A Class of Multivalent Analytic Functions Defined by a New Linear Operator

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Abstract
The main object of the present paper is to derive some results for multivalent analytic functions defined by a linear operator. Making use of a certain operator, which is defined here by means of Hadamard product, we introduce a subclasses $S_{A,B}^{p,y}(\alpha,\lambda,\mu,\nu,a,c)$ of the class $A(p)$ of normalized p-valent analytic functions on the open unit disk. Also we have extended some of the previous results and have given necessary and sufficient condition for this class.

Keywords: Analytic functions, Multivalent functions, Hadamard product, Subordination, Linear operators.

1. Introduction
Let $A(p)$ denote the class of functions $f$ of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$$  \hspace{1cm} (1.1)

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(p ∈ N = \{1,2,...\}) which are analytic in the open unit disk \( \Delta = \{ z ∈ \mathbb{C} : |z| < 1 \}\). We write \( A(I) = A \). If \( f \) and \( g \) are analytic in \( \Delta \), we say that \( f \) is subordinate to \( g \) in \( \Delta \), written \( f < g \), if there exists Schwarz function \( w \), analytic in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \Delta \) such that \( f(z) = g(w(z)) \), \( z \in \Delta \). If \( g \) is univalent and \( g(0) = f(0) \), then \( f(\Delta) \subset g(\Delta) \) it follows \( f < g \). For two functions \( f \) given by (1.1) and \( g \) given by

\[
g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k},
\]

their Hadamard product (or convolution) is defined by

\[
(f \ast g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}.
\]

For \( a \in \mathbb{R}, c \in \mathbb{R}\setminus\mathbb{Z}_0^- \), where \( \mathbb{Z}_0^- := \{..., -2, -1, 0\} \), we introduce a linear operator

\[
\mathcal{J}_{\mu, \nu}^{\lambda, p}(a, c): A(p) \to A(p)
\]

defined by

\[
\mathcal{J}_{\mu, \nu}^{\lambda, p}(a, c) f(z) = \phi_{\mu, \nu}^{\lambda, p}(a, c; z) \ast f(z), \quad (f \in A(p), z \in \Delta),
\]

where

\[
\phi_{\mu, \nu}^{\lambda, p}(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k (p + 1)_k (p + 1 - \mu + \nu)_k}{(c)_k (p + 1 - \mu)_k (p + 1 - \lambda + \nu)_k} z^{p+k},
\]

and \((d)_k\) is the Pochhammer symbol defined by

\[
(d)_k = \begin{cases} 1 & k = 0 \\ d(d+1)(d+2) \ldots (d+k-1) & k \in \mathbb{N}. \end{cases}
\]

Also \( 0 \leq \lambda < 1, \mu, \nu \in \mathbb{R} \) and \( \mu - \nu - p < 1 \). We note that:

(i) If \( \lambda = \mu = 0 \) in (1.2), then we have a linear operator was introduced by Saitoh [19].

(ii) If \( a = c = 1 \) in (1.2), then \( \phi_{\mu, \nu}^{\lambda, p}(a, c; z) \ast f(z) \equiv \Delta_{z p}^{\lambda, \mu, \nu} f(z) \), where \( \Delta_{z p}^{\lambda, \mu, \nu} f(z) \) is the fractional operator introduced by Choi[6].
We now introduce the following family of linear operators $L_{\mu, \nu}^{\lambda, p, \alpha} f(z)$ analogous to
$
\mathcal{D}_{\mu, \nu}^{\lambda, p} (a, c):
$\n$L_{\mu, \nu}^{\lambda, p, \alpha} (a, c): A(p) \to A(p)$

which defined as

$L_{\mu, \nu}^{\lambda, p, \alpha} (a, c) f(z) := \psi_{\mu, \nu}^{\lambda, p, \alpha} (a, c; z) * f(z),
$\n(1.4)

where $\psi_{\mu, \nu}^{\lambda, p, \alpha} (a, c; z)$ is the function defined in terms of the Hadamard product by the following condition:

$\phi_{\mu, \nu}^{\lambda, p} (a, c; z) * \psi_{\mu, \nu}^{\lambda, p, \alpha} (a, c; z) = \frac{z^p}{(1 - z)^{\alpha + p}}, \quad (\alpha > -p).
$\n(1.5)

We can easily find from (1.3)-(1.5) that

$L_{\mu, \nu}^{\lambda, p, \alpha} (a, c) f(z) = z^p + \sum_{k=1}^{\infty} \frac{(c)_k (p + l - \mu)_k (p + l + \alpha + p)_k}{(a)_k (p + l)_k (p + l + \alpha + p)_k} a_{p + k} z^{p + k}.
$\n(1.6)

By definition and specializing the parameters $\lambda, \mu, p, a, c$ and $\alpha$ we obtain:

$L_{0, \nu}^{0, p, l} (p + l) f(z) = f(z)$ and $L_{0, \nu}^{0, l, p} (p, l) f(z) = zf^{(l)}(z)/p$. It should be remarked that the linear operator $L_{\mu, \nu}^{\lambda, p, \alpha} (a, c) f(z)$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A(p)$, we have the following observations:

- $L_{0, \nu}^{0, p, \alpha} (a, c) f(z) \equiv J_p^a (a, c) f(z)$, the Cho-Kwon-Srivastava operator [5].
- $L_{0, \nu}^{0, p, \alpha} (a, a) f(z) \equiv D^{\alpha + p - l} f(z)$, where $D^{\alpha + p - l}$ is the Well-known Ruscheweyh derivative of $(\alpha + p - l)$-th order was studied by Goel and Sohi [9].
- $L_{0, \nu}^{0, p, l} (p + l - \lambda, l) f(z) \equiv \Omega_{z}^{(\lambda, p)} = \frac{\Gamma(p + l + \lambda)}{\Gamma(p + l)} z^\lambda D_z^\lambda f(z)$, where $\Omega_{z}^{(\lambda, p)}$ is the fractional derivative operator defined by Srivastava and Aouf [20] and $D_z^\lambda f(z)$ is the fractional derivative of $f(z)$ of order $\lambda$ [12, 15,17].
- $L_{0, \nu}^{\alpha, l, \alpha - l} (a, c) f(z) \equiv J_{c}^{a, \alpha} f(z)$, the linear operator investigated by Hohlov [11].
• \( L_{0,v}^{0,1-\alpha,\alpha}(a, c) f(z) \equiv L_p(a, c) f(z) \), the linear operator studied by Saitoh [19] which yields the operator \( L(a, c) f(z) \) introduced by Carleson and Shaffer for \( p = 1 \) [3].

• \( L_{0,v}^{0,p,\alpha}(\alpha + p + 1, l) f(z) \equiv F_{\alpha,p}(f)(z) = \frac{\alpha+p}{z^\alpha} \int_0^z t^{\alpha-l} f(t) dt, (\alpha > -p) \), the generalized Bernardi-Libera-Livingston integral operator [7].

• \( L_{0,v}^{0,1-\alpha,\alpha}(\lambda + 1, \mu) f(z) \equiv J_{\lambda,\mu} f(z), (\lambda > -1, \mu > 0) \), the Choi-Saigo-Srivastava operator which is closely related to the Carleson-Shaffer operator \( L(\mu, \lambda + 1) f(z) \) [3].

• \( L_{0,v}^{0,p,l}(p + \alpha, l) f(z) \equiv J_{\alpha,p} f(z), (\alpha \in \mathbb{Z}, \alpha > -p) \), the operator considered by Liu and Noor [13].

Now by using the linear operator \( L_{\mu,v}^{\lambda,p,\alpha}(a, c) f(z) \), defined by (1.6), we introduce the new class \( S_{A,B}^{p,\gamma}(\alpha, \lambda, \mu, \nu, a, c) \) as follows:

**Definition 1.1.** We say that a function \( f \in A(p) \) is in the class \( S_{A,B}^{p,\gamma}(\alpha, \lambda, \mu, \nu, a, c) \), if it satisfies the following condition:

\[
\frac{1}{p - \gamma} \left[ \frac{(L_{\mu,v}^{\lambda,p,\alpha}(a, c) f(z))'}{z^{p-l}} - \gamma \right] < \frac{l + A z}{l + B z}, \quad z \in \Delta, \quad (1.7)
\]

where \(-l \leq B < A \leq l\) and \(0 \leq \gamma < p\).

By specializing the parameters \( A, B, \alpha, \lambda, \mu, \nu, a, c \) and \( p \), we obtain many classes which were studied by authors earlier. For more details see [1,2,4,10,12,14,15].

The aim of this paper, is to give more results the above class of multivalent functions. Also we continue and extended some of the previous results and have given other properties of this class.

2. Main Results

**Theorem 2.1.** A function \( f \in A(p) \) belongs to the class \( S_{A,B}^{p,\gamma}(\alpha, \lambda, \mu, \nu, a, c) \) if and only if

\[
\frac{(f * X)(z)}{z^p} + \frac{z}{p} \left[ \frac{(f * X)(z)}{z^p} \right]' < \frac{l + A z}{l + B z}, \quad z \in \Delta, \quad (2.1)
\]

where
\[
X := z^p + \frac{p}{p - \gamma} \sum_{k=1}^{\infty} \left( \frac{c_k (p + 1 - \mu)_k (p + 1 - \lambda + \nu)_k (\alpha + p)_k}{(a)_k (p + 1)_k (p + 1 - \mu + \nu)_k k!} z^{p+k} \right) \tag{2.2}
\]

**Proof.** Let \( f \in A(p) \). From (1.7) we have

\[
\frac{l}{p - \gamma} \left[ \left( \frac{c_{\mu,v}^{\alpha} (a, c) f(z)}{z^{p-l}} \right)' - \gamma \right]
\]

\[
= \frac{l}{p - \gamma} \left[ p z^{p-l} + \sum_{k=1}^{\infty} \frac{(c)_k (p+l-\mu)_k (p+l-\lambda+\nu)_k (\alpha+p)_k (p+k) a_{p+k} z^{p+k-l}}{(a)_k (p+l)_k (p+l-\mu+\nu)_k k!} \right]
\]

\[
= l + \sum_{k=1}^{\infty} \frac{(c)_k (p+l-\mu)_k (p+l-\lambda+\nu)_k (\alpha+p)_k (p+k)}{(p-\gamma) (a)_k (p+l)_k (p+l-\mu+\nu)_k k!} a_{p+k} z^{k}
\]

\[
= \left[ l + \sum_{k=1}^{\infty} a_{p+k} z^k \right] \ast \left[ l + \sum_{k=1}^{\infty} \frac{(c)_k (p+l-\mu)_k (p+l-\lambda+\nu)_k (\alpha+p)_k (p+k)}{(p-\gamma) (a)_k (p+l)_k (p+l-\mu+\nu)_k k!} z^k \right]
\]

\[
= \frac{f(z)}{z^p} \ast \left[ \frac{X}{z^p} + z p \left( \frac{X}{z^p} \right)' \right]
\]

\[
= \frac{(f \ast X)(z)}{z^p} + \frac{z}{p} \left[ \left( \frac{f \ast X)(z)}{z^p} \right)' \right] < \frac{l+Az}{l+Bz}, \quad z \in \Delta,
\]

and therefore the left-hand side of (1.7) and of (2.1) are the same. \( \Box \)

If we put \( \alpha = c = 1, \lambda = \mu = 0 \) and \( a = p + l \) in theorem 2.1, we have:

**Corollary 2.1.** A function \( f \in A(p) \) belong to \( S_p(A, B, \gamma) \) (see [2]) if and only if

\[
l + \frac{l}{p - \gamma} \sum_{k=1}^{\infty} (p+k) a_{p+k} z^k < \frac{l+Az}{l+Bz}.
\]

If we put \( \alpha = A = c = 1, \lambda = \mu = 0, B = -1 \) and \( a = p + l \) in theorem 2.1, we have:

**Corollary 2.2.** A function \( f \in A(p) \) belong to \( S_p(\gamma) \) (see [15]) if and only if

\[
l + \frac{l}{p - \gamma} \sum_{k=1}^{\infty} (p+k) a_{p+k} z^k < \frac{l+z}{l+z}.
\]

If we put \( \alpha = c = 1, \lambda = \mu = \gamma = 0 \) and \( a = p + l \) in theorem 2.1, we have:
Corollary 2.3. A function $f \in A(p)$ belong to $S_p(A, B)$ (see [4]) if and only if 
\[
1 + \frac{1}{p} \sum_{k=1}^{\infty} (p + k)a_{p+k}z^k < \frac{1 + Az}{1 + Bz}.
\]

If we put $A = a = c = l, \lambda = \mu = 0, B = -l$ and $\alpha = n$ in theorem 2.1, we have:

Corollary 2.4. A function $f \in A(p)$ belong to $T_{n+p-l}(\gamma)$ (see [10]) if and only if 
\[
1 + \frac{1}{p - \gamma} \sum_{k=1}^{\infty} \frac{(p + k)(n + p)_k}{k!} a_{p+k}z^k < \frac{1 + z - Az}{1 + z - Bz}.
\]

If we put $a = c = l, \lambda = \mu = \gamma = 0, A \rightarrow -A, B \rightarrow -B$ and $\alpha = n$ in theorem 2.1, we have:

Corollary 2.5. A function $f \in A(p)$ belong to $V_{n,p}(A, B)$ (see [12]) if and only if 
\[
1 + \frac{1}{p - \gamma} \sum_{k=1}^{\infty} (p + k)(n + p)_k \frac{a_{p+k}z^k}{k!} < \frac{1 - Az}{1 - Bz}.
\]

If we put $a = c = l, \lambda = \mu = \gamma = 0, A \rightarrow -A, B \rightarrow -B$ and $\alpha = n$ in theorem 2.1, we have:

Corollary 2.6. A function $f \in A(p)$ belong to $V_{n,p}(A, B, \gamma)$ (see [1]) if and only if 
\[
1 + \frac{1}{p - \gamma} \sum_{k=1}^{\infty} (p + k)(n + p)_k \frac{a_{p+k}z^k}{k!} < \frac{1 - Az}{1 - Bz}.
\]

If we put $a = c = p = l, \lambda = \mu = \gamma = 0, A \rightarrow -A, B \rightarrow -B$ and $\alpha = p + l$ in theorem 2.1, we have:

Corollary 2.7. A function $f \in A(p)$ belong to $\mathcal{K}(A, B)$ (see [14]) if and only if 
\[
1 + \sum_{k=1}^{\infty} (l + k)a_{l+k}z^k < \frac{1 - Az}{1 - Bz}.
\]

Theorem 2.2. A function $f$ belongs to the class $S_{A,B}^{p,\gamma}(\alpha, \lambda, \mu, \nu, a, c)$ if and only if there exists a Schwarz function $w(z)$ such that 
\[
f(z) = \left[\sum_{k=1}^{\infty} \frac{l}{Q(k)} \left(\frac{k}{p + k}\right)z^{p+k}\right] \ast z^p \left[\int_z \left(\frac{\gamma + p - \gamma l + Aw(t)}{t} \frac{l + Bw(t)}{l + Bw(t)}\right) dt - \ln z^p\right]
\]

where
\[ Q(k) := \frac{(c)_k (p + l - \mu)_k (p + l - \lambda + \nu)_k (\alpha + p)_k}{(a)_k (p + l)_k (p + l - \mu + \nu)_k k!}. \]

**Proof.** Let \( f \in S_{A,B}^{p,\gamma} (\alpha, \lambda, \mu, \nu, a, c) \). Then from the definition 1.1, we have

\[
\frac{l}{p - \gamma} \left[ (L_{\mu,v}^{\lambda,p,\alpha} (a, c) f(z))' \right] - \gamma = \frac{l + A w(z)}{l + B w(z)}, \quad z \in \Delta,
\]

where \(|w(z)| < 1 \) in \( \Delta \) with \( w(0) = 0 \). Therefore

\[
(L_{\mu,v}^{\lambda,p,\alpha} (a, c) f(z))' = \gamma + \frac{(p - \gamma) l + A w(z)}{z} l + B w(z)
\]

\[
\Rightarrow \int \left( L_{\mu,v}^{\lambda,p,\alpha} (a, c) f(t) \right)' dt = \int \left[ \gamma + \frac{(p - \gamma) l + A w(t)}{t} l + B w(t) \right] dt
\]

\[
\Rightarrow L_{\mu,v}^{\lambda,p,\alpha} (a, c) f(z) = \frac{l}{z^p} + \ln z + \sum_{k=1}^{\infty} Q(k) \frac{p}{k} a_{p+k} z^k
\]

\[
= \int \left[ \gamma + \frac{(p - \gamma) l + A w(t)}{t} l + B w(t) \right] dt.
\]

Thus from (1.4) and (1.6) we obtain

\[
f(z) \sum_{k=1}^{\infty} Q(k) \left[ \frac{p+k}{k} \right] z^{p+k} = z^p \left[ \int \left[ \gamma + \frac{(p - \gamma) l + A w(t)}{t} l + B w(t) \right] dt - \ln z^p \right],
\]

and our assertion follows immediately. \( \blacksquare \)

**Remark 2.1.** By specializing the parameters \( A, B, \alpha, \lambda, \mu, \gamma, a, c, p \) and by using Theorem 2.2 and Corollaries 2.1-2.7, we get necessary and sufficient condition for functions belongs to classes were studied in [1,2,4,10,12,14,15].

**References**


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