Zweier Ideal Convergent Sequence Spaces Defined By Orlicz Functions

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Abstract

An ideal $I$ is a family of subsets of positive integers $N$ which is closed under taking finite unions and subsets of its elements. In this article we introduce ideal convergent sequence spaces using Zweier transform and Orlicz function. We study some topological and algebraic properties. Further we prove some inclusion relations related to these new spaces.

Keywords: Ideal, $I$-convergence, Zweier sequence, Orlicz function.

1. Introduction

The concept of ideal convergence as a generalization of statistical convergence, and any concept involving ideal convergence plays a vital role not only in pure mathematics but also in other branches of
science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modelling, and motion planning in robotics.

Kostyrko, et. al [16] was initially introduced the notion of $I$-convergence based on the structure of the admissible ideal $I$ of subset of natural numbers $N$. Further details on ideal convergence, we refer to ([2-3], [7-13], [17], [21], [25-26], [29-33]), and many others.

Let $X$ be a non-empty set. Then a family of set $I \subseteq 2^X$ (power sets of $X$) is said to be an ideal if $I$ is additive i.e. $A, B \in I \Rightarrow A \cup B \in I$ and hereditary i.e. $A \in I, B \subseteq A \Rightarrow B \in I$.

A non-empty family of sets $F \subset 2^X$ is said to be a filter on $N$ if and only if $\phi \notin F(I)$, for $A, B \in F(I)$ we have $A \cap B \in F(I)$ and for each $A \in F(I)$ and $B \supseteq A$ implies $B \in F(I)$.

An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$.

A non trivial ideal $I \subset 2^X$ is called admissible if $I \supset \{\{x\}: x \in X\}$.

A non trivial ideal $I$ is maximal if there exist any non–trivial ideal $J \neq I$ containing $I$ as a subset.

For each ideal $I$ there is a filter $F(I)$ corresponding to $I$ i.e. $F(I) = \{ K \subseteq N: K^c \in I \}$, where $K^c = N - K$.

A subset $A$ of $N$ is said to have asymptotic density $\delta(A)$ if $\delta(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{A}(k)$ exists, where $\chi_{A}$ is the characteristic function of $A$.

**Remark 1.** If we take $I = I_f = \{ A \subseteq N: A$ is a finite subset\}. Then $I_f$ is a nontrivial admissible ideal of $N$ and the corresponding convergence coincide with the usual convergence.

If we take $I = I_{\delta} = \{ A \subseteq N: \delta(A) = 0 \}$ where $\delta(A)$ denote the asymptotic density of the set $A$. Then $I_{\delta}$ is a non-trivial admissible ideal of $N$ and the corresponding convergence coincide with the statistical convergence.

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of the regular function $M$ is replaced by

$$M(x + y) \leq M(x) + M(y),$$

Then this function is called modulus function. The notion of modulus function was introduced by Nakano [22]. Ruckle [24] and Maddox [19] further investigated the modulus functions with application to sequence spaces.
Remark 2. If $M$ is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all $\lambda$ with $0 < \lambda < 1$.

An Orlicz function $M$ is said to satisfy $\Delta_2$-condition for all values of $u$ if there exists a constant $K > 0$ such that $M(Lu) \leq KLM(u)$ for all values of $L > 1$ (see [15]).

Lindenstrauss and Tzafriri [18] used the idea of Orlicz functions to construct the sequence space

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space $l_M$ becomes a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\},$$

which is called an Orlicz sequence space. The space $l_M$ is closely related to the space $l_p$ which is an Orlicz sequence space with $M(t) = t^p$ for $1 \leq p < \infty$.

Later on Orlicz sequence spaces were investigated by Parashar and Chaudhary [23], Esi [5], Tripathy et al. [28], Bhardwaj and Singh [1], Et [4], Esi and Et [6] and many others.

The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Malkowsky [20] and many others. Sengonul [27] defined the sequence $y = (y_i)$ which is frequently used as the $Z^p$-transformation of the sequence $x = (x_i)$ i.e.

$$y_i = px_i + (1-p)x_{i-1}$$

Where $x_{-1} = 0, p \neq 0, 1 < p < \infty$ and $Z^p$ denotes the matrix $Z^p = (z_{ik})$ defined by

$$z_{ik} = \begin{cases} p, & (i = k) \\ 1-p, & (i-1 = k); (i, k \in N) \\ 0, & \text{otherwise}. \end{cases}$$

Sengonul [27] introduced the Zweier sequence spaces $Z$ and $Z_0$ as follows

$$Z = \left\{ x = (x_k) \in w : Z^p x \in c \right\},$$

$$Z_0 = \left\{ x = (x_k) \in w : Z^p x \in c_0 \right\}.$$
2. Definitions and Preliminaries

We assume throughout this paper that the symbols $R$ and $N$ as the set of real and natural numbers, respectively. Throughout the paper, we also denote $I$ is an admissible ideal of subsets of $N$, unless otherwise stated.

A sequence $(x_k) \in w$ is said to be $I$-convergent to the number $L$ if for every $\varepsilon > 0$, \[ \{ k \in N : |x_k - L| \geq \varepsilon \} \in I. \] In this case we write $I - \lim x_k = L$.

A sequence $(x_k) \in w$ is said to be $I$-null if $L = 0$. In this case we write $I - \lim x_k = 0$.

Let $I$ be an admissible ideal. A sequence $(x_k) \in w$ is said to be $I$-Cauchy if for every $\varepsilon > 0$ there exists a number $m = m(\varepsilon)$ such that \[ \{ k \in N : |x_k - x_m| \geq \varepsilon \} \in I. \]

A sequence $(x_k) \in w$ is said to be $I$-bounded if there exists $M > 0$ such that \[ \{ k \in N : |x_k| > M \} \in I. \]

A sequence space $E$ is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ and for all sequence $(\alpha_k)$ of scalars with $|\alpha_k| \leq 1$ for all $k \in N$.

A sequence space $E$ is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi$ is a permutation of $N$.

A sequence space $E$ is said to be sequence algebra if $(x_k)^*(y_k) = (x_k y_k) \in E$ whenever $(x_k), (y_k) \in E$.

A sequence space $E$ is said to be convergence free if $(y_k) \in E$ whenever $(x_k) \in E$ and $x_k = 0$ implies $y_k = 0$.

Let $K = \{ k_1 < k_2 < ... \} \subset N$ and let $E$ be a sequence space. A $K$-step space of $E$ is a sequence space $\lambda_K^E = \{ (x_k) \in w : (x_k) \in E \}$.

A canonical preimage of a sequence $(x_k) \in \lambda_K^E$ is a sequence $(y_k) \in w$ defined by

$$ y_k = \begin{cases} x_k, & \text{if } k \in K \\ 0, & \text{otherwise.} \end{cases} $$

A canonical preimage of a step space $\lambda_K^E$ is a set of canonical preimages of all the elements in $\lambda_K^E$, i.e. $y$ is in the canonical preimage of $\lambda_K^E$ if and only if $y$ is a canonical preimage of some $x \in \lambda_K^E$. 

310
A sequence space $E$ is said to be *monotone* if it contain the canonical preimages of its step spaces.

Throughout the article $Z^I, Z^I_0, Z^I_{\infty}, m^I_Z$ and $m^I_{Z_0}$ represents Zweier $I$-convergent, Zweier $I$-null, Zweier bounded $I$-convergent and Zweier bounded $I$-null sequence space, respectively.

The following results will be used for establishing some result of this article.

**Lemma 1** [14]. The sequence space $E$ is solid implies that $E$ is monotone.

### 3. Main Results

In this article, we introduce the following classes of sequences:

$$Z^I(M) = \left\{ x = (x_k) : \left\{ k \in N : \sum_{n=1}^{\infty} M \left( \left( \frac{Z^p x}{\rho} \right)_n - L \right) \geq \varepsilon \right\} \in I \right\} \text{ for some } L \in C$$

$$Z^I_0(M) = \left\{ x = (x_k) : \left\{ k \in N : \sum_{n=1}^{\infty} M \left( \left( \frac{Z^p x}{\rho} \right)_n \right) \geq \varepsilon \right\} \in I \right\}$$

$$Z^I_{\infty}(M) = \left\{ x = (x_k) : \left\{ k \in N : \exists K > 0, \sum_{n=1}^{\infty} M \left( \left( \frac{Z^p x}{\rho} \right)_n \right) \geq K \right\} \in I \right\}.$$

Also we write

$$m^I_Z(M) = Z^I(M) \cap Z^I_0(M), \quad m^I_{Z_0}(M) = Z^I_0(M) \cap Z^I_{\infty}(M).$$

**Theorem 1.** For any Orlicz function $M$, the classes of sequence $Z^I(M), Z^I_0(M)$ and $Z^I_{\infty}$ are linear spaces.

Proof: We shall prove that result for the space $Z^I(M)$.

Let $(x_k), (y_k) \in Z^I(M)$ and let $\alpha, \beta$ be scalars. Then there exists positive numbers $\rho_1$ and $\rho_2$ such that

$$\left\{ k \in N : \sum_{n=1}^{\infty} M \left( \left( \frac{Z^p x}{\rho} \right)_n - L_1 \right) \geq \varepsilon \right\} \in I \text{ for some } L_1 \in C.$$
\[
\left\{ k \in N : \sum_{n=1}^{\infty} M \left( \frac{(Z^p y_n - L_2)}{\rho_2} \right) \geq \varepsilon \right\} \in I \text{ for some } L_2 \in C.
\]

That is

\[
A_1 = \left\{ k \in N : \sum_{n=1}^{\infty} M \left( \frac{(Z^p x_n - L_1)}{\rho_1} \right) \geq \frac{\varepsilon}{2} \right\} \in I
\]

and

\[
A_2 = \left\{ k \in N : \sum_{n=1}^{\infty} M \left( \frac{(Z^p y_n - L_2)}{\rho_2} \right) \geq \frac{\varepsilon}{2} \right\} \in I
\]

Let \( \rho_3 = \max \{2|\alpha|\rho_1, 2|\beta|\rho_2\} \). Since \( M \) is non-decreasing and convex function, we have

\[
M \left( \frac{(\alpha(Z^p x_n + \beta(Z^p y_n)) - (\alpha L_1 + \beta L_2))}{\rho_3} \right) \leq M \left( \frac{\|\alpha(Z^p x_n - L_1)\|}{\rho_3} + \frac{\|\beta(Z^p y_n - L_2)\|}{\rho_3} \right)
\]

\[
\leq M \left( \frac{(Z^p x_n - L_1)}{\rho_1} \right) + M \left( \frac{(Z^p y_n - L_2)}{\rho_2} \right).
\]

Now, from (1) and (2), we have

\[
\left\{ k \in N : \sum_{n=1}^{\infty} M \left( \frac{(\alpha(Z^p x_n + \beta(Z^p y_n)) - (\alpha L_1 + \beta L_2))}{\rho_3} \right) \geq \varepsilon \right\} \subset A_1 \cup A_2 \in I.
\]

Therefore \( (\alpha(Z^p x_n + \beta(Z^p y_n)) \in Z^I(M) \). Hence \( Z^I(M) \) is a linear space.

**Theorem 2.** The space \( Z^I_0(M) \) and \( Z^I(M) \) are Banach spaces normed by

\[
\left\| (Z^p x)_n \right\| = \inf \left\{ \rho > 0 : \sup_k M \left( \frac{(Z^p x_n)}{\rho} \right) \leq 1 \right\}.
\]

The proof of the theorem is easy, so omitted.
Theorem 3. Let $M_1, M_2$ be Orlicz functions that satisfy the $\Delta_2$-condition. Then

(i) $W(M_2) \subseteq W(M_1, M_2),$

(ii) $W(M_1) \cap W(M_2) = W(M_1 + M_2)$ for $W = Z^I, Z^I_0, m^I_Z, m^I_{Z_0}.$

Proof. (i) Let $(x_k) \in Z^I_0(M_2).$ Then there exists $\rho > 0$ such that

$$\left\{ k \in N : \sum_{n=1}^{\infty} M_2 \left( \frac{(Z^p_n x_n)}{\rho} \right) \geq \varepsilon \right\} \subseteq I \quad (3)$$

Let $\varepsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $M_1(t) < \varepsilon$ for $0 \leq t \leq \delta.$

Write $\left( Z^p_n y_n \right) = M_2 \left( \frac{(Z^p_n x_n)}{\rho} \right)$ and consider

$$\lim_{0 \leq (Z^p_n y_n) \leq \delta, k \in N} M_1(Z^p_n y_n) = \lim_{(Z^p_n y_n) \leq \delta, k \in N} M_1(Z^p_n y_n) + \lim_{(Z^p_n y_n) > \delta, k \in N} M_1(Z^p_n y_n).$$

We have

$$\lim_{(Z^p_n y_n) \leq \delta, k \in N} M_1(Z^p_n y_n) = M_1(2 \lim_{(Z^p_n y_n) \leq \delta, k \in N} \left( Z^p_n y_n \right). \quad (4)$$

For $Z^p_n y > \delta,$ we have

$$\left( Z^p_n y \right) < \frac{(Z^p_n y_n)}{\delta} < 1 + \frac{(Z^p_n y_n)}{\delta}.$$ 

Since $M_1$ is non-decreasing and convex, it follows that

$$M_1(Z^p_n y_n) < M_1 \left( 1 + \frac{(Z^p_n y_n)}{\delta} \right) < \frac{1}{2} M_1(2) + \frac{1}{2} M_1 \left( \frac{(Z^p_n y_n)}{\delta} \right).$$

Since $M_1$ satisfies the $\Delta_2$-condition, we have

$$M_1(Z^p_n y_n) < \frac{1}{2} K \frac{(Z^p_n y_n)}{\delta} M_1(2) + \frac{1}{2} K \frac{(Z^p_n y_n)}{\delta} M_1(2) = K \frac{(Z^p_n y_n)}{\delta} M_1(2).$$

Hence
\[
\lim_{(Z^p y)_n} M_1 \left( Z^p y \right)_n \leq \max \left( 1, K \delta^{-1} M_1(2) \right) \lim_{(Z^p y)_n} \left( Z^p y \right)_n.
\]  

From (3), (4) and (5), we have

\[ (x_k) \in Z^l_0 (M_1, M_2). \]

Thus \( Z^l_0 (M_2) \subseteq Z^l_0 (M_1, M_2). \) The other case can be proved similarly.

(ii) Let \((x_k) \in Z^l_0 (M_1) \cap Z^l_0 (M_2).\) Then there exists \( \rho > 0 \) such that

\[
\left\{ k \in N : \sum_{n=1}^{\infty} M_1 \left( \frac{(Z^p x)_n}{\rho} \right) \geq \varepsilon \right\} \in I
\]

and

\[
\left\{ k \in N : \sum_{n=1}^{\infty} M_2 \left( \frac{(Z^p x)_n}{\rho} \right) \geq \varepsilon \right\} \in I.
\]

The rest of the proof follows the following equality

\[
\lim_{k \in N} (M_1 + M_2) \left( \frac{(Z^p x)_n}{\rho} \right) = \lim_{k \in N} M_1 \left( \frac{(Z^p x)_n}{\rho} \right) + \lim_{k \in N} M_2 \left( \frac{(Z^p x)_n}{\rho} \right).
\]

Taking \( M_2 (x) = x \) and \( M_1 (x) = M(x) \) for all \( x \in [0, \infty) \) in Theorem 3(i). We have the following result.

**Corollary 1.** \( W \subseteq W(M) \) for \( W = Z^l, Z^l_0, m^l_0, m^l_0. \)

**Proposition 1.** The space \( Z^l_0 (M) \) and \( m^l_0 (M) \) are solid and monotone.

Proof: We shall prove the result for \( Z^l_0 (M). \) For \( m^l_0 (M) \), the result can be proved similarly.

Let \((x_k) \in Z^l_0 (M).\) then there exists \( \rho > 0 \) such that

\[
\left\{ k \in N : \sum_{n=1}^{\infty} \left( \frac{(Z^p x)_n}{\rho} \right) \geq \varepsilon \right\} \in I.
\]

Let \((\alpha_k)\) be a sequence scalar with \( |\alpha_k| \leq 1 \) for all \( k \in N. \) Then the result follows from (6) and the following inequality.
\[ M \left( \frac{\alpha_k (Z^p x)_n}{\rho} \right) \leq \alpha_k \left( M \left( \frac{(Z^p x)_n}{\rho} \right) \right) \leq M \left( \frac{(Z^p x)_n}{\rho} \right) \text{ for all } k \in N, \]

which follows from the remark.

That the space \( Z^I_0 (M) \) is monotone follows from the Lemma 1.

**Proposition 2.** The space \( Z^I (M) \) and \( m^I_k (M) \) are neither monotone nor solid in general.

Proof: The proof of this result follows from the following example.

**Example 1.** Let \( I = I_f \) and \( M (x) = x \) for all \( x \in [0, \infty) \). Consider the K-step space \( T_k \) of \( T \) defined as follows.

Let \( (x_k) \in T \) and let \( (y_k) \in T_k \) be such that \( y_k = \begin{cases} x_k, & \text{if } k \text{ is odd} \\ 0, & \text{otherwise} \end{cases} \).

Consider the sequence \( (x_k) \) defined by \( x_k = \frac{1}{2} \) for all \( k \in N \). Then \( (x_k) \in Z^I (M) \) but its K-step space preimage does not belong to \( Z^I (M) \). Thus \( Z^I (M) \) is not monotone. Hence \( Z^I (M) \) is not solid by Lemma 1.

**Proposition 3.** The space \( Z^I (M) \) and \( Z^I_0 (M) \) are not convergence free in general.

Proof: The proof of this result follows from the following example.

**Example 2.** Let \( I = I_f \) and \( M (x) = x^2 \) for all \( x \in [0, \infty) \). Consider the sequence \( (x_k) \) and \( (y_k) \) defined by \( x_k = \frac{1}{k^2} \) and \( y_k = k^2 \) for all \( k \in N \).

Then \( (x_k) \) belongs to \( Z^I (M) \) and \( Z^I_0 (M) \), but \( (y_k) \) does not belong to both \( Z^I (M) \) and \( Z^I_0 (M) \).

Hence the spaces are not convergence free.

**Proposition 4.** The spaces \( Z^I (M) \) and \( Z^I_0 (M) \) are sequence algebra.

Proof: We prove that \( Z^I_0 (M) \) is sequence algebra. For the space \( Z^I (M) \), the result can be proved similarly.

Let \( (x_k), (y_k) \in Z^I_0 (M) \). Then
Let \( \rho = \rho_1 \rho_2 > 0 \). Then we can show that
\[
\left \{ k \in \mathbb{N} : \sum_{n=1}^{\infty} M \left( \frac{[Z^n x]_n}{\rho_1} \right) \geq \varepsilon \right \} \in I \text{ for some } \rho_1 > 0
\]
and
\[
\left \{ k \in \mathbb{N} : \sum_{n=1}^{\infty} M \left( \frac{[Z^n y]_n}{\rho_2} \right) \geq \varepsilon \right \} \in I \text{ for some } \rho_2 > 0.
\]

Thus \((x_k, y_k) \in Z_0^I(M)\). Hence \(Z_0^I(M)\) is sequence algebra.

**Theorem 4:** Let \( M \) be a Orlicz function. Then \( Z_0^I(M) \subset Z^I(M) \subset Z_\infty^I(M) \) and the inclusions are proper.

**Proof:** Let \((x_k) \in Z^I(M)\). Then there exist \( L \in C \) and \( \rho > 0 \) such that
\[
\left \{ k \in \mathbb{N} : \sum_{n=1}^{\infty} M \left( \frac{(Z^n x)_n - L}{\rho} \right) \geq \varepsilon \right \} \in I.
\]
We have
\[
M \left( \frac{(Z^n x)_n}{2\rho} \right) \leq \frac{1}{2} M \left( \frac{(Z^n x)_n - L}{\rho} \right) + M \left( \frac{|L|}{\rho} \right).
\]
Taking supremum over \( k \) on both sides we get \((x_k) \in Z^I_\infty(M)\). The inclusion \(Z^I_0(M) \subset Z^I(M)\) is obvious.

That the inclusion is proper follows from the following example.

**Example 3.** Let \( I = I_d, M(x) = x^2 \) for all \( x \in [0, \infty) \).

(a) Consider the sequence \((x_k)\) defined by \( x_k = 1 \) for all \( k \in \mathbb{N} \). Then \((x_k) \in Z^I(M)\), but \((x_k) \not\in Z^I_0(M)\).
(b) Consider the sequence \( (y_k) \) defined as
\[
y_k = \begin{cases} 
2, & \text{if } k \text{ is even} \\
0, & \text{otherwise}
\end{cases}
\]

Then \( (y_k) \in Z^I_e(M) \), but \( (y_k) \notin Z^I(M) \).

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5. References


