On \((\alpha, p)\)-convex contraction and asymptotic regularity

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Abstract

In this paper, we present the notions of \((\alpha, p)\)-convex contraction (resp. \((\alpha, p)\)-contraction) and asymptotically \(T^2\)-regular (resp. \((T, T^2)\)-regular) sequences, and prove fixed point theorems in the setting of metric spaces.

Keywords: Approximate fixed point, fixed point, \((\alpha, p)\)-convex contraction, asymptotically regular sequence, asymptotically \(T\) (resp. \(T^2\) and \((T, T^2)\))-regular sequences.

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1. Introduction and preliminaries

Let \((X, d)\) be a metric space, and \(C\) a nonempty set of \(X\). A mapping \(T: C \to C\) is called nonexpansive if \(d(Tx, Ty) \leq d(x, y)\) for all \(x, y \in C\). In 2007, Goebel and Japón Pineda \cite{8} introduced the class of mean nonexpansive mappings, an extension for the class of nonexpansive mappings. A mapping \(T: C \to C\) is called mean nonexpansive (or \(\alpha\)-nonexpansive) if, for some \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\) with \(\sum_{i=1}^{n} \alpha_i = 1, \alpha_i \geq 0\) for all \(i\), and \(\alpha_1, \alpha_n > 0\), we have

\[
\sum_{i=1}^{n} \alpha_i d(T^i x, T^i y) \leq d(x, y)
\]

for all \(x, y \in C\). Further, Goebel and Japón Pineda \cite{8} introduced the class of \((\alpha, p)\)-nonexpansive

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mappings. A mapping \( T : C \to C \) is called \((\alpha, p)\)-nonexpansive, if for some \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) with \( \sum_{i=1}^{n} \alpha_i = 1 \), \( \alpha_i \geq 0 \) for all \( i \), and \( \alpha_1, \alpha_n > 0 \), and for some \( p \in [1, \infty) \), we have
\[
\sum_{i=1}^{n} \alpha_i d^p(T^i x, T^i y) \leq d^p(x, y)
\]
for all \( x, y \in C \). In particular, for \( n = 2 \), the above inequality reduces to
\[
\alpha_1 d^p(Tx, Ty) + \alpha_2 d^p(T^2 x, T^2 y) \leq d^p(x, y)
\]
for all \( x, y \in C \), we say that \( T \) is \(((\alpha_1, \alpha_2), p)\)-nonexpansive.

**Example 1.1.** Let \( X = [0, \infty) \subset \mathbb{R} \) with usual metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Define a translation function \( T : X \to X \) by the formula \( Tx = x + a \) for any fixed \( a > 0 \). Now, setting \( \alpha_1 = \alpha_2 = 1/2 \) and \( p \geq 1 \), we have
\[
|Tx - Ty|^p + |T^2 x - T^2 y|^p = 2|x - y|^p,
\]
that is,
\[
\frac{1}{2} |Tx - Ty|^p + \frac{1}{2} |Tx - Ty|^p = |x - y|^p.
\]
Therefore, \( T \) is \(((\alpha_1, \alpha_2), p)\)-nonexpansive mapping.

**Example 1.2.** Let \( X = [0, 1, 2] \) with usual metric \( d(x, y) = |x - y| \) for all \( x, y \in X \). Define the mapping
\[
T : X \to X, \quad Tx = \begin{cases} 1, & x \neq 0, \\ 0, & x = 0. \end{cases}
\]
Setting \( \alpha = (\alpha_1, \alpha_2) \), \( \alpha_1, \alpha_2 > 0 \) and \( \alpha_1 + \alpha_2 = 1 \), for any \( p \geq 1 \), we have
\[
\alpha_1 |Tx - Ty|^p + \alpha_2 |T^2 x - T^2 y|^p \leq |x - y|^p.
\]
Therefore, \( T \) is \(((\alpha_1, \alpha_2), p)\)-nonexpansive mapping.

In 1982, Istrătescu [10] introduced the class of convex contraction mappings in the setting of metric space and generalized the well known Banach’s contraction principle [2]. Some works have appeared recently on generalization of such class of mappings in the setting of metric, ordered metric, and cone metric, \( b \)-metric and \( 2 \)-metric spaces (for example, Alghamdi et al. [1], Ghorbanian et al. [7], Miandaragh et al. [14], Miculescu and Mihail [15], Khan et al. [12], etc.).

Let \( (X, d) \) be a metric space and \( T : X \to X \) be a mapping. Given \( \varepsilon > 0 \), then \( x_0 \in X \) is said to be an \( \varepsilon \)-fixed point of \( T \) on \( X \), whenever \( d(x_0, Tx_0) < \varepsilon \). Note that every fixed point is \( \varepsilon \)-fixed point but the converse need not be true. We denote the set of all \( \varepsilon \)-fixed points of \( T \) for a given \( \varepsilon > 0 \) by \( F_\varepsilon(T) = \{ x \in X | d(Tx, x) < \varepsilon \} \) and \( \text{Fix}(T) \), the set of all fixed points of \( T \).

We say that \( T \) has the approximate fixed point property (AFPP) if for all \( \varepsilon > 0 \), there exists an \( \varepsilon \)-fixed point of \( T \) i.e., for all \( \varepsilon \), \( F_\varepsilon(T) \neq \emptyset \), or equivalently, \( \inf_{x \in X} d(Tx, x) = 0 \).

For details we refer to Berinde [3], Kohlenbach and Leuştean [13], Reich and Zaslavski [16], Tijjs et al. [17].

**Example 1.3** ([12]). If \( X = [0, \infty) \), let \( T : X \to X, \ Tx = x + \frac{1}{2x+T} \) for all \( x \in X \). Setting \( 0 < \varepsilon < \frac{1}{2} \) and taking \( x_0 \in X \) such that \( x_0 > \frac{1-\varepsilon}{2\varepsilon} \), we obtain,
\[
d(Tx_0, x_0) = |Tx_0 - x_0| = \left| \frac{1}{2x_0+1} \right| < \varepsilon.
\]
This shows that \( T \) has an \( \varepsilon \)-fixed point, so \( F_\varepsilon(T) \neq \emptyset \). Note that \( T \) has no fixed point in \( X \).
Definition 1.4 ([4]). A self mapping $T$ on $X$ is said to be asymptotically regular at a point $x \in X$ if 
$$\lim_{n \to \infty} d(T^nx, T^{n+1}x) = 0.$$ 

Definition 1.5 ([5]). A sequence $\{x_n\}$ in $X$ is called an asymptotically $T$-regular, if 
$$\lim_{n \to \infty} d(x_n, Tx_n) = 0.$$

Lemma 1.6 ([3]). If $(X, d)$ is a metric space and $T$ is an asymptotically regular self mapping on $X$, that is 
$$d(T^mx, T^{m+1}x) \to 0 \text{ for all } x \in X,$$ 
then $T$ has the AFPP.

In the next section, we discuss the notions of $(\alpha, p)$-convex contraction (resp. $(\alpha, p)$-contraction) and asymptotically $T^2$-regular (resp. $(T, T^2)$-regular) sequences. Further, we show with examples that the notions of asymptotically $T$-regular and $T^2$-regular sequences are independent to each other.

2. $(\alpha, p)$-convex contraction and asymptotic regularity

Let $T$ be a self mapping on a metric space $(X, d)$.

Definition 2.1. A self mapping $T$ on $X$ is said to be $(\alpha, p)$-contraction, if for some $\alpha \in (0, 1)$ and $p \geq 1$, 
there exists $0 \leq k < 1$ satisfying the following inequality 
$$\alpha d^p(Tx, Ty) + (1 - \alpha)d^p(T^2x, T^2y) \leq kd^p(x, y)$$
(2.1) 
for all $x, y \in X$.

Note that if we set $\alpha = \alpha_1, \alpha_2 = 1 - \alpha$, and $k = 1$ in the inequality (2.1), then $T$ reduces to $(\alpha_1, \alpha_2, p)$-nonexpansive (see [8]). Further, if $p = 1$ and $k < 1$ (resp. $k = 1$) in the inequality (2.1), then $T$ reduces to 
$\alpha$-contraction (resp. $\alpha$-nonexpansive) with multi-index length 2 (see [9]).

Definition 2.2. A self mapping $T$ on $X$ is said to be $(\alpha, p)$-convex contraction, if for some $\alpha \in (0, 1)$ and 
$p \geq 1$, there exist $k_i \geq 0$ for all $i \in \{1, 2, \ldots, 5\}$ such that 
$$\sum_{i=1}^{5} k_i < 1$$ 
satisfying the following inequality 
$$\alpha d^p(Tx, Ty) + (1 - \alpha)d^p(T^2x, T^2y) \leq k_1d^p(x, y) + k_2d^p(x, Tx)$$
$$+ k_3d^p(Tx, T^2x) + k_4d^p(y, Ty) + k_5d^p(Ty, T^2y)$$
(2.2) 
for all $x, y \in X$.

Obviously, if $k_i = 0$ for all $i \in \{2, 3, 4, 5\}$, then the inequality (2.2) reduces to $(\alpha, p)$-contraction. We 
shall call $\alpha$-contraction and $\alpha$-convex contraction, if $p = 1$ in the inequalities (2.1) and (2.2). If $\alpha = k_1 = 0$ 
and $p = 1$ in (2.2), then it reduces to two-sided convex contraction [10].

Example 2.3. On $X = [0, 1]$, consider $T: X \to X$, endowed with usual metric $d(x, y) = |x - y|$. We define 
$$Tx = \frac{1-x^2}{2},$$ for all $x \in X$. Then, we obtain $T^2x = \frac{3+2x^2-x^4}{8}$. Now, we have 
$$|Tx - Ty| = \frac{1}{2}|x^2 - y^2| = \frac{(x+y)}{2}|x-y| \leq |x-y|.$$ 
Also, 
$$|T^2x - T^2y| = \frac{1}{8}|(2x^2 - x^4) - (2y^2 - y^4)| \leq \frac{1}{4}|x^2 - y^2| + \frac{1}{8}|x^4 - y^4| \leq |x-y|.$$ 
Therefore, for $\alpha = \frac{1}{2}$ and $p = 1$, we obtain 
$$\alpha|Tx - Ty| + (1 - \alpha)|T^2x - T^2y| \leq |x-y|.$$ 
This shows that $T$ is nonexpansive and $\alpha$-nonexpansive for $p = 1$.

Further, for $p = 2$ and $\alpha = \frac{1}{2}$, we obtain 
$$\alpha|Tx - Ty|^2 + (1 - \alpha)|T^2x - T^2y|^2 \leq \frac{1}{2}|x-y|^2 + \frac{1}{8}|x-y|^2 = \frac{5}{8}|x-y|^2.$$ 
This shows that $T$ is $(\alpha, p)$-contraction for $p = 2 > 1$. 

In [6], Gallagher mentioned that all nonexpansive mappings are mean nonexpansive, but the converse is not true. That is, there exists a mean nonexpansive mapping which is not nonexpansive (see [6, Examples 2.3 and 2.4]). However, it may happen that a nonexpansive mapping need not necessarily be a mean nonexpansive.

**Example 2.4.** Let $T: X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. We define $Tx = \frac{x^2}{2}$ for all $x \in X$. Setting $\alpha = \frac{1}{2}$ and $p = 1$. Now, we have

$$|Tx - Ty| = \frac{1}{2}|x^2 - y^2| \leq |x - y|.$$  

Also, we have,

$$|T^2 x - T^2 y| = \frac{1}{8}|x^4 - y^4| = \frac{(x^2 + y^2)(x + y)}{8}|x - y| \leq \frac{1}{2}|x - y|.$$  

Therefore,

$$\frac{1}{2}|Tx - Ty| + \frac{1}{2}|T^2 x - T^2 y| \leq \frac{3}{4}|x - y|,$$

where $k = \frac{3}{4}, \alpha = \frac{1}{2}$. This shows that $T$ is nonexpansive but not mean nonexpansive.

Now, we introduce the notions of asymptotically $T^2$-regular (resp. $(T, T^2)$-regular) sequences.

**Definition 2.5.** A sequence $\{x_n\}$ is called an asymptotically $T^2$-regular, if $\lim_{n \to \infty} d(x_n, T^2 x_n) = 0$.

**Example 2.6.** Let $X = \mathbb{R}$ endowed with usual metric $d(x, y) = |x - y|$. We define

$$T: X \to X, \quad Tx = \begin{cases} 1 - x^2, & x \neq 1, \\ 2, & x = 1. \end{cases}$$  

Choose a sequence $\{x_n\}$ in $X$ such that $x_n \to 1$ as $n \to \infty$, except the constant sequence $x_n = 1$. Then, $Tx_n = (1 - x_n^2) \to 0$ as $n \to \infty$. Therefore, $\lim_{n \to \infty} |Tx_n - x_n| = 1 \neq 0$. Also, we have $T^2 x_n = T(Tx_n) = T(1 - x_n^2) = [1 - (1 - x_n^2)^2] \to 1$. Consequently, $|x_n - T^2 x_n| \to 0$. Therefore, $\{x_n\}$ is asymptotically $T^2$-regular sequence but not asymptotically $T$-regular sequence in $X$.

**Example 2.7.** Let $T: X \to X$, where $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define

$$Tx = \begin{cases} \frac{x^2}{2}, & x < 2, \\ 0, & x = 2, \\ 2, & x > 2. \end{cases}$$  

Consider a sequence $\{x_n\}$ in $X$ such that $x_n \to 2$ as $n \to \infty$, except the constant sequence $x_n = 2$. Then, $Tx_n \to 2$ as $n \to \infty$. Therefore, $\lim_{n \to \infty} |Tx_n - x_n| = 0$. Further, we have $T^2 x_n = T(Tx_n) \to 2$ or $0$, according as $x_n \to 2$ from left or right. So, $\lim_{n \to \infty} T^2 x_n$ does not exist. Therefore, $|x_n - T^2 x_n|$ does not tend to $0$ as $n \to 0$. It shows that $\{x_n\}$ is asymptotically $T$-regular sequence, but not asymptotically $T^2$-regular sequence in $X$.

It may be observed from Examples 2.6 and 2.7, that the notions of asymptotically $T$-regular and $T^2$-regular sequences are independent to each other.

**Definition 2.8.** A sequence $\{x_n\}$ in $X$ is called an asymptotically $(T, T^2)$-regular, if $\lim_{n \to \infty} d(x_n, T x_n) = 0$ and $\lim_{n \to \infty} d(x_n, T^2 x_n) = 0$. 
Obviously, if \( \{x_n\} \) is an asymptotically \((T, T^2)\)-regular sequence, then it satisfies both asymptotically \( T \) and \( T^2 \)-regular conditions.

**Example 2.9.** Let \( T: X \to X \), where \( X = \mathbb{R} \) with usual metric \( d(x, y) = |x - y| \). Define

\[
T_x = \begin{cases} 
4 - x, & x < 2, \\
0, & x = 2, \\
\frac{x^2}{2}, & x > 2.
\end{cases}
\]

Consider a sequence \( \{x_n\} \) in \( X \) such that \( x_n \to 2 \) as \( n \to \infty \), except the constant sequence \( x_n = 2 \). Then, \( T_{x_n} \to 2 \) as \( n \to \infty \) and \( T^2 x_n = T(T_{x_n}) \to 2 \). Therefore, \( |x_n - T_{x_n}| \to 0 \) and \( |x_n - T^2 x_n| \to 0 \) as \( n \to \infty \). So, \( \{x_n\} \) is both asymptotically \( T \)-regular and \( T^2 \)-regular sequence in \( X \). Therefore, \( \{x_n\} \) is asymptotically \((T, T^2)\)-regular sequence in \( X \).

**Lemma 2.10.** If a sequence \( \{x_n\} \) in \( X \) is asymptotically \((T, T^2)\)-regular in \( X \), then

\[
\lim_{n \to \infty} d(Tx_n, T^2x_n) = 0.
\]

**Proof.** By the triangle inequality, we obtain

\[
d(Tx_n, T^2x_n) \leq d(Tx_n, x_n) + d(x_n, T^2x_n).
\]

Hence, \( d(Tx_n, T^2x_n) \to 0 \) as \( n \to \infty \). \( \square \)

The converse of Lemma 2.10 is not true. In support of this, we have the following example.

**Example 2.11.** Let \( T: X \to X \), where \( X = \mathbb{R} \) with usual metric \( d(x, y) = |x - y| \). We consider

\[
T_x = \begin{cases} 
1, & x \neq 0, \\
0, & x = 0.
\end{cases}
\]

Choose a sequence \( \{x_n\} \) in \( X \) such that \( x_n \to 0 \) as \( n \to \infty \). Then, \( T_{x_n} \) and \( T^2 x_n \) converge to 1 as \( n \to \infty \). Therefore, \( |T_{x_n} - x_n| \to 1 \neq 0 \) and \( |x_n - T^2 x_n| \to 1 \neq 0 \) as \( n \to \infty \). It shows that \( d(Tx_n, T^2x_n) \to 0 \) as \( n \to \infty \), but the sequence \( \{x_n\} \) is neither asymptotically \( T \)-regular nor asymptotically \( T^2 \)-regular in \( X \). Therefore, the sequence \( \{x_n\} \) is not asymptotically \((T, T^2)\)-regular.

3. Fixed point results

**Theorem 3.1.** Let \((X, d)\) be a metric space and \( T: X \to X \) be a \((\alpha, p)\)-contraction such that \( k + \alpha < 1 \). Then, \( T \) has the AFPP. Further, if \((X, d)\) is a complete metric space, then \( T \) has a unique fixed point.

**Proof.** Let \( x_0 \in X \). Now, we define a sequence \( \{x_n\} \) by \( x_{n+1} = T^{n+1}x_0 \) for all \( n \geq 0 \). If \( x_n = x_{n+1} \) i.e., \( T^n x_0 = T(T^n x_0) \) for some \( n \), then the conclusion follows immediately. Without loss of generality, we assume that \( x_n \neq x_{n+1} \) for all \( n \geq 0 \). Setting \( v = d(x_0, T_{x_0}) + d(T_{x_0}, T^2 x_0) \) we have \( d(x_0, Tx_0) \leq v \) and \( d(Tx_0, T^2 x_0) \leq v \). Taking \( x = x_0 \) and \( y = Tx_0 \) in the inequality (2.1), we obtain

\[
(1 - \alpha)d^p(T^2 x_0, T^3 x_0) \leq \alpha d^p(Tx_0, T^2 x_0) + (1 - \alpha)d^p(T^2 x_0, T^3 x_0)
\]

\[
\leq k d^p(x_0, Tx_0) = kv^p \Rightarrow d^p(T^2 x_0, T^3 x_0) \leq \frac{k}{1 - \alpha} v^p \Rightarrow d(T^2 x_0, T^3 x_0) \leq hv,
\]

where \( h^p = \frac{k}{1 - \alpha} \) and since \( k + \alpha < 1 \) \( \Rightarrow h^p < 1 \).

Again, taking \( x = T_{x_0} \) and \( y = T^2 x_0 \) in relation (2.1), we obtain

\[
(1 - \alpha)d^p(T^3 x_0, T^4 x_0) \leq \alpha d^p(T^2 x_0, T^3 x_0) + (1 - \alpha)d^p(T^3 x_0, T^4 x_0)
\]

\[
\leq k d^p(x_{n+1}, x_n) = kv^p \Rightarrow d^p(T^3 x_0, T^4 x_0) \leq \frac{k}{1 - \alpha} v^p \Rightarrow d(T^3 x_0, T^4 x_0) \leq hv,
\]

where \( h^p = \frac{k}{1 - \alpha} \) and since \( k + \alpha < 1 \) \( \Rightarrow h^p < 1 \).
\[ \leq kd^p(Tx_0, T^2x_0) \Rightarrow d^p(T^3x_0, T^4x_0) \leq \eta^p v^p \Rightarrow d(T^3x_0, T^4x_0) \leq hv. \]

And
\[ (1 - \alpha)d^p(T^4x_0, T^5x_0) \leq \alpha d^p(T^3x_0, T^4x_0) \leq (1 - \alpha)d^p(T^4x_0, T^5x_0) \leq kd^p(T^2x_0, T^3x_0) \Rightarrow d(T^4x_0, T^5x_0) \leq h^2v. \]

Also, we obtain
\[ d(T^5x_0, T^6x_0) \leq h^2v. \]

Following similar arguments as in ([12, 14]), we obtain \( d(T^m x_0, T^{m+1} x_0) \leq h^1v \), whenever \( m = 2l \) or \( m = 2l + 1 \). Therefore, \( d(T^m x_0, T^{m+1} x_0) \to 0 \) as \( m \to \infty \), i.e., \( T \) is asymptotically regular at \( x_0 \). By Lemma 1.6, \( T \) has an approximate fixed point. Now, suppose that \( T \) is continuous and \( (X,d) \) is a complete metric space. In order to show that \( \{x_n\} \) is a Cauchy sequence in \( X \), fix a nonzero positive integer \( m \).

Case (i). For \( m = 2l \) with \( l, q \geq 1 \), then
\[
\begin{align*}
d(T^m x_0, T^{m+q} x_0) &= d(T^{2l} x_0, T^{2l+q} x_0) \\
&\leq d(T^{2l} x_0, T^{2l+1} x_0) + d(T^{2l+1} x_0, T^{2l+2} x_0) \\
&+ d(T^{2l+2} x_0, T^{2l+3} x_0) + d(T^{2l+3} x_0, T^{2l+4} x_0) + \ldots \\
&+ d(T^{2l+q-2} x_0, T^{2l+q-1} x_0) + d(T^{2l+q-1} x_0, T^{2l+q} x_0) \\
&\leq h^1v + h^1v + h^1v + h^1v + \ldots \\
&\leq 2h^1 \left(1 + h + h^2 + h^3 + \ldots\right) v \leq 2h^1 \frac{1}{1-h} v.
\end{align*}
\]

Case (ii). Similarly, for \( m = 2l + 1 \) with \( l, q \geq 1 \), we obtain
\[
\begin{align*}
d(T^m x_0, T^{m+q} x_0) &= d(T^{2l+1} x_0, T^{2l+q+1} x_0) \\
&\leq d(T^{2l+1} x_0, T^{2l+2} x_0) + d(T^{2l+2} x_0, T^{2l+3} x_0) \\
&+ d(T^{2l+3} x_0, T^{2l+4} x_0) + d(T^{2l+4} x_0, T^{2l+5} x_0) + \ldots \\
&+ d(T^{2l+q-1} x_0, T^{2l+q} x_0) + d(T^{2l+q} x_0, T^{2l+q+1} x_0) \\
&\leq h^1v + h^1v + h^1v + h^1v + \ldots \\
&\leq 2h^1 \left(1 + h + h^2 + h^3 + \ldots\right) v \leq 2h^1 \frac{1}{1-h} v.
\end{align*}
\]

Taking \( l \to \infty \) in all cases, since \( h < 1 \), we obtain, \( d(T^m x_0, T^n x_0) \to 0 \). Therefore, \( \{x_n\} \) is a Cauchy sequence in \( X \). Since, \( X \) is complete, there exists a point \( z \in X \) such that \( x_n = T^nx_0 \to z \in X \) as \( n \to \infty \). This shows that \( z \) is a fixed point of \( T \). Now, we prove that \( T \) has a unique fixed point in \( X \). Let \( z^* \in X \) be another fixed point of \( T \). Using (2.1) for \( x = z \) and \( y = z^* \), we obtain
\[
\alpha d^p(Tz, Tz^*) + (1 - \alpha)d^p(Tz, T^2z^*) \leq kd^p(z, z^*) \Rightarrow (1 - k)d^p(z, z^*) \leq 0
\]
leading to \( d(z, z^*) = 0 \), a contradiction. Hence, \( T \) has a unique fixed point in \( X \). \( \square \)

We have the following example for the validity of Theorem 3.1.

**Example 3.2.** Let \( T : X \to X \), where \( X = [0,1] \) with usual metric \( d(x,y) = |x-y| \). Define \( Tx = \frac{1-x^2}{2} \) for all \( x \in X \). Setting \( \alpha = \frac{1}{8} \) and \( p = 2 \), we obtain
\[
\alpha|Tx - Ty|^2 + (1 - \alpha)|T^2x - T^2y|^2 \leq \alpha|x-y|^2 + \frac{(1 - \alpha)}{2}|x-y|^2 = \frac{1 + \alpha}{2}|x-y|^2 = \frac{7}{12}|x-y|^2.
\]

This shows that \( T \) is \( (\alpha, p) \)-contraction with \( \alpha + k = \frac{3}{4} < 1 \). Moreover, \( x = -1 + \sqrt{2} \) is the unique fixed point of \( T \) in \( X \).
**Theorem 3.3.** Let \((X, d)\) be a metric space and \(T: X \to X\) be a \((\alpha, p)\)-convex contraction such that \(\left(\sum_{i=1}^{5} k_i\right) + \alpha < 1\). Then, \(T\) has the AFPP. Further, if \((X, d)\) is a complete metric space, then \(T\) has a unique fixed point.

**Proof.** We define a sequence \(\{x_n\}\) by \(x_{n+1} = T^{n+1}x_0\) for all \(n \geq 0\) and continue the same arguments as in Theorem 3.1, setting \(v = d(x_0, Tx_0) + d(Tx_0, T^2x_0)\). Now, using (2.2) for \(x = x_0\) and \(y = Tx_0\), we obtain

\[
(1 - \alpha)d^p(T^2x_0, T^3x_0) \leq (1 - \alpha)d^p(Tx_0, T^2x_0) + (1 - \alpha)d^p(T^2x_0, T^3x_0)
\]

\[
\leq (k_1 + k_2 + k_3 + k_4)d^p(x_0, Tx_0) + (k_3 + k_4)d^p(Tx_0, T^2x_0) + k_5d^p(T^2x_0, T^3x_0)
\]

Therefore,

\[
d^p(T^2x_0, T^3x_0) \leq \frac{k_1 + k_2 + k_3 + k_4}{1 - \alpha - k_5}v^p = h^p v^p \Rightarrow d(T^2x_0, T^3x_0) \leq hv
\]

for \(h^p = \frac{(k_1 + k_2 + k_3 + k_4)}{1 - \alpha - k_5}\); moreover, since \(\left(\sum_{i=1}^{5} k_i\right) + \alpha < 1\) \(\Rightarrow h^p < 1\).

Similarly, one can obtain

\[
d(T^3x_0, T^4x_0) \leq hv, \quad d(T^4x_0, T^5x_0) \leq h^2v, \quad d(T^5x_0, T^6x_0) \leq h^2v.
\]

Following similar arguments as in Theorem 3.1, we obtain \(d(T^mx_0, T^{m+1}x_0) \to 0\) as \(m \to \infty\), i.e., \(T\) is asymptotically regular at \(x_0\). By Lemma 1.4, \(T\) has AFPP. Further, by assuming the continuity of \(T\) and the completeness of \(X\), the existence of a fixed point \(z\) can be proved, using similar arguments as in Theorem 3.1.

Now, we show that \(T\) has a unique fixed point in \(X\). Let \(z^* \in X\) be another fixed point of \(T\). Using (2.2) for \(x = z\) and \(y = z^*\), we obtain

\[
\alpha d^p(Tz, Tz^*) + (1 - \alpha)d^p(T^2z, T^2z^*) \leq k_1d^p(z, z^*) + k_2d^p(z, Tz) + k_3d^p(Tz, T^2z)
\]

\[
+ k_4d^p(z^*, Tz^*) + k_5d^p(Tz^*, T^2z^*) \Rightarrow (1 - k_1)d^p(z, z^*) \leq 0,
\]

which gives \(d(z, z^*) = 0\), a contradiction and hence, \(T\) has a unique fixed point in \(X\). \(\square\)

One can verify the validity of Theorem 3.3 with Example 3.2 taking with \(\alpha = \frac{1}{5}\), \(k_1 = \frac{7}{17}\), \(k_2 = k_3 = k_4 = k_5 = 0\), and \(p = 2\).

**Theorem 3.4.** Let \((X, d)\) be a complete metric space and \(T: X \to X\) be a \((\alpha, p)\)-contraction such that \(0 \leq k < \alpha\) or \(k + \alpha < 1\). If \(T\) is asymptotically regular at some point \(x_0 \in X\), then there exists a unique fixed point of \(T\).

**Proof.** Let \(T\) be an asymptotically regular mapping at \(x_0 \in X\). Consider a sequence \(\{T^nx_0\}\) in \(X\) and for any two non zero positive integers \(m, n \geq 1\) such that \(m > n\), let us analyze the following two situations:

Case(i). When \(0 \leq k < \alpha\). Using the inequality (2.1), we obtain

\[
\alpha d^p(T^m x_0, T^n x_0) \leq \alpha d^p(T^m x_0, T^n x_0) + (1 - \alpha)d^p(T^{m+1}x_0, T^{n+1}x_0)
\]

\[
\leq kd^p(T^{m-1}x_0, T^{n-1}x_0) \leq k \left[ d(T^{m-1}x_0, T^m x_0) + d(T^m x_0, T^n x_0) + d(T^n x_0, T^{n-1}x_0) \right]^p
\]

Taking \(n, m \to \infty\) and using the asymptotically regularity of \(T\) at \(x_0\), the above inequality gives

\[
\alpha \lim_{n \to \infty} d^p(T^m x_0, T^n x_0) \leq k \lim_{n \to \infty} d^p(T^m x_0, T^n x_0)
\]

that is,

\[
(\alpha - k) \lim_{n \to \infty} d^p(T^m x_0, T^n x_0) \leq 0.
\]

Since \(0 \leq k < \alpha\), it follows \(\lim_{n \to \infty} d(T^m x_0, T^n x_0) = 0\).
Case (ii). When $0 < k + \alpha < 1$. Using the inequality (2.1), we obtain

$$(1 - \alpha) d^p(T^m x_0, T^n x_0) \leq \alpha d^p(T^{m-1} x_0, T^{n-1} x_0) + (1 - \alpha) d^p(T^m x_0, T^n x_0)$$

$$\leq k d^p(T^{m-2} x_0, T^{n-2} x_0)$$

$$\leq k [d(T^{m-2} x_0, T^{m} x_0) + d(T^{n} x_0, T^{n-1} x_0) + d(T^{m} x_0, T^{n} x_0)]^p$$

$$\leq k [d(T^{m-2} x_0, T^{m} x_0) + d(T^{n} x_0, T^{n-1} x_0) + d(T^{m} x_0, T^{n} x_0)]^p.$$

Taking $n, m \to \infty$, we find

$$(1 - \alpha) \lim_{n \to \infty} d^p(T^m x_0, T^n x_0) \leq k \lim_{n \to \infty} d^p(T^m x_0, T^n x_0) \Rightarrow (1 - \alpha - k) \lim_{n \to \infty} d^p(T^m x_0, T^n x_0) \leq 0.$$

Therefore, $\lim_{n \to \infty} d(T^m x_0, T^n x_0) = 0$ as $0 < k + \alpha < 1$. Consequently, $\{T^n x_0\}$ is a Cauchy sequence in $X$. Since $X$ is complete, it follows $T^n x_0 \to z$ as $n \to \infty$ for some $z \in X$. Now, we show that $Tz = z$, i.e., $z$ is a fixed point of $T$. For this, using again the inequality (2.1), we find

$$\alpha d^p(Tz, T^n x_0) \leq \alpha d^p(Tz, T^n x_0) + (1 - \alpha) d^p(T^{2} z, T^{n+1} x_0) \leq kd^p(z, T^{n-1} x_0).$$

As $n \to \infty$, we obtain

$$\alpha d^p(Tz, z) \leq 0,$$

which leads to $d(Tz, z) = 0$, that is $Tz = z$. Therefore, $z$ is a fixed point of $T$. The uniqueness of the fixed point follows immediately as in Theorem 3.1.

**Example 3.5.** Let $T: X \to X$, where $X = [0, 1]$ with usual metric $d(x, y) = |x - y|$. Define $T x = \frac{1 + x}{2}$ for all $x \in X$. For any arbitrary $x_0 \in X$, we have $T x_0 = \frac{1 + x_0}{2}$ and $T^n x_0 = \frac{2^n - 1 + x_0}{2^n}$, where $T^n$ denotes the $n$th iterate of $T$. Also, we have

$$\lim_{n \to \infty} d(T^n x_0, T^{n+1} x_0) = \lim_{n \to \infty} \left| \frac{2^n - 1 + x_0}{2^n} - \frac{2^{n+1} - 1 + x_0}{2^{n+1}} \right| = 0.$$ 

This shows that $T$ is asymptotically regular at all points in $X$. Obviously, $\{T^n x_0\}$ is a sequence in $X$ such that $T^n x_0 \to 1 \in X$ as $n \to \infty$. Taking $\alpha = \frac{1}{4}$, $k = \frac{1}{2}$, and $p = 2$, then $T$ is $(\alpha, p)$-contraction for all $x, y \in X$ such that $k < \alpha$ or $k + \alpha < 1$. Thus, all the conditions of Theorem 3.4 are satisfied and hence, 1 is the unique fixed point of $T$.

**Theorem 3.6.** Let $(X, d)$ be a complete metric space and $T: X \to X$ be a $\alpha$-contraction such that $k < \alpha$. If there exists an asymptotically $T$-regular sequence in $X$, then $T$ has a unique fixed point.

**Proof.** Let $\{x_n\}$ be an asymptotically $T$-regular sequence in $X$. Then, for any two non zero positive integers $m, n$ such that $m > n$, we obtain

$$\alpha d(x_m, x_n) \leq \alpha \left[ d(x_m, Tx_m) + d(Tx_m, Tx_n) + d(Tx_n, x_n) \right]$$

$$= \alpha \left[ d(x_m, Tx_m) + d(Tx_n, x_n) \right] + \alpha d(Tx_m, Tx_n)$$

$$\leq \alpha \left[ d(x_m, Tx_m) + d(Tx_n, x_n) \right] + \alpha d(Tx_m, Tx_n) + (1 - \alpha) d(T^2 x_m, T^2 x_n)$$

$$\leq \alpha \left[ d(x_m, Tx_m) + d(Tx_n, x_n) \right] + kd(x_m, x_n),$$
that is,

\[ d(x_m, x_n) \leq \alpha \left[ d(x_m, T_xm) + d(Tx_n, x_n) \right]. \]

Taking \( n, m \to \infty \) and using the fact that the sequence \( \{x_n\} \) is asymptotically \( T \)-regular, we obtain

\[ \lim_{n \to \infty} d(x_m, x_n) = 0. \]

This shows that \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, there exists a point \( z \in X \) such that \( x_n \to z \in X \) as \( n \to \infty \).

Now, we show that \( Tz = z \), i.e., \( z \) is a fixed point of \( T \).

\[ \alpha d(Tz, x_n) \leq \alpha \left[ d(Tz, Tx_n) + d(Tx_n, x_n) \right] \]
\[ \leq \alpha d(Tz, x_n) + (1 - \alpha) d(Tz^2, Tz x_n) + \alpha d(Tx_n, x_n) \leq kd(z, x_n) + \alpha d(Tx_n, x_n). \]

As \( n \to \infty \) and since \( \{x_n\} \) is asymptotically \( T \)-regular, we obtain

\[ \alpha d(Tz, z) \leq 0 \]

leading to \( Tz = z \). Therefore, \( z \) is a fixed point of \( T \). The uniqueness of the fixed point follows immediately.

**Example 3.7.** Let \( T: X \to X \), where \( X = [0, 1] \) with usual metric \( d(x, y) = |x - y| \). Define \( T = \frac{x}{2} \) for all \( x \in X \). Consider a sequence \( \{x_n\} \) in \( X \) such that \( x_n \to 0 \), then \( Tx_n \to 0 \), i.e., \( \lim_{n \to \infty} |x_n - Tx_n| = 0 \). So, \( \{x_n\} \) is asymptotically \( T \)-regular in \( X \). Setting \( \alpha = \frac{1}{2} \), \( k = \frac{3}{2} \), then \( T \) is \( \alpha \)-contraction for all \( x, y \in X \) such that \( k < \alpha \). Thus, all the conditions of Theorem 3.6 are satisfied and hence, \( 0 \) is the unique fixed point of \( T \).

**Theorem 3.8.** Let \( (X, d) \) be a complete metric space and \( T: X \to X \) be a \( \alpha \)-contraction such that \( k + \alpha < 1 \). If there exists an asymptotically \( T^2 \)-regular sequence in \( X \), then \( T \) has a unique fixed point.

**Proof.** Let \( \{x_n\} \) be an asymptotically \( T^2 \)-regular sequence in \( X \). Then, for any two non zero positive integers \( m, n \), such that \( m > n \), we obtain

\[
(1 - \alpha)d(x_m, x_n) \leq (1 - \alpha) \left[ d(x_m, T^2x_m) + d(T^2x_m, T^2x_n) + d(T^2x_n, x_n) \right]
= (1 - \alpha) \left[ d(x_m, T^2x_m) + d(T^2x_n, x_n) \right] + (1 - \alpha)d(T^2x_m, T^2x_n)
\leq (1 - \alpha) \left[ d(x_m, T^2x_m) + d(T^2x_n, x_n) \right] + \alpha d(Tx_m, x_n) + (1 - \alpha)d(T^2x_m, T^2x_n)
\leq (1 - \alpha) \left[ d(x_m, T^2x_m) + d(T^2x_n, x_n) \right] + kd(x_m, x_n),
\]

that is,

\[
d(x_m, x_n) \leq \frac{1 - \alpha}{1 - \alpha - k} \left[ d(x_m, T^2x_m) + d(T^2x_n, x_n) \right].
\]

Since \( \{x_n\} \) is asymptotically \( T^2 \)-regular sequence, by taking \( n, m \to \infty \), we obtain

\[
\lim_{n \to \infty} d(x_m, x_n) = 0,
\]

which proves that \( \{x_n\} \) is a Cauchy sequence. Since, \( X \) is complete, there exists a point \( z \in X \) such that \( x_n \to z \in X \) as \( n \to \infty \).

In order to show that \( z \) is a fixed point of \( T \) in \( X \), we make several steps.
First, we show that $T^2z = z$. Using inequality (2.1), we obtain
\[
(1 - \alpha)d(T^2z, x_n) \leq (1 - \alpha)\left[ d(T^2z, T^2x_n) + d(T^2x_n, x_n) \right]
\leq \alpha d(Tz, x_n) + (1 - \alpha)d(T^2z, T^2x_n) + (1 - \alpha)d(T^2x_n, x_n)
\leq k d(z, x_n) + (1 - \alpha)d(T^2x_n, x_n).
\]
Taking $n \to \infty$, and using the asymptotically $T^2$-regularity of the sequence \( \{x_n\} \), we obtain
\[
(1 - \alpha)d(T^2z, z) \leq 0,
\]
which gives $T^2z = z$. Therefore, one can obtain inductively that $T^{2n}z = z$ and $T^{2n+1}z = Tz$ for $n \geq 1$.

We show that $Tz = z$, i.e., $z$ is a fixed point of $T$.

Using the inequality (2.1), we obtain
\[
(1 - \alpha)d(z, Tz) = (1 - \alpha)d(T^2z, T^3z) \leq \alpha d(Tz, T^2z) + (1 - \alpha)d(T^2z, T^3z) \leq kd(z, Tz),
\]
that is,
\[
(1 - \alpha - k)d(z, Tz) \leq 0
\]
a contradiction, if $Tz \neq z$. Therefore, $z$ is a fixed point of $T$. Using the inequality (2.1), one can obtain the uniqueness of fixed point. \( \square \)

**Example 3.9.** Let $T: X \to X$, where $X = \{0, 1, 2\}$ and $A = \{0, 1\} \subset X$ with usual metric $d(x, y) = |x - y|$. Define
\[
Tx = \begin{cases} 
1, & x \notin A, \\
0, & x \in A.
\end{cases}
\]
Consider a sequence $\{x_n\}$ in $X$ such that $x_n \to 0$, then $Tx_n \to 1$ and $T^2x_n \to 0$ as $n \to \infty$. Consequently, $|x_n - T^2x_n| \to 0$ as $n \to \infty$. So, $\{x_n\}$ is asymptotically $T^2$-regular in $X$. Setting $\alpha = k = \frac{1}{2}$, then $T$ is a contraction for all $x, y \in X$ such that $k + \alpha < 1$. Thus, all the conditions of Theorem 3.8 are satisfied and hence, 0 is the unique fixed point of $T$.

The following Theorems 3.10 and 3.12 are motivated by Theorems 3.1 and 3.4 of Khan and Jhade [11].

**Theorem 3.10.** Let $(X, d)$ be a complete metric space and $T: X \to X$ be an $\alpha$-convex contraction such that $0 < k_1 + \alpha < 1$ and $\mu, h < 1$, where $\mu = \max\left(\frac{k_3}{\alpha - k_2 - k_3}, \frac{k_3}{\alpha - k_4 - k_3}\right)$ and $h = \max\left(\frac{k_2 + k_3}{1 - \alpha - k_3}, \frac{k_4 + k_5}{1 - \alpha - k_5}\right)$. If there exists an asymptotically $(T, T^2)$-regular sequence in $X$, then $T$ has a unique fixed point.

**Proof.** Let $\{x_n\}$ be an asymptotically $(T, T^2)$-regular sequence in $X$. Then, for any non-zero positive integers $m, n$ such that $m > n$, we obtain
\[
(1 - \alpha)d(x_m, x_n) \leq (1 - \alpha)\left[ d(x_m, T^2x_m) + d(T^2x_m, T^2x_n) + d(T^2x_n, x_n) \right]
\leq (1 - \alpha)\left[ d(x_m, T^2x_m) + d(T^2x_m, T^2x_n) \right] + (1 - \alpha)d(T^2x_m, T^2x_n)
\leq (1 - \alpha)\left[ d(x_m, T^2x_m) + d(T^2x_m, x_m) \right] + \alpha d(Tx_m, x_n) + (1 - \alpha)d(T^2x_m, T^2x_n)
\leq (1 - \alpha)\left[ d(x_m, T^2x_m) + d(T^2x_m, x_m) \right] + k_1d(x_m, x_n) + k_2d(x_m, Tx_m) + k_3d(Tx_m, T^2x_m) + k_4d(x_n, Tx_n) + k_5d(Tx_n, T^2x_n),
\]
that is,

\[
(1 - \alpha - k_1) d(x_m, x_n) \leq (1 - \alpha) \left[ d(x_m, T^2 x_m) + d(T^2 x_n, x_n) \right] \\
+ k_2 d(x_m, T x_m) + k_3 d(T x_m, T^2 x_m) + k_4 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n).
\]

Since \( \{x_n\} \) is asymptotically \((T, T^2)\)-regular sequence. Letting \( n, m \to \infty \) and using Lemma 2.10, we obtain \( \lim_{n \to \infty} d(x_n, x_m) = 0 \). This shows that \( \{x_n\} \) is a Cauchy sequence in \( X \). Since, \( X \) is complete, there exists a point \( z \in X \) such that \( x_n \to z \in X \) as \( n \to \infty \). Now, we show that \( z \) is a fixed point of \( T \) in \( X \). For this, first we show that \( T^2 z = z \). Using inequality (2.1), we obtain

\[
(1 - \alpha) d(T^2 z, x_n) \leq (1 - \alpha) \left[ d(T^2 z, T^2 x_n) + d(T^2 x_n, x_n) \right] \\
\leq k_1 d(z, x_n) + k_2 d(z, T z) + k_3 d(T z, T^2 z) \\
+ k_4 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n) + (1 - \alpha) d(T^2 x_n, x_n)
\]

\[
\leq k_1 d(z, x_n) + k_2 d(z, T z) + k_3 \left[ d(T z, x_n) + d(T^2 z, x_n) \right] \\
+ k_4 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n) + (1 - \alpha) d(T^2 x_n, x_n),
\]

that is,

\[
(1 - \alpha - k_3) d(T^2 z, x_n) \leq k_1 d(z, x_n) + k_2 d(z, T z) + k_3 d(T z, x_n) \\
+ k_4 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n) + (1 - \alpha) d(T^2 x_n, x_n).
\]

Taking \( n \to \infty \) and using Lemma 2.10, we obtain

\[
(1 - \alpha - k_3) d(T^2 z, z) \leq (k_2 + k_3) d(z, T z),
\]

that is,

\[
d(T^2 z, z) \leq \frac{k_2 + k_3}{1 - \alpha - k_3} d(T z, z).
\]

Similarly, by symmetry of the \( \alpha \)-convex contraction, one can obtain

\[
d(T^2 z, z) \leq \frac{k_4 + k_5}{1 - \alpha - k_5} d(T z, z).
\]

Since, \( h = \max \left\{ \frac{k_2 + k_3}{1 - \alpha - k_3}, \frac{k_4 + k_5}{1 - \alpha - k_5} \right\} < 1 \). This shows that \( d(T^2 z, z) \leq h d(T z, z) \).

Now, we show that \( T z = z \), i.e., \( z \) is a fixed point of \( T \).

\[
\alpha d(T z, x_n) \leq \alpha \left[ d(T z, T x_n) + d(T x_n, x_n) \right] + (1 - \alpha) d(T^2 z, T^2 x_n) \\
= \alpha d(T z, T x_n) + (1 - \alpha) d(T^2 z, T^2 x_n) + \alpha d(T x_n, x_n) \\
\leq k_1 d(z, x_n) + k_2 d(z, T z) + k_3 d(T z, T^2 z) \\
+ k_4 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n) + \alpha d(T x_n, x_n) \\
\leq k_1 d(z, x_n) + k_2 d(z, T z) + k_3 d(T z, x_n) \\
+ k_4 d(T^2 z, z) + k_5 d(x_n, T x_n) + k_5 d(T x_n, T^2 x_n) + \alpha d(T x_n, x_n).
\]

As \( n \to \infty \), we obtain

\[
\alpha d(T z, z) \leq (k_2 + k_3) d(T z, z) + k_3 d(T^2 z, z),
\]
that is,
\[ d(Tz, z) \leq \frac{k_3}{\alpha - k_2 - k_3}d(T^2z, z). \]

Similarly, based on the symmetry of \(\alpha\)-convex contractions, one can prove
\[ d(Tz, z) \leq \frac{k_5}{\alpha - k_4 - k_5}d(T^2z, z). \]

Since \(\mu = \max\{\frac{k_3}{\alpha - k_2 - k_3}, \frac{k_5}{\alpha - k_4 - k_5}\} < 1\), we find
\[ d(Tz, z) \leq \mu d(T^2z, z) \leq h\mu d(Tz, z), \]

that is,
\[ (1 - h\mu)d(Tz, z) \leq 0 \]
leading to \(d(Tz, z) = 0\) as \(h\mu < 1\). Therefore, \(z\) is a fixed point of \(T\). For uniqueness, let \(z^* \in X\) be another fixed point of \(T\). Using (2.1) for \(x = z\) and \(y = z^*\), we obtain
\[ \alpha d(Tz, Tz^*) + (1 - \alpha)d(T^2z, T^2z^*) \leq k_1d(z, z^*) + k_2d(z, Tz) + k_3d(Tz, T^2z) + k_4d(z^*, Tz^*) + k_5d(Tz^*, T^2z^*), \]

that is,
\[ (1 - k_1)d(z, z^*) \leq 0, \]

which in turn gives \(d(z, z^*) = 0\) and hence, \(T\) has a unique fixed point in \(X\).

**Example 3.11.** Let \(T : X \to X\), where \(X = [0, 1]\). Define \(T_x = \frac{1+x}{2}\) for all \(x \in X\). Consider a sequence \(\{x_n\}\) in \(X\) such that \(x_n \to \frac{1}{2}\) as \(n \to \infty\). Consequently, \(T_x, T^2x_n \to \frac{1}{2}\) as \(n \to \infty\). Therefore, the sequence \(\{x_n\}\) is asymptotically \((T, T^2)\)-regular in \(X\). Setting \(\alpha = \frac{1}{2}, k_1 = \frac{5}{2}, k_2 = k_3 = k_4 = k_5 = 0\), then \(T\) is \(\alpha\)-convex contraction such that \(k_1 + \alpha < 1\), \(\mu = 0 < 1\) and \(h = 0 < 1\). Thus, all the conditions of Theorem 3.10 are satisfied and hence, \(\frac{1}{3}\) is the unique fixed point of \(T\).

**Theorem 3.12.** Let \((X, d)\) be a complete metric space and \(T : X \to X\) be a \(\alpha\)-convex contraction such that \(k_1 < \alpha\) or, \(0 < k_1 + \alpha < 1\) and \(\mu, h < 1\), where \(\mu = \max\{\frac{k_3}{\alpha - k_2 - k_3}, \frac{k_5}{\alpha - k_4 - k_5}\}\) and \(h = \max\{\frac{k_2+k_3}{1-\alpha-k_2}, \frac{k_4+k_5}{1-\alpha-k_4}\}\). If \(T\) is asymptotically regular at some point \(x_0\) in \(X\), then there exists a unique fixed point of \(T\).

**Proof.** Let \(T\) be an asymptotically regular mapping at \(x_0 \in X\). Consider a sequence \(\{T^n x_0\}\) and for any two non zero positive integers \(m, n \geq 1\) such that \(m > n\).

We analyze the following cases.

Case (i). When \(k_1 < \alpha\). We obtain
\[
\begin{align*}
\alpha d(T^m x_0, T^n x_0) &\leq \alpha d(T^m x_0, T^n x_0) + (1 - \alpha)d(T^{m+1} x_0, T^{n+1} x_0) \\
&\leq k_1 d(T^{m-1} x_0, T^{n-1} x_0) + k_2 d(T^{m-1} x_0, T^m x_0) \\
&\quad + k_3 d(T^m x_0, T^{m+1} x_0) + k_4 d(T^{n-1} x_0, T^n x_0) + k_5 d(T^n x_0, T^{n+1} x_0) \\
&\leq k_1 \left[d(T^{m-1} x_0, T^m x_0) + d(T^m x_0, T^n x_0) \\
&\quad + d(T^n x_0, T^{n+1} x_0)\right] + k_2 d(T^{m-1} x_0, T^m x_0) \\
&\quad + k_3 d(T^m x_0, T^{m+1} x_0) + k_4 d(T^{n-1} x_0, T^n x_0) + k_5 d(T^n x_0, T^{n+1} x_0),
\end{align*}
\]
that is,
\[
(\alpha - k_1) d(T^m x_0, T^n x_0) \leq (k_1 + k_2) d(T^{m-1} x_0, T^{m-1} x_0)
+ (k_1 + k_4) d(T^{n-1} x_0, T^{n-1} x_0)
+ k_3 d(T^m x_0, T^{m+1} x_0) + k_5 d(T^n x_0, T^{n+1} x_0).
\]

Taking \(n, m \to \infty\) and using the asymptotically regularity of \(T\) at \(x_0\), we obtain
\[
\lim_{n \to \infty} d(T^m x_0, T^n x_0) = 0.
\]

Case (ii). When \(0 < k_1 + \alpha < 1\), we obtain
\[
(1 - \alpha) d(T^m x_0, T^n x_0) \leq \alpha d(T^{m-1} x_0, T^{n-1} x_0)
+ (1 - \alpha) d(T^m x_0, T^n x_0)
\leq k_1 d(T^{m-1} x_0, T^{n-1} x_0)
+ k_2 d(T^{m-2} x_0, T^{n-2} x_0)
+ k_3 d(T^{n-1} x_0, T^{n-1} x_0)
+ k_4 d(T^{m-2} x_0, T^{m-2} x_0) + k_5 d(T^n x_0, T^n x_0)
\leq k_1 d(T^{m-1} x_0, T^{n-1} x_0)
+ d(T^m x_0, T^n x_0)
+ d(T^{n-1} x_0, T^{m-1} x_0)
+ k_2 d(T^{m-1} x_0, T^m x_0)
+ k_3 d(T^m x_0, T^{m+1} x_0) + k_4 d(T^{n-1} x_0, T^n x_0) + k_5 d(T^n x_0, T^{n+1} x_0).
\]

Taking \(n, m \to \infty\), we obtain
\[
(1 - \alpha - k_1) \lim_{n \to \infty} d(T^{m+1} x_0, T^{n+1} x_0) \leq 0,
\]

which gives \(\lim_{n \to \infty} d(T^{m+1} x_0, T^{n+1} x_0) = 0\).

In both cases it follows that \(\{T^n x_0\}\) is a Cauchy sequence in \(X\). Since \(X\) is complete, so \(T^n x_0 \to z\) as \(n \to \infty\) for some \(z \in X\). Thus, by following the same argument as in Theorem 3.10, one can obtain the unique fixed point of \(T\).

One can check the validity of Theorem 3.12 with Example 3.5 setting with \(\alpha = \frac{2}{5}, k_1 = \frac{7}{27}, k_2 = k_3 = k_4 = k_5 = 0,\) and \(p = 1\).

**Corollary 3.13.** Let \((X, d)\) be a metric space and \(T : X \to X\) be a two-sided convex contraction mapping. Then, \(T\) has AFPP. Further, if \((X, d)\) is a complete metric space, then \(T\) has a unique fixed point.

**References**


