Model Reduction by Hermite Polynomials and Genetic Algorithm

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Abstract

The present paper attempts to develop order reduction methods where the suggested reduction model consists of two phases. First, full order system is expanded by Hermite polynomials, then a set of parameters in a fixed structure are determined, whose values define the reduced order system. The values are obtained using Genetic Algorithm (GA) by minimizing the errors between the first coefficients of Hermite polynomials expansion of full and reduced systems. To satisfy the stability, Routh criterion is used as constraints in optimization problem. To present the ability of the proposed method, three test systems are reduced. The results obtained are compared with other existing techniques. The results obtained show the accuracy and efficiency of the proposed method.

Keywords: Hermite polynomials, genetic algorithm, Routh array, order reduction, stability constraints.

1. Introduction

Various methods are reported in the literature for order reduction in time domain and frequency domain. Model reduction started by Davison in 1966 [1] and followed by Chidambaram by suggestion several modifications to Davison’s approach [2]-[4]. After that different approaches proposed using dominant pole retention or dominant eigenvalue retention [5], Routh approximation [6], Hurwitz polynomial approximation [7], [8], stability equation method [9], [10], moments matching [11]-[14], continued fraction method [15]-[17] and Pade approximation [18] and etc.

The issue of optimality in model reduction was considered by Wilson [19], [20] who suggested an optimization approach based on minimization of the integral squared impulse response error between full and reduced-order models. This attempt was continued by other researches through other approaches [21]-[24].
In 1981 [25], the controllability and observability of the states was considered in model reduction by Moore. The suggested approach suffered from steady state errors but the stability of the reduced model was assured if the original system was also stable [26]. Furthermore, the concept of $H_\infty$, $H_2$, $L_2$ and $L_\infty$ were used for model reduction in [27]-[30].

In recent decade, the evolutionary techniques such as Particle Swarm Optimization (PSO) and Genetic Algorithm (GA) are used for order reduction of systems [31]-[33]. In these approaches, the reduced order model’s parameters are achieved by minimizing a fitness function which is often Integral Square Error (ISE), Integral Absolute Error (IAE), $H_2$ norm or $H_\infty$ norm [34]-[36].

This paper introduces a new alternative method for order reduction using orthogonal polynomials through Hermite polynomials. In this method, the full order system is expanded by Hermite polynomials and then the $l$ first coefficients of Hermite polynomials are obtained. A desire fixed structure for reduced order model is considered and a set of parameters are defined, whose values determine the reduced order system. These unknown parameters are determined using GA by minimizing the errors between the $l$ first coefficients of Hermite polynomials expansion of full and reduced systems. To satisfy the stability, Routh criterion is applied as it is used in [37] where, it states in optimization problem as constraints and subsequently, optimization problem converted to a constrained optimization problem. To show the accuracy of the proposed method, three systems are reduced by the proposed method and compared with those available in the literature.

To make a proper background, Hermite polynomials and GA are briefly explained in Sections 2 and 3, respectively. The proposed method is explained in Section 4. The ability of the proposed approach is shown in Section 5. The paper is concluded in Section 6. Finally, the references are introduced.

2. The Hermite polynomials

The Hermite polynomials are a class of orthogonal polynomials [38]. A Hermite polynomial $H_n$ in $x$ of degree on $n$ is defined as:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left( e^{-x^2} \right)$$

(1)

where $\frac{d^n}{dx^n}$ represents differential operator.

Also, the Hermite polynomials can be obtained by recursive formula as follows:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

(2)

in which $H_0(x) = 1$ and $H_1(x) = 2x$.

The Hermite polynomials are orthogonal on the interval $(-\infty, +\infty)$ with respect to the weight function $W(x)$, where $W(x)$ is expressed as

$$W(x) = e^{-x^2}$$

(3)

By the above definitions, we have the following:

$$\int_{-\infty}^{\infty} W(x) H_m(x) H_n(x) dx = \begin{cases} 2^n n! \sqrt{\pi} & m = n \\ 0 & m \neq n \end{cases}$$

(4)

Therefore, a piecewise continuous function, $f(x)$, can be expanded as:

$$\sum_{n=0}^{\infty} C_n H_n(x) = \begin{cases} f(x) & f(x) \text{ is continuous} \\ \frac{f(x^-) + f(x^+)}{2} & \text{at discontinuouos points} \end{cases}$$

(5)
where $C_n$ are coefficients of Hermite polynomial that are obtained by following equation:

$$C_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x) \, dx$$  \hspace{1cm} (6)$$

Thus, by considering the first $l$ terms of (5), a good approximant of $f(x)$ is obtained.

3. Genetic algorithm

GA is a search algorithm based on the mechanism of genetic and natural selection. The GAs start with random generation of initial population and then the selection, crossover and mutation are preceded until the maximal generation is reached. A typical simple genetic algorithm is described in detail in [39]. The principle of GA is shown by Fig. 1.

4. The Proposed model reduction method

Let

$$G(s) = \frac{a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n}{s^n + b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}$$  \hspace{1cm} (7)$$
be an \( n \)th order single-input single-output (SISO) system which is strictly proper and asymptotically stable where \( a_i \) and \( b_i \) are constants. It is desired to have a stable \( r \)th order approximant (that \( r \) is smaller than \( n \)) to (7) as

\[
G_r(s) = \frac{c_1s^{r-1} + c_2s^{r-2} + \ldots + c_r}{s^r + d_1s^{r-1} + d_2s^{r-2} + \ldots + d_r},
\]

(8)

such that the principal and important specifications of the full order system are retained in the reduced order model. In (8) \( c_i \) and \( d_i \) are unknown constants.

To obtain the reduced model by the proposed method, firstly, the full order system is expanded based on Hermite polynomials expansion. Then the first coefficients of Hermite polynomials expansion of original system are obtained and shown by \( C_i \), \( i = 0, 1, \ldots, l \). Then a desired fixed structure is considered for reduced order model as defined in (8) where \( c_i \) and \( d_i \) are unknown parameters of reduced order model that are obtained by GA. The goal of the optimization is to find the best parameters for \( G_r(s) \). Therefore, each chromosome is a \( d \)-dimensional vector in which \( d = c_i + d_i \). Each chromosome is a solution to \( G_r \) and for each solution (chromosome), the Hermite polynomials are obtained. Each chromosome in the population is evaluated by minimizing the following fitness function:

\[
J^* = \sum_{i=0}^{l} \left| C_i - \hat{C}_i \right|
\]

(9)

in which \( \hat{C}_i \) are the coefficients of Hermite polynomials of reduced order system. The algorithm searches for the best chromosome until the termination criteria are met. At this stage the best parameters are given as parameters of reduced order model.

Furthermore, the reduced model must be stable if the original system is stable. Therefore, the Routh criterion is applied to assure the stability. For specifying the stability conditions, first, the denominator of reduced order model in (8) is shown as below [40]:

\[
s^r + h_1s^{r-1} + (h_2 + h_3 + \ldots + h_r)s^{r-2} + h_1(h_3 + h_4 + \ldots + h_r)s^{r-3} + \ldots = \sum_{i=1}^{r} (h_i s^{r-i}) + \sum_{i=1}^{r} (h_i h_{i+1} s^{r-i-1}) + \ldots
\]

(10)

which is constructed by taking the coefficients of the first two rows of the Routh array with the elements of its first column given by

\[
1, h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8, \ldots, h_{r+k}, h_{r+k+1}, h_{r+k+2}, \ldots, h_{r+2k}
\]

where \( k \) is equal to 1 for even \( r \) and \( k \) is equal to 0 for odd \( r \).

Comparing the entries of the first row with \( e_1, d_2, d_3, \ldots \) and those of the second row with \( d_1, d_3, d_5, \ldots \), the following relations are obtained:

\[
\begin{align*}
d_1 &= h_1 \\
d_2 &= (h_2 + h_3 + \ldots + h_r) \\
d_3 &= h_1(h_3 + h_4 + \ldots + h_r) \\
&\vdots \\
d_r &= (h_{r+k} h_{r+k+1} h_{r+k+2} \ldots h_{r+2k})
\end{align*}
\]

(12)

Substituting the above relations in reduced order model’s denominator, (10) is achieved. Therefore, the necessary and sufficient condition for all the poles of the reduced system to be strictly in the left-half plane is
\[ h_i > 0 \]
\[ h_2 > 0 \]
\[ h_r > 0 \]

and subsequently
\[ d_1 > 0 \]
\[ d_2 > 0 \]
\[ d_r > 0 \]

Thus, to have a stable reduced system, the reduced order model’s parameters are determined by minimizing the following fitness function:
\[
J = \sum_{i=0}^{l} \left| C_i - \hat{C}_i \right|
\]

subject to \( d_j > 0 \) for \( j = 1, \ldots, r \)

5. Simulations and results

To assess the efficiency of the proposed approach, it has been applied on three test systems, where a step-by-step procedure is given for the first test system.

Test system 1: The first system to be reduced is a system of order 6, given by Mukherjee where a procedure was presented in [41] to obtain reduced system using response- matching technique. The system is as follows:
\[
G(s) = \frac{s^3 + 1014s^4 + 14069s^5 + 69140s^6 + 140100s^7 + 100000}{s^6 + 222s^5 + 14541s^4 + 248420s^3 + 1454100s^2 + 2220000s + 1000000}
\]

The reduced order model can be achieved by the following steps, using Hermite polynomials and GA:

Step 1: Based on section 2, the Hermite polynomials of the full order system in (16) are obtained as:
\[
G(s) = (0.0387) \times 1 + (0.0197) \times (2s) + (-0.0014) \times (4s^2 - 2) + (-0.0019) \times (8s^3 - 12s) + \\
(1.1702 \times 10^{-4}) \times (16s^4 - 48s^5 + 12) + (1.5412 \times 10^{-4}) \times (32s^5 - 160s^6 + 120s) + \cdots
\]

Step 2: The full order of the system represented in (16) is going to be reduced to a third-order system with the following transfer function:
\[
G_r(s) = \frac{c_0s^2 + c_1s + c_2}{s^3 + d_0s^2 + d_1s + d_2}
\]

where \( c_i \) and \( d_i \) are unknown parameters of reduced order model.

Step 3: GA is applied to obtain the unknown parameters in (18). Since, the goal of the optimization is to find the best parameters for \( G_r(s) \), therefore, a configuration is considered with six genes \( (c_i + d_i) \). The number of chromosomes for a population is set to be 100.

The chromosomes evolve through successive iterations, called generations. During each generation, the chromosomes are evaluated with some measure of fitness, which is calculated from the objective function (15).

Moving to a new generation is done from the results obtained for the old generation. A based roulette wheel is created from the obtained values of the objective function of the current population. To create the next generation, new chromosomes, called off spring, are formed using a crossover operator and a mutation operator. In this paper, one point crossover is applied with the crossover probability \( p_c = 0.9 \).
and the mutation probability is selected to be changed linearly from $p_m = 0.05$ to $p_m = 0.005$. Also, the number of iteration is considered to be 100.

Each chromosome is a solution to $G_r$ and for each solution (chromosome), the Hermite polynomials are obtained. Each chromosome in the population is evaluated using the objective function defined by (15) searching for the best $J$ until the termination criteria is met. At this stage the best parameters are given for reduced order model where, the following reduced order model is obtained:

$$G_{\text{Hermite}} = \frac{4.56s^2 + 26.96s + 50.74}{s^3 + 53.49s^2 + 690.6s + 507.02}$$

(19)

The Hermite polynomials of obtained reduced order model are as:

$$G_{\text{Hermite}}(s) = (0.0387)s + (0.0197)\times(2s) + (-0.0014)\times(4s^2 - 2) + (-0.0019)\times(8s^3 - 12s) +$$

$$\left(1.693 \times 10^{-4}\right)\times(16s^4 - 48s^2 + 12) + \left(1.5410 \times 10^{-4}\right)\times(32s^5 - 160s^3 + 120s) + \ldots$$

(20)

Comparing (17) and (20) shows that the best approximant of $G(s)$ is achieved. The step response of the full order system and that of the system with third-order reduced models are shown in Fig. 2. This figure shows that the obtained reduced order model is an adequate low-order model that retains the characteristics of full order model. Also, to show the efficiency of the proposed method, the step and frequency responses of the obtained reduced model are compared with those available in the literature.

Figs. 3 - 4, show the comparison of the results obtained with the proposed method by Mukherjee [41], Optimal Hankel norm approximation (HSV) [42] and Balanced Truncation (BT) [42], respectively.

Figure 2: The step response of full order and reduced order model by the proposed method for test system 1.
Figure 3: The step response of full order and reduced order model by the proposed method and other methods for test system 1.

Figure 4: The frequency response of full order and reduced order model by the proposed method and other methods for test system 1.
These figures show that the achieved results from the proposed method and the suggested method by Mukherjee are very similar to original system comparing to HSV and BT methods. The steady-state gains of full and reduced systems should be equal since it is a very important requirement for model reduction. Also, the frequency responses of full and reduced systems are the same, which will make the stability and performance characteristics of both systems to be the same.

Furthermore, the specifications of the proposed method such as steady state value and maximum overshoot are obtained and compared with the suggested method by Mukherjee [41], HSV and BT. The results are shown in Table 1. Also, $H_\infty$ norm of the error between the step responses of full order and reduced order models $e = |y - y_r|$ is given in Table 1. It is clearly seen that the specifications of reduced order model that is achieved by the proposed method and the one by Mukherjee are close to the specifications of original system.

Table1. Comparison of methods for test system 1.

<table>
<thead>
<tr>
<th>Model</th>
<th>Steady state value</th>
<th>Overshoot (%)</th>
<th>ISE</th>
<th>Infinity norm of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original system</td>
<td>0.1</td>
<td>0</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Hermite - GA</td>
<td>0.1</td>
<td>0</td>
<td>2.5x10^-6</td>
<td>0.0069</td>
</tr>
<tr>
<td>Proposed by Mukherjee</td>
<td>0.1</td>
<td>0</td>
<td>1.34x10^-6</td>
<td>0.0074</td>
</tr>
<tr>
<td>BT</td>
<td>0.114</td>
<td>0</td>
<td>0.0379</td>
<td>0.0141</td>
</tr>
<tr>
<td>HSV</td>
<td>0.114</td>
<td>0</td>
<td>0.0375</td>
<td>0.0139</td>
</tr>
</tbody>
</table>

Also, the plot of $e = |y - y_r|$ is given for the full order and reduced systems in Fig. 5. This figure illustrates that the obtained error by the proposed method and the one by Mukherjee is very similar and less than HSV and BT methods.

Figure 5: The plot of $e = |y - y_r|$ for the full order and reduced systems by the proposed method and other methods for test system 1.
Test system 2: In [40], a procedure is presented to obtain a reduced order system by Routh-Pade approximation using Luus-Jaakola algorithm. To compare the proposed method with Luus-Jaakola algorithm, the system given in [40] is adopted which is a third-order system:

\[ G = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2} \]  

(21)

Based on the explanations given for test system 1, the obtained reduced system by the proposed method is as follows:

\[ G_{Hermite} = \frac{5.97s + 4.172}{s^2 + 2.58s + 4.176} \]  

(22)

The step and frequency responses of the original system and the obtained reduced model are shown in Figs. 6-7. In these figures, the responses of the system with second-order primary reduced models obtained by other methods are also included for comparison. Also, the plot of \( e = |y - \hat{y}| \) is given for the full order and reduced systems in Fig. 8.

![Step Response](image)

Figure 6: The step response of full order and reduced order model by the proposed method and other methods for test system 2.
Figure 7: The frequency response of full order and reduced order model by the proposed method and other methods for test system 2.

Figure 8: The plot of $e = |y - y_r|$ for the full order and reduced systems by the proposed method and other methods for test system 2.
Furthermore, the specifications of the proposed method such as maximum overshoot, steady state value, ISE and $H_{\infty}$ norm of the error between the step responses of full order and reduced order models ($e = |y - y_r|$) are given in Table 2.

Table 2. Comparison of methods for test system 2.

<table>
<thead>
<tr>
<th>Model</th>
<th>Steady state value</th>
<th>Overshoot (%)</th>
<th>ISE</th>
<th>Infinity norm of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original system</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hermite - GA</td>
<td>0.999</td>
<td>90.2</td>
<td>0.0382</td>
<td>0.1824</td>
</tr>
<tr>
<td>Proposed by Luss</td>
<td>1</td>
<td>66.1</td>
<td>0.1404</td>
<td>0.3425</td>
</tr>
<tr>
<td>BT</td>
<td>0.836</td>
<td>123</td>
<td>0.3802</td>
<td>0.1635</td>
</tr>
<tr>
<td>HSV</td>
<td>0.836</td>
<td>115</td>
<td>0.4043</td>
<td>0.1635</td>
</tr>
</tbody>
</table>

Once again, the results obtained confirm that a satisfactory approximation has been achieved. It is clearly seen that the specifications of reduced order model that is achieved by the proposed method are close to the specifications of original system and better than other methods.

**Test system 3:** The third system to be reduced is a system given in [31] by Mukherjee, where a procedure is presented to obtain reduced system. The system is as follows:

$$G(s) = \frac{18s^7 + 514s^6 + 5982s^5 + 3638s^4 + 122664s^3 + 222088s^2 + 185760s + 40320}{s^8 + 36s^7 + 546s^6 + 4536s^5 + 22449s^4 + 67284s^3 + 118124s^2 + 109584s + 40320}$$  \hspace{1cm} (23)

Based on the explanations given for test system 1, the obtained reduced system by the proposed method is as follows:

$$G_{Hermite} = \frac{16.68s + 5.43}{s^2 + 6.67s + 5.43}$$ \hspace{1cm} (24)

The comparison of the proposed method with the method suggested by Mukherjee in [31], HSV and BT methods are shown in Figs. 9-11 and Table 3, which illustrate that the achieved results from the proposed method is very similar to original system comparing to other methods.
Figure 10: The frequency response of full order and reduced order model by the proposed method and other methods for test system 3.

Figure 11: The plot of $e = |y - y_r|$ for the full order and reduced systems by the proposed method and other methods for test system 3.
Table 3. Comparison of methods for test system 3.

<table>
<thead>
<tr>
<th>Model</th>
<th>Steady state value</th>
<th>Overshoot (%)</th>
<th>ISE</th>
<th>Infinity norm of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original system</td>
<td>1</td>
<td>120</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Hermite - GA</td>
<td>1</td>
<td>125</td>
<td>0.0022</td>
<td>0.0437</td>
</tr>
<tr>
<td>Proposed by Mukherjee</td>
<td>1</td>
<td>129</td>
<td>0.0569</td>
<td>0.3361</td>
</tr>
<tr>
<td>BT</td>
<td>0.94</td>
<td>134</td>
<td>0.0314</td>
<td>0.0595</td>
</tr>
<tr>
<td>HSV</td>
<td>0.944</td>
<td>132</td>
<td>0.0326</td>
<td>0.0559</td>
</tr>
</tbody>
</table>

6. Conclusion

In this paper, an approach based on orthogonal polynomials using Hermite polynomials and GA is investigated for order reduction. Routh array is applied to determine the stability conditions. To present the accuracy and efficiency of the method, three systems are reduced by the proposed method. The proposed method is compared with some order reduction techniques where the results obtained show that the proposed approach has high accuracy whose results in an adequate low-order model that retains the characteristics of full order model.

References


