal{U}(g) \cap \mathcal{J} \) such that \( \mathcal{C}_{q_h} \) is rational rotation which is centered at \( q_{h_n} \), where \( \mathcal{U}(g) \) is a \( C^1 \)-neighborhood of \( g \), and \( q_{h_n} \) is the
continuation of $q$ for $h$. Then there is $k > 0$ such that $h^k : \mathcal{C}_{qh} \to \mathcal{C}_{qh}$ is the identity map. Then we define a measure $\mu \in \mathcal{M}^s(M)$ by

$$
\mu(B) = \frac{1}{k} \sum_{i=0}^{k-1} \eta(h^i(B \cap h^{-1}(\mathcal{C}_{qh})))
$$

for any Borel set $B$ of $M$, where $\eta$ is a normalized Lebesgue measure on $\mathcal{C}_{qh}$. Then as in the proof of previous argument, we can derive a contradiction.

By [14, Proposition], $g$ is measure expansive if and only if $g^n$ is measure expansive for $n \in \mathbb{Z} \setminus \{0\}$. Theorem 2.4 can be rewritten as the following.

**Theorem 2.5.** Let the homoclinic class $H_f(p)$ be $R$-robustly measure expansive. Then there exist $0 < \lambda < 1$ and $L \leq 1$ such that $q$ is a hyperbolic periodic point of period $\pi(q)$ with $L > \pi(q)$ and $q \sim p$, then

$$
\prod_{i=0}^{\pi(q)-1} \|Df|_{E^s(f^i(q))}\| < \lambda^{\pi(q)} \quad \text{and} \quad \prod_{i=0}^{\pi(q)-1} \|Df^{-1}|_{E^u(f^{-i}(q))}\| < \lambda^{\pi(q)}.
$$

### 3. Local product structure

Let $\Lambda$ be a closed, $f$-invariant set. We say that $\Lambda$ has a **local product structure** if for given $\epsilon > 0$ there exists a $\delta > 0$ such that if $d(x, y) < \delta$ and $x, y \in \Lambda$, then

$$
\emptyset \neq W^s_\epsilon(x) \cap W^u_\epsilon(y) \subset \Lambda.
$$

By the uniqueness of the dominated splitting, if $q \in H_f(p)$ is a periodic point with $q \sim p$ then we have $E(q) = E^s(q)$ and $F(q) = E^u(q)$. Let $\dim E = s$ and $\dim F = u$, and put $D^s_t = \{x \in \mathbb{R}^t : \|x\| \leq r\} \ (r > 0)$, for $j = s, u$. Let $\text{Emb}_\Lambda(D^s_1, M)$ be the space of $C^1$ embeddings $\beta : D^s_1 \to M$ such that $\beta(0) \in \Lambda$ endowed with the $C^1$ topology. Then we have the following.

**Proposition 3.1 ([4, 12]).** Let $H_f(p)$ be the homoclinic class of $f$ associated to a hyperbolic periodic point $p$, and let $\Lambda = H_f(p)$. Suppose that $\Lambda$ has a dominated splitting $E \oplus F$. Then there exist sections $\phi^s : \Lambda \to \text{Emb}_\Lambda(D^s_1, M)$ and $\phi^u : \Lambda \to \text{Emb}_\Lambda(D^u_1, M)$ such that by defining $W^s_\epsilon(x) = \phi^s(x)D^s_\epsilon$ and $W^u_\epsilon(x) = \phi^u(x)D^u_\epsilon$, for each $x \in \Lambda$, we have

1. $T_xW^s_\epsilon(x) = E(x)$ and $T_xW^u_\epsilon(x) = F(x)$;
2. for every $0 < \epsilon_1 < 1$ there exists $0 < \epsilon_2 < 1$ such that $f(W^s_{\epsilon_1}(x)) \subset W^s_{\epsilon_2}(f(x))$ and $f^{-1}(W^u_{\epsilon_2}(x)) \subset W^u_{\epsilon_1}(f^{-1}(x))$;
3. for every $0 < \epsilon_1 < 1$ there exists $0 < \delta < 1$ such that if $d(x, y) < \delta$ $(x, y \in \Lambda)$ then $W^s_{\epsilon_1}(x) \cap W^u_{\epsilon_1}(y) \neq \emptyset$, and this intersection is transverse.

The sets $W^s_{\epsilon_1}(x)$ and $W^u_{\epsilon_1}(x)$ are called the **local center stable** and **local unstable manifolds** of $x$, respectively. The following lemma can be proved similarly to that of Lemma 4 in [20].

**Lemma 3.2.** Let $H_f(p)$ be the homoclinic class of $f$ associated to a hyperbolic periodic point $p$, and suppose that $H_f(p)$ is $R$-robustly measure expansive. Then for $C, \lambda$ as in Theorem 3.1 and $\delta > 0$ satisfying $\lambda^t < 1$ and $q \sim p$, there exists $0 < \epsilon_1 < \epsilon$ such that if for all $0 \leq n \leq (\pi(q)$ it holds that for some $\epsilon_2 > 0$, $f^n(W^s_{\epsilon_1}(q)) \subset W^s_{\epsilon_2}(f^n(q))$, then

$$
f^{-\pi(q)}(W^u_{\epsilon_2}(q)) \subset W^u_{\epsilon_2}(f^{-\pi(q)}(q)) \subset W^u_{\epsilon_2}(f^{-\pi(q)}(q)).
$$

Similarly, if $f^{-n}(W^u_{\epsilon_2}(q)) \subset W^u_{\epsilon_1}(f^{-n}(q))$, then

$$
f^{-\pi(q)}(W^u_{\epsilon_2}(q)) \subset W^u_{\epsilon_2}(f^{-\pi(q)}(q)) \subset W^u_{\epsilon_2}(f^{-\pi(q)}(q)).
$$
Recall that by using the Smale’s transverse theorem, we have $H_f(p) = \overline{\text{homo}_p}$, where $\text{homo}_p = \{q \in P_h(f) : q \sim p\}$.

**Lemma 3.3.** Let $H_f(p)$ be the homoclinic class of $f$ associated to a hyperbolic periodic point $p$, and let $\epsilon > 0$ be a measure expansive constant. Suppose that $H_f(p)$ is $R$-robustly measure expansive. Then

(a) for any hyperbolic periodic point $q \in \text{homo}_p$ and $0 < \epsilon_1 < \epsilon$, there is $\epsilon_2 > 0$ such that

$$f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ and } f^{-n}(W^{cu}_{\epsilon_2}(q)) \subset W^{cu}_{\epsilon_1}(f^{-n}(q)) \text{ for all } n \geq 0.$$  

(b) for any $y \in W^{cs}_{\epsilon_2}(q)$ and $q \in \text{homo}_p$ we have

$$\lim_{n \to \infty} d(f^n(q), f^n(y)) = 0.$$  

**Proof.** Let $f \in S = S_1$ and let $H_f(p)$ is $R$-robustly measure expansive. To prove (a), it is enough to show that $f^n(W^{cs}_{\epsilon_2}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q))$. Let $\sup\{\dim W^{cs}_{\epsilon_2}(q) : q \in \text{homo}_p\} < \epsilon$. Since $q \in \text{homo}_p$, we define

$$\epsilon(q) = \sup\{\epsilon > 0 : f^n(W^{cs}_{\epsilon}(q)) \subset W^{cs}_{\epsilon_1}(f^n(q)) \text{ for all } n \geq 0\}.$$  

By Proposition 3.1 and Lemma 3.2, $\epsilon(q) > 0$. Let $\epsilon_0 = \inf(\epsilon(q) : q \in \text{homo}_p)$. If $\epsilon_0 > 0$ then it is a proof of (a). Suppose, by contradiction, that there is a sequence $\{q_n\} \subset \text{homo}_p$ such that $\epsilon(q_n) \to 0$ as $n \to \infty$. Then we have $0 < m_n < \pi(q_n)$ and $y_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $d(f^{m_n}(q_n), f^{m_n}(y_n)) = \epsilon_1$ for $f^{m_n}(q_n), f^{m_n}(y_n) \in W^{cs}_{\epsilon(q_n)}(q_n)$. Let $I_n$ be a closed connected arc joining $f^{m_n}(q_n)$ with $f^{m_n}(y_n)$. Then we know that

(i) $I_n \subset W^{cs}_{\epsilon(q_n)}(q_n);$

(ii) $f^i(I_n) \subset W^{cs}_{\epsilon_1}(f^i(q_n))$ for $0 \leq i \leq \pi(q_n);$

(iii) $\text{diam}(I_n) = \epsilon_1.$

By Lemma 3.2, we know $f^{\pi(q_n)}(W^{cs}_{\epsilon(q_n)}(q_n)) \subset W^{cs}_{\epsilon_1(\pi(q_n))}(q_n)$. Observe that if $n \to \infty$ then $m_n \to \infty$ and $\pi(q_n) - m_n \to \infty$. Suppose that $f^{m_n}(q_n) \to x$ and $f^{m_n}(y_n) \to y$ as $n \to \infty$. Then $I_n \to I$, where $I$ is a close connected arc joining $x$ with $y$. It means that $\text{diam}(f^i(I)) \leq \epsilon_1$ for all $j \in \mathbb{Z}$, and $x \in \overline{\text{homo}_p} = H_f(p)$. We show that the closed connected arc $I \subset H_f(p)$. Since $f \in S$, $H_f(p) = C_f(p)$. For any $a \in I$, take $a_n \in W^{cs}_{\epsilon(q_n)}(q_n)$ such that $f^{m_n}(a_n) \to a$ as $n \to \infty$. As in the proof of [21, Lemma 2.6], let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $\epsilon(q_n) < \epsilon$. Then for $n$ sufficiently large, $\{q_n, f(a_n), \ldots, f^{m_n-1}(a_n), a, f^{m_n+1}(a_n), \ldots, f^{\pi(q_n)-1}(a_n), q_n\}$ is a periodic $\epsilon$-chain through $a$ and having a point in $H_f(p)$. Since $q_n \in \text{homo}_p, H_f(q_n) = H_f(p) = C_f(q_n) = C_f(p)$ and so the closed connected arc $I \subset H_f(p)$. We define a measure $\mu \in \mathcal{M}^*(M)$ by $\mu(C) = \mu_1(C \cap I)$ for any Borel set $C$ of $M$, where $\mu_1$ is a normalized Lebesgue measure on $I$. Let

$$\Gamma_\epsilon(x) = \{y \in H_f(p) : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}.$$  

Since for all $i \in \mathbb{Z}$, $\text{diam}(f^i(I)) \leq \epsilon$, we can construct the set

$$\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\} \subset \Gamma_\epsilon(x).$$  

Thus we have

$$0 < \mu(\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}) \leq \mu(\Gamma_\epsilon(x)).$$  

Since $H_f(p)$ is measure expansive, $\mu(\Gamma_\epsilon(x)) = 0$. Thus $\mu(\{y \in I : d(f^i(x), f^i(y)) \leq \epsilon \text{ for } i \in \mathbb{Z}\}) = 0$ which is a contradiction. The proof of (b) is similar as in the proof of item (b) of [21, Lemma 2.6].
Remark 3.4. In the Lemma 3.3, we consider $q \in \text{homo}_p$. Then we can extend $x \in H_f(p)$, that is, for any $x \in H_f(p)$ and $\epsilon_1 > 0$ there exists $\epsilon_2 > 0$ such that $f^n(W^{cs}_{\epsilon_2}(x)) \subset W^{cs}_{\epsilon_1}(f^n(x))$ for all $n \geq 0$. And if $z \in W^{cs}_{\epsilon_2}(x)$ and $z \in H_f(p)$, then $d(f^i(z), f^i(x)) \to 0$ as $n \to \infty$.

**Proposition 3.5.** Suppose that the homoclinic class $H_f(p)$ is R-robustly measure expansive. Then $H_f(p)$ has a local product structure.

**Proof.** By Lemma 3.3, there is $\epsilon_2 > 0$ such that for any $q \in \text{homo}_p$,

$$W^{cs}_{\epsilon_2}(q) = W^s_{\epsilon_2}(q) \text{ and } W^{cu}_{\epsilon_2}(q) = W^u_{\epsilon_2}(q).$$

By Proposition 3.1 (3), there is $\delta > 0$ such that for any $q, r \in \text{homo}_p$,

$$W^{cs}_{\epsilon_2}(q) \cap W^{cu}_{\epsilon_2}(r) \neq \emptyset.$$

By $\lambda$-lemma, $W^s_{\epsilon_2}(q) \subset \overline{W^s(p)}$ and $W^u_{\epsilon_2}(r) \subset \overline{W^u(p)}$. Thus we know that $W^s_{\epsilon_2}(q) \cap W^u_{\epsilon_2}(r) \subset H_f(p)$. This means that $H_f(p)$ has a local product structure. $\Box$

**Corollary 3.6.** Suppose that the homoclinic class $H_f(p)$ is R-robustly measure expansive. Then for any hyperbolic periodic point $q \in H_f(p)$, $\text{index}(p) = \text{index}(q)$.

**Proof.** The proof is directly obtained by Proposition 3.1 (3), Lemma 3.3, and Proposition 3.5. Thus for any hyperbolic periodic point $q \in H_f(p)$,

$$W^s(p) \cap W^u(q) \neq \emptyset \text{ and } W^u(p) \cap W^s(q) \neq \emptyset.$$

Thus we have $\text{index}(p) = \text{index}(q)$. $\Box$

4. **Proof of Theorem 1.2**

For any $\delta > 0$, a sequence $(x_i)_{i \in \mathbb{Z}}$ is a $\delta$-pseudo orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Let $\Lambda$ be a closed $f$-invariant set. We say that $f$ has the shadowing property on $\Lambda$ such that for any $\epsilon > 0$ there is $\delta > 0$ such that for any $\delta$-pseudo orbit $(x_i)_{i \in \mathbb{Z}} \subset \Lambda$ there is $z \in M$ such that $d(f^i(z), x_i) < \epsilon$ for all $i \in \mathbb{Z}$. The following proposition is a very useful result for proving of Theorem 1.2.

**Proposition 4.1** ([23, Proposition 3.3]). Let $p$ be a hyperbolic periodic point, and let $H_f(p)$ be the homoclinic class of $f$ containing $p$. Let $0 < \lambda < 1$ and $L \geq 1$ be given. Assume that $H_f(p)$ satisfies the following properties.

1. There is a continuous $Df$-invariant splitting $T_{H_f(p)}M = E \oplus F$ with $\dim E = \text{index}(p)$ such that for any $x \in H_f(p)$,

$$\|Df|_{E(x)}\|/m(Df|_{F(x)}) < \lambda^2,$$

where $m(A) = \inf \{|\|A\| : \|v\| = 1\}$ denotes the minimin norm of a linear map $A$.

2. For any $q \in H_f(p) \cap P(f)$, if $q$ is hyperbolic and $\pi(q) > L$, then $\text{index}(p) = \text{index}(q)$ and

$$\prod_{i=0}^{\pi(q)-1} \|Df|_{E(f^i(q))}\| < \lambda^{\pi(q)}, \quad \prod_{i=0}^{\pi(q)-1} \|Df|_{E_u(f^{-i}(q))}\| < \lambda^{\pi(q)}.$$

3. $f$ has the shadowing property on $H_f(p)$.

Then $H_f(p)$ is hyperbolic.

**End of the Proof of Theorem 1.2.** Since $H_f(p)$ is R-robustly measure expansive, by Theorems 2.3 and 2.5, propositions (1) and (2) hold. By Proposition 3.5 and Bowen’s result [3, Proposition 3.6], if the homoclinic class $H_f(p)$ is R-robustly measure expansive then $f$ has the shadowing property on $H_f(p)$, and so, proposition (3) also holds. Thus if $H_f(p)$ is R-robustly measure expansive then it is hyperbolic. $\Box$
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References

[23] X. Wen, S. Gan, L. Wen, $C^1$-stably shadowable chain components are hyperbolic, J. Differential Equations, 246 (2009), 340–357. 4.1