Polyharmonic functions with negative coefficients

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Abstract

A $2p$ times continuously differentiable complex-valued mapping $F = u + iv$ in a domain $D \subset \mathbb{C}$ is polyharmonic if $F$ satisfies the polyharmonic equation $\Delta \cdots \Delta F = 0$, where $p \in \mathbb{N}^+$ and $\Delta$ represents the complex Laplacian operator. The main aim of this paper is to introduce a subclasses of polyharmonic mappings. Coefficient conditions, distortion bounds, extreme points, of the subclasses are obtained. ©2017 All rights reserved.

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1. Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unite disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $S$ denote the subclasses of $A$ consisting of functions which are univalent in $U$. A continuous mapping $f = u + iv$ is a complex-valued harmonic mapping in a domain $D \subset \mathbb{C}$ if both $u$ and $v$ are real harmonic in $D$, i.e., $\Delta u = \Delta v = 0$, where $\Delta$ is the complex Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$ 

In any simply connected domain $D \subset \mathbb{C}$ we can write $f = h + \bar{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ for all $z \in D$. See Clunie and Sheil-Small [2].

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Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{SH}$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$ 

Observe that $\mathcal{H}$ reduces to $S$, the class of normalized univalent analytic functions, if the co-analytic part of $f$ is zero. Denote by $\mathcal{HS}^*$ and $\mathcal{HC}$ the subclasses of $\mathcal{HS}$ consisting of functions $f$ that map $U$ onto starlike and convex domain, respectively.

In 1984 Clunie and Sheil-Small [2] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{H}$ and its subclasses such that Silverman [6], Silverman and Silvia [7], and Jahangiri [3] studied the harmonic univalent functions.

2. Preliminaries

A continuous complex-valued mapping $F$ in $D$ is biharmonic if the Laplacian of $F$ is harmonic, i.e., $F$ satisfies the equation $\Delta(\Delta F) = 0$. It can be shown that in a simply connected domain $D$, every biharmonic mapping has the representation

$$F(z) = G_1(z) + |z|^2 G_2(z), \quad (2.1)$$

where both $G_1$ and $G_2$ are harmonic in $D$.

More generally, a complex-valued mapping $F$ of a domain $D$ is called polyharmonic (or $p$-harmonic) if $F$ satisfies the equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ for $p \in \mathbb{N}^+$. In a simply connected domain, a mapping $F$ is polyharmonic if and only if $F$ has the following representation:

$$F(z) = H(z) + \overline{G(z)} = \sum_{k=1}^{p} |z|^{2(k-1)} J_{p-k+1}(z), \quad (2.2)$$

where $\Delta|J_{p-k+1}(z)| = 0$ and for each $J_{p-k+1} = h_{p-k+1} + \overline{g}_{p-k+1}$, $(k \in \{1, ..., p\})$ is harmonic in $D$, where

$$h_{p-k+1}(z) = \sum_{n=1}^{\infty} a_{n,p-k+1} z^n, \quad g_{p-k+1}(z) = \sum_{n=1}^{\infty} b_{n,p-k+1} z^n, \quad (a_{1,p} = 1, |b_{1,p}| < 1).$$

Denote by $\mathcal{H}_p^0 (b_{1,p} = 0, a_{1,p-k+1} = b_{1,p-k+1} = 0)$ the subclass of $\mathcal{H}_p$ the class of function $F$ of the form $(2.1)$ that are harmonic, univalent, and sense-preserving in the unit disk. Obviously, if $p = 1$ and $p = 2$, $F$ is harmonic and biharmonic, respectively. Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology.

In [5], Qiao and Wang introduced the class $\mathcal{HS}_p$ of polyharmonic mappings $F$ given by $(2.1)$ satisfying the condition

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n] (|a_{n,p-k+1}| + |b_{n,p-k+1}|) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|), \quad (2.3)$$

where $0 \leq |b_{1,1}| + \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|) < 1$, and the subclass $\mathcal{HC}_p$ of $\mathcal{HS}_p$, where

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n^2] (|a_{n,p-k+1}| + |b_{n,p-k+1}|) \leq 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|). \quad (2.4)$$
The classes of all mappings \( F \) in \( \mathcal{HS}_p \), which are of the form (2.1), and subject the conditions (2.3) and (2.4) are denoted by \( \mathcal{HS}_p^0, \mathcal{HC}_p^0 \), respectively.

Now we introduce new classes of polyharmonic mappings, denoted by \( \mathcal{HS}_p(\alpha) \) and \( \mathcal{HC}_p(\alpha) \) as follows:

Denote by \( \mathcal{HS}_p(\alpha) \) the class of all functions of the form (2.1) that satisfy the condition

\[
\frac{\partial}{\partial \theta}(\arg F(re^{i\theta})) \geq \alpha, \quad (0 \leq \alpha < 1, \ |z| = r < 1).
\] (2.5)

Also, denote by \( \mathcal{HC}_p(\alpha) \) the subclass of \( \mathcal{HS}_p(\alpha) \) such that the functions \( H \) and \( G \) in \( F = H + \overline{G} \) are of the form:

\[
H(z) = z - \sum_{n=2}^{\infty} |a_{n,1}|z^n - \sum_{k=2}^{p} \sum_{n=2}^{\infty} |z|^{2(k-1)}|a_{n,p-k+1}|z^n,
\]

\[
G(z) = \sum_{n=2}^{\infty} |b_{n,1}|z^n + \sum_{k=2}^{p} \sum_{n=1}^{\infty} |z|^{2(k-1)}|b_{n,p-k+1}|z^n.
\] (2.6)

3. Main results

**Theorem 3.1.** Let \( F \) be given by (2.1) and

\[
\sum_{k=1}^{p} \sum_{n=2}^{\infty} \left\{ \frac{2(k-1) + n - \alpha}{1 - \alpha} |a_{n,p-k+1}| + \frac{2(k-1) + n + \alpha}{1 - \alpha} |b_{n,p-k+1}| \right\} \leq 1 - \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| - \sum_{k=2}^{p} \left\{ \frac{2k - 1 - \alpha}{1 - \alpha} |a_{1,p-k+1}| + \frac{2k - 1 + \alpha}{1 - \alpha} |b_{1,p-k+1}| \right\},
\] (3.1)

where \(0 \leq \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| + \sum_{k=2}^{p} \left\{ \frac{2k - 1 - \alpha}{1 - \alpha} |a_{1,p-k+1}| + \frac{2k - 1 + \alpha}{1 - \alpha} |b_{1,p-k+1}| \right\} < 1\). Then \( F \) is univalent and sense preserving in \( U \) and \( F \in \mathcal{HS}_p(\alpha) \).

**Proof.** First, we note that \( F \) is locally univalent and sense-preserving in \( U \). This is because

\[
|H'(z)| > 1 - \sum_{k=2}^{p} (2k-1)|a_{1,p-k+1}|r^{2(k-1)} - \sum_{k=1}^{p} \sum_{n=2}^{\infty} (2k-1 + n)|a_{n,p-k+1}|r^{2(k-1)+n-1} > 1 - \sum_{k=2}^{p} (2k-1)|a_{1,p-k+1}| - \sum_{k=1}^{p} \sum_{n=2}^{\infty} (2k-1 + n)|a_{n,p-k+1}|
\]
To show that $F$ is univalent in $U$ we notice that for $|z_1| < 1$, and by (3.1), we have

\[
|F(z_1) - F(z_2)| \geq |H(z_1) - H(z_2)| - |G(z_1) - G(z_2)|
\]

\[
= \left| (z_1 - z_2) + \sum_{k=1}^{p} \sum_{n=2}^{\infty} a_{n,p-k+1} (z_1^n - z_2^n) \right| - \left| \sum_{k=1}^{p} \sum_{n=1}^{\infty} b_{n,p-k+1} (z_1^n - z_2^n) \right|
\]

\[
\geq |z_1 - z_2| \left\{ 1 - \sum_{n=2}^{\infty} a_{n,p} \frac{z_1^n - z_2^n}{z_1 - z_2} + \sum_{n=1}^{\infty} b_{n,p} \frac{z_1^n - z_2^n}{z_1 - z_2} \right\}
\]

\[
- \left| \sum_{k=2}^{p} \left( \sum_{n=1}^{\infty} a_{n,p-k+1} \frac{|z_1|^{2(k-1)} (z_1^n - z_2^n)}{z_1 - z_2} + \sum_{n=1}^{\infty} b_{n,p-k+1} \frac{|z_1|^{2(k-1)} (z_1^n - z_2^n)}{z_1 - z_2} \right) \right|
\]

\[
\geq |z_1 - z_2| \left\{ 1 - |b_{1,1}| - |z_2| \sum_{n=2}^{\infty} n(|a_{n,p}| + |b_{n,p}|) \right\}
\]

\[
- |z_2| \sum_{k=2}^{p} \sum_{n=1}^{\infty} (2(k-1) + n)(|a_{n,p-k+1}| + |b_{n,p-k+1}|) \right\}
\]

\[
\geq |z_1 - z_2| \left\{ (1 - |b_{1,1}| - |z_2| \sum_{n=2}^{\infty} \left\{ n - \frac{\alpha}{1 - \alpha} |a_{n,p}| + n + \frac{\alpha}{1 - \alpha} |b_{n,p}| \right\} \right\}
\]

\[
- |z_2| \sum_{k=2}^{p} \sum_{n=1}^{\infty} \left\{ (2(k-1) + n - \frac{\alpha}{1 - \alpha}) |a_{n,p-k+1}| + (2(k-1) + n + \frac{\alpha}{1 - \alpha}) |b_{n,p-k+1}| \right\}
\]

\[
\geq |z_1 - z_2| (1 - |b_{1,1}|) (1 - |z_2|) > 0.
\]

Consequently, $F$ is univalent in $U$.

Now we show that $F \in \mathcal{H}(\delta_p(\alpha))$. According to the condition (2.4) we only need to show that if (3.1) holds, then

\[
\frac{\partial}{\partial \theta} \left( \text{arg} F(re^{i\theta}) \right) = \mathcal{H} \left( \frac{\partial}{\partial \theta} \log F(re^{i\theta}) \right) = \partial \left( \frac{zH'(z) - zG'(z)}{H(z) + G(z)} \right) > \alpha,
\]

where $z = re^{i\theta}$, $0 \leq \theta < 2\pi$, $0 \leq r < 1$, and $0 \leq \alpha < 1$.

Using the fact that $\Re w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

\[
\left| A(z) + (1 - \alpha) B(z) \right| - \left| A(z) - (1 + \alpha) B(z) \right| \geq 0,
\]

where $B(z) = H(z) + G(z)$ and $zH'(z) - zG'(z)$. 

\[
\text{(3.2)}
\]
Substituting for $B(z)$ and $A(z)$ in (3.2),

\[
|A(z) - (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \\
= \left| (1 - \alpha)H(z) + zH'(z) + \frac{1}{1 - \alpha}G(z) - zG'(z) \right| - \left| (1 + \alpha)H(z) - zH'(z) + \frac{1}{1 + \alpha}G(z) + zG'(z) \right| \\
= \sum_{k=1}^{p} \sum_{n=1}^{\infty} [2(k-1) + n + 1 - \alpha]|a_{n,p-k+1}|z^{2(k-1)+n} \\
- \sum_{k=1}^{p} \sum_{n=1}^{\infty} [2(k-1) + n - 1 + \alpha]|b_{n,p-k+1}|z^{2(k-1)+n} \\
+ \sum_{k=1}^{p} \sum_{n=1}^{\infty} [2(k-1) + n - 1 - \alpha]|a_{n,p-k+1}|z^{2(k-1)+n} \\
- \sum_{k=1}^{p} \sum_{n=1}^{\infty} [2(k-1) + n + 1 + \alpha]|b_{n,p-k+1}|z^{2(k-1)+n} \\
\geq (2 - \alpha)|z| + \sum_{k=2}^{p} [2(k-1) + 2 - \alpha]|a_{1,p-k+1}| + \sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n + 1 - \alpha]|a_{n,p-k+1}|z^{2(k-1)+n} \\
- \alpha|b_{1,1}|z - \sum_{k=2}^{p} [2(k-1) + \alpha]|b_{1,p-k+1}|z^{2(k-1)+1} \\
- \sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n + 1 - \alpha]|b_{n,p-k+1}|z^{2(k-1)+n} \\
- \alpha|z| + \sum_{k=2}^{p} [2(k-1) - \alpha]|a_{1,p-k+1}| + \sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n - 1 - \alpha]|a_{n,p-k+1}|z^{2(k-1)+n} \\
- (2 + \alpha)|b_{1,1}|z - \sum_{k=2}^{p} [2(k-1) + 2 + \alpha]|b_{1,p-k+1}|z^{2(k-1)+1} \\
- \sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n + 1 + \alpha]|b_{n,p-k+1}|z^{2(k-1)+n} \\
\geq 2(1 - \alpha)|z| \left\{ 1 - \frac{1 + \alpha}{1 - \alpha}|b_{1,1}| - \sum_{k=2}^{p} \frac{2(k-1) - \alpha}{1 - \alpha}|a_{1,p-k+1}|z^{2k+n-3} \right\} \\
- \sum_{k=2}^{p} \frac{2(k-1) + \alpha}{1 - \alpha}|b_{1,p-k+1}|z^{2k+n-3} - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \frac{2(k-1) + n - \alpha}{1 - \alpha}|a_{n,p-k+1}|z^{2k+n-3} \\
- \sum_{k=1}^{p} \sum_{n=2}^{\infty} \frac{2(k-1) + n + \alpha}{1 - \alpha}|b_{n,p-k+1}|z^{2k+n-3} \right\} \\
\geq 2(1 - \alpha)|z| \left\{ 1 - \frac{1 + \alpha}{1 - \alpha}|b_{1,1}| - \sum_{k=2}^{p} \frac{2(k-1) - \alpha}{1 - \alpha}|a_{1,p-k+1}| - \sum_{k=2}^{p} \frac{2(k-1) + \alpha}{1 - \alpha}|b_{1,p-k+1}| \right\} - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \frac{2(k-1) + n - \alpha}{1 - \alpha}|a_{n,p-k+1}| \right\} - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \frac{2(k-1) + n + \alpha}{1 - \alpha}|b_{n,p-k+1}| \right\} \geq 0, \text{ by (3.1).}
The starlike polyharmonic mappings

\[ F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \left\{ \sum_{n=1}^{\infty} \left\{ \frac{1-\alpha}{2(k-1)+n-\alpha} x_{n,p-k+1} z^{n} + \frac{1-\alpha}{2(k-1)+n+\alpha} y_{n,p-k+1} \right\} \right\} , \quad (3.3) \]

where \( \sum_{k=1}^{p} \left\{ \sum_{n=1}^{\infty} \left\{ |x_{n,p-k+1}| + |y_{n,p-k+1}| \right\} \right\} = 1 \), show that the coefficient bound given by (3.1) is sharp. The functions of form (3.3) are in \( \mathcal{HS}_{p}(\alpha) \) because

\[ \sum_{k=1}^{p} \sum_{n=1}^{\infty} \left\{ \frac{2(k-1)+n-\alpha}{1-\alpha} |a_{n,p-k+1}| + \frac{2(k-1)+n+\alpha}{1-\alpha} |b_{n,p-k+1}| \right\} = 1 + \sum_{k=1}^{p} \left\{ \sum_{n=1}^{\infty} \left\{ |x_{n,p-k+1}| + |y_{n,p-k+1}| \right\} \right\} = 1 - \frac{1}{1-\alpha} |b_{1,1}| - \sum_{k=2}^{p} \left\{ \frac{2(k-1)-\alpha}{1-\alpha} |a_{1,p-k+1}| + \frac{2(k-1)+\alpha}{1-\alpha} |b_{1,p-k+1}| \right\} . \]

The restriction placed in Theorem 3.1 on the moduli of the coefficients of \( F = H + \mathcal{G} \) enables us to conclude for arbitrary rotation of the coefficients of \( F \) that the resulting functions would still be harmonic univalent and \( F \in \mathcal{HS}_{p}(\alpha) \).

Next, we discuss the geometric properties of mappings belonging to \( \mathcal{HS}_{p}(\alpha) \).

**Theorem 3.2.** Each mapping in \( \mathcal{HS}_{p}(\alpha) \) maps \( \mathbb{U} \) onto a starlike domain with respect to the origin.

**Proof.** Let \( r \in (0,1) \) be a fixed number and

\[ F_{r}(z) = z + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{p} r^{2(k-1)} a_{n,p-k+1} \right) z^{n} + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{p} r^{2(k-1)} b_{n,p-k+1} \right) z^{n} . \]

Obviously, \( F_{r} \) is a harmonic mapping. Since

\[ F_{r}(z) = \sum_{n=2}^{\infty} \left[ \sum_{k=1}^{p} r^{2(k-1)} a_{n,p-k+1} \right] z^{n} + \sum_{n=2}^{\infty} \left[ \sum_{k=1}^{p} r^{2(k-1)} b_{n,p-k+1} \right] z^{n} \]

\[ \leq \sum_{n=2}^{\infty} \sum_{k=1}^{p} \left( 2(k-1)+n \right) \left( |a_{n,p-k+1}| + |b_{n,p-k+1}| \right) \]

\[ \leq \sum_{n=2}^{\infty} \sum_{k=1}^{p} \left( \frac{2(k-1)+n-\alpha}{1-\alpha} |a_{n,p-k+1}| + \frac{2(k-1)+n+\alpha}{1-\alpha} |b_{n,p-k+1}| \right) \leq 1 , \]

it follows that \( F_{r} \in \mathcal{HS}_{p}(0) \). By (2.5), we know that \( F_{r} \) maps \( \mathbb{U} \) onto a starlike domain with respect to the origin for each \( r \in (0,1) \), we show that \( F \) is starlike with respect to the origin. \( \square \)

**Example 3.3.** Let \( F_{1}(z) = z + \frac{1}{2} z^{2} + \frac{1}{2} z^{2} \). Then \( F_{1} \in \mathcal{HS}_{1}(\frac{1}{2}) \) is a univalent, sense preserving polyharmonic mapping. In particular, \( F_{1} \) maps \( \mathbb{U} \) onto a starlike domain with respect to the origin (see Figure 1).

**Example 4.4.** Let \( F_{2}(z) = z + \frac{1}{3} z^{2} + \frac{2}{3} z^{2} \). Then \( F_{2} \in \mathcal{HS}_{1}(\frac{1}{3}) \) is a univalent, sense preserving polyharmonic mapping. In particular, \( F_{1} \) maps \( \mathbb{U} \) onto a starlike domain with respect to the origin (see Figure 1).
We next show that the condition (3.1) is also necessary for functions in $\mathcal{HT}_p(\alpha)$.

**Theorem 3.5.** Let $F = H + G$ with $H$ and $G$ are given by (2.6). Then $F \in \mathcal{HT}_p(\alpha)$ if and only if

$$
\sum_{k=1}^p \sum_{n=2}^{\infty} \left\{ \frac{2(k-1) + n - \alpha}{1 - \alpha} \right\} a_{n,p-k+1} + \left\{ \frac{2(k-1) + n + \alpha}{1 - \alpha} \right\} b_{n,p-k+1}
\leq 1 - \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| - \sum_{k=2}^p \left\{ \frac{2k-1 - \alpha}{1 - \alpha} |a_{n,p-k+1}| + \frac{2k-1 + \alpha}{1 - \alpha} |b_{n,p-k+1}| \right\},
$$

(3.4)

where $0 \leq \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| + \sum_{k=2}^p \left\{ \frac{2k-1 - \alpha}{1 - \alpha} |a_{n,p-k+1}| + \frac{2k-1 + \alpha}{1 - \alpha} |b_{n,p-k+1}| \right\} < 1$.

**Proof.** We first suppose that $F \in \mathcal{HT}_p(\alpha)$, then by (2.5) we have

$$\Re\left\{ \frac{zH'(z) - G'(z)}{H(z) + G(z)} \right\} - \alpha \geq 0.$$

The above condition must hold for all values of $z$, $|z| = r < 1$. Upon choosing the values of $z$ on the positive real axis where $0 \leq z = r < 1$ we must have

$$
\left( [(1 - \alpha) - ((1 + \alpha)|b_{1,p}|)] - \sum_{k=2}^p \left\{ (2k-1 - \alpha)|a_{n,p-k+1}| + (2k-1 + \alpha)|b_{n,p-k+1}| r^{2(k-1)} \right\} \right.
$$

$$
- \sum_{k=1}^p \sum_{n=2}^{\infty} \left\{ 2(k-1) + n - \alpha |a_{n,p-k+1}| + 2(k-1) + n + \alpha |b_{n,p-k+1}| r^{2k+n-3} \right\} r^{2k+n-3} \geq 0.
$$

(3.5)

If the condition (3.4) does not hold then the numerator in (3.5) is negative for $r$ sufficiently close to 1. Thus there exits a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (3.5) is negative. This contradicts the required condition for $F \in \mathcal{HT}_p(\alpha)$ and so the proof is complete. \qed

**Example 3.6.** Let $F_1(z) = z - \frac{1}{4} z^2 + \frac{5}{8} z^2$. Then $F_1 \in \mathcal{HT}_1\left(\frac{10}{11}\right)$ is a univalent, sense preserving polyharmonic mapping. In particular, $F_1$ maps $U$ onto a starlike domain with respect to the origin (see Figure 2).

**Example 3.7.** Let $F_2(z) = z - \frac{1}{10} z^2 + \frac{5}{10} z^2$. Then $F_2 \in \mathcal{HT}_1\left(\frac{5}{11}\right)$ is a univalent, sense preserving polyharmonic mapping. In particular, $F_1$ maps $U$ onto a starlike domain with respect to the origin (see Figure 2).
Theorem 3.8. \( F \in \mathcal{F}(\mathcal{J}_p(\alpha)) \) if and only if \( F \) can be expressed as

\[
F(z) = \sum_{k=1}^{p} \sum_{n=1}^{\infty} \left( X_{n,p-k+1} H_{n,p-k+1}(z) + Y_{n,p-k+1} G_{n,p-k+1}(z) \right),
\]

where

\[
H_{1,1}(z) = z, \quad H_{n,1}(z) = z - \frac{1 - \alpha}{n - \alpha} z^n \quad (n = 2, 3, \ldots),
\]

\[
H_{n,p-k+1}(z) = z - |z|^{2(k-1)} \frac{1 - \alpha}{2(k-2) + n - \alpha} z^n \quad (n = 1, 2, \ldots, 2 \leq k \leq p),
\]

\[
G_{1,1}(z) = z + \frac{1 - \alpha}{n + \alpha} z^{2\pi} \quad (n = 1, 2, \ldots),
\]

\[
G_{n,p-k+1}(z) = z + |z|^{2(k-1)} \frac{1 - \alpha}{2(k-2) + n + \alpha} z^{2\pi} \quad (n = 1, 2, \ldots, 2 \leq k \leq p),
\]

and

\[
\sum_{k=1}^{p} \sum_{n=1}^{\infty} (X_{n,p-k+1} + Y_{n,p-k+1}) = 1, \quad (X_{n,p-k+1} \geq 0, Y_{n,p-k+1} \geq 0).
\]

In particular, the extreme points of \( \mathcal{F}(\mathcal{J}_p(\alpha)) \) are \( \{H_{n,p-k+1}\} \) and \( \{G_{n,p-k+1}\} \).

Proof. Note that for \( F \) we may write

\[
F(z) = \sum_{k=1}^{p} \sum_{n=1}^{\infty} \left( X_{n,p-k+1} H_{n,p-k+1}(z) + Y_{n,p-k+1} G_{n,p-k+1}(z) \right)
\]

\[
= \sum_{n=1}^{\infty} X_{n,1} H_{n,1}(z) + Y_{n,1} G_{n,1}(z) + \sum_{k=2}^{p} \sum_{n=1}^{\infty} X_{n,p-k+1} H_{n,p-k+1}(z) + Y_{n,p-k+1}
\]

\[
= z - \sum_{k=2}^{p} \sum_{n=2}^{\infty} \left| z \right|^{2(k-1)} \frac{1 - \alpha}{2(k-1) + n - \alpha} X_{n,p-k+1} z^n
\]

\[+ \sum_{k=2}^{p} \sum_{n=1}^{\infty} \left| z \right|^{2(k-1)} \frac{1 - \alpha}{2(k-1) + n + \alpha} Y_{n,p-k+1} z^{2\pi} - \sum_{n=2}^{\infty} \frac{1 - \alpha}{n - \alpha} z^n + \sum_{n=1}^{\infty} \frac{1 - \alpha}{n + \alpha} z^{2\pi}.\]
Then, by Theorem 3.5 we have

\[
\sum_{k=1}^{p} \sum_{n=2}^{\infty} \left\{ \frac{1 - \alpha}{2(k-1)+n-\alpha} \left( 2(k-1) + n - \alpha X_{n,p-k+1} \right) + \frac{1 - \alpha}{2(k-1)+n+\alpha} \left( 2(k-1) + n + \alpha Y_{n,p-k+1} \right) \right\} \\
+ Y_{1,1} + \sum_{k=2}^{p} \left\{ \frac{2k-1 - \alpha}{2k-1-\alpha} \left( 2k-1 + \alpha Y_{1,p-k+1} \right) \right\} \\
\leq \sum_{k=1}^{p} \sum_{n=2}^{\infty} \left( X_{n,p-k+1} + Y_{n,p-k+1} \right) + \sum_{k=1}^{p} \left( X_{1,p-k+1} + Y_{1,p-k+1} \right) - Y_{1,1} \leq 1 - Y_{1,1} \leq 1,
\]

so \( F \in \mathcal{HT}_p(\alpha) \). Conversely, suppose that \( F \in \mathcal{HT}_p(\alpha) \). Then

\[
\sum_{k=1}^{p} \sum_{n=2}^{\infty} \left\{ \frac{2(k-1) + n - \alpha}{1-\alpha} |a_{n,p-k+1}| + \frac{2(k-1) + n + \alpha}{1-\alpha} |b_{n,p-k+1}| \right\} \\
\leq 1 - \frac{1 + \alpha}{1-\alpha} |b_{1,1}| - \sum_{k=2}^{p} \left\{ \frac{2k-1 - \alpha}{1-\alpha} |a_{n,p-k+1}| + \frac{2k-1 + \alpha}{1-\alpha} |b_{n,p-k+1}| \right\}.
\]

Setting

\[
X_{n,p-k+1} = \left( \frac{2(k-1) + n - \alpha}{1-\alpha} \right) |a_{n,p-k+1}| \quad (2 \leq k \leq p, \ n = 1, 2, \ldots), \\
X_{n,1} = \left( \frac{n - \alpha}{1-\alpha} \right) |a_{n,1}| \quad (n = 2, 3, \ldots), \\
Y_{n,p-k+1} = \left( \frac{2(k-1) + n + \alpha}{1-\alpha} \right) |b_{n,p-k+1}| \quad (1 \leq k \leq p, \ n = 1, 2, \ldots),
\]

and

\[
X_{1,1} = 1 - \sum_{k=1}^{p} \sum_{n=2}^{\infty} \left( X_{n,p-k+1} + Y_{n,p-k+1} \right) - \sum_{k=2}^{p} \left( X_{1,p-k+1} + Y_{1,p-k+1} \right) - Y_{1,1},
\]

we obtain

\[
F(z) = \sum_{k=1}^{p} \sum_{n=1}^{\infty} (X_{n,p-k+1}H_{n,p-k+1}(z) + Y_{n,p-k+1}G_{n,p-k+1}(z)),
\]

as required. \( \square \)

Finally, we give the distortion bounds for functions in \( \mathcal{HT}_p(\alpha) \), which yields a covering result for \( \mathcal{HT}_p(\alpha) \).

**Theorem 3.9.** If \( F \in \mathcal{HT}_p(\alpha) \), then

\[
|f(z)| \leq \left( 1 + |b_{1,1}| \right) r + \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_{1,1}| \right) r^2, \quad |z| = r < 1,
\]

and

\[
|f(z)| \geq \left( 1 - |b_{1,1}| \right) r - \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_{1,1}| \right) r^2, \quad |z| = r < 1.
\]
Proof. We only prove the first inequality. The argument for second inequality is similar and will be omitted. Let \( F \in \mathcal{HC}_p(\alpha) \). Taking the absolute value of \( F \), we obtain

\[
|F(z)| \leq (1 + |b_{1,1}|)|z| + \left( \sum_{k=1}^{p} \sum_{n=2}^{\infty} (|a_{n,p-k+1}| + |b_{n,p-k+1}|) \right) |z|^n
\]

\[
\leq (1 + |b_{1,1}|)r + \left( \sum_{k=1}^{p} \sum_{n=2}^{\infty} |a_{n,p-k+1}| + |b_{n,p-k+1}| \right) r^{\alpha}
\]

\[
= (1 + |b_{1,1}|)r + \frac{1 - \alpha}{2 - \alpha} \left( \sum_{k=1}^{p} \sum_{n=2}^{\infty} \left( \frac{1 - \alpha}{2 - \alpha} |a_{n,p-k+1}| + \frac{1 - \alpha}{2 - \alpha} |b_{n,p-k+1}| \right) \right)^2
\]

\[
\leq (1 + |b_{1,1}|)r + \frac{1 - \alpha}{2 - \alpha} \left( \frac{2(k-1) + n - \alpha}{1 - \alpha} |a_{n,p-k+1}| + \frac{2(k-1) + n + \alpha}{1 - \alpha} |b_{n,p-k+1}| \right) r^{\alpha}
\]

\[
\leq (1 + |b_{1,1}|)r + \frac{1 - \alpha}{2 - \alpha} \left( 1 - \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| \right) r^{\alpha} \quad \text{(by (3.4))}
\]

\[
= (1 + |b_{1,1}|)r + \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_{1,1}| \right) r^{\alpha}.
\]

The bounds given in Theorem 3.5 for the functions \( F = H + \mathcal{G} \) of the form (2.6) also hold for functions of the form (2.2) if the coefficient condition (3.1) is satisfied. The functions \( F \) given by

\[
F(z) = z + |b_{1,1}|z + \left( \frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_{1,1}| \right) z^2
\]

for \(|b_{1,1}| \leq (1 - \alpha)/(1 + \alpha) \) show that the bounds given in Theorem 3.5 are sharp. \( \square \)

The following covering result follows from the second inequality in Theorem 3.5.

**Corollary 3.10.** If \( F \in \mathcal{HC}_p(\alpha) \), then

\[
\left\{ w : |w| < \frac{1}{2 - \alpha} (1 - |b_{1,1}|) \left[ 1 + (2\alpha - 1)|b_{1,1}| \right] \right\} \subset F(U).
\]

The corresponding definition for polyharmonic convex function of order \( \alpha \) leads to the following corollary.

**Corollary 3.11.** Let \( F \) be given by (2.1) and

\[
\sum_{k=1}^{p} \sum_{n=2}^{\infty} \left\{ \frac{2(k-1) + n(\alpha - n)}{1 - \alpha} |a_{n,p-k+1}| + \frac{2(k-1) + n(\alpha + n)}{1 - \alpha} |b_{n,p-k+1}| \right\}
\]

\[
\leq 1 - \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| - \sum_{k=2}^{p} \left\{ \frac{2k - 1 - \alpha}{1 - \alpha} |a_{1,p-k+1}| + \frac{2k - 1 + \alpha}{1 - \alpha} |b_{1,p-k+1}| \right\},
\]

where \( 0 \leq \frac{1 + \alpha}{1 - \alpha} |b_{1,1}| + \sum_{k=2}^{p} \left\{ \frac{2k-1-\alpha}{1-\alpha} |a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha} |b_{1,p-k+1}| \right\} < 1 \). Then \( F \) is univalent and sense preserving in \( U \) and \( F \in \mathcal{HC}_p(\alpha) \).
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References