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Polyharmonic functions with negative coefficients

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Abstract

A 2p times continuously differentiable complex-valued mapping $F = u + iv$ in a domain $\mathcal{D} \subset \mathbb{C}$ is polyharmonic if F satisfies the polyharmonic equation $\underbrace{\Delta \cdots \Delta}_{p} F = 0$, where $p \in \mathbb{N}^{+}$ and Δ represents the complex Laplacian operator. The main aim of this paper is to introduce a subclasses of polyharmonic mappings. Coefficient conditions, distortion bounds, extreme points, of the subclasses are obtained.

Keywords: Univalent functions, polyharmonic mappings, extreme points.
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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unite disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $\mathcal{S}$ denote the subclasses of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{U}$. A continuous mapping $f = u + iv$ is a *complex-valued* harmonic mapping in a domain $\mathcal{D} \subset \mathbb{C}$ if both u and v are real harmonic in $\mathcal{D}$, i.e., $\Delta u = \Delta v = 0$, where Δ is the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In any simply connected domain $\mathcal{D} \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ for all $z \in \mathcal{D}$. See Clunie and Sheil-Small [2].

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Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1.$$

Observe that $\mathcal{H}$ reduces to $\mathcal{S}$, the class of normalized univalent analytic functions, if the co-analytic part of f is zero. Denote by $\mathcal{HS}^*$ and $\mathcal{HC}$ the subclasses of $\mathcal{HS}$ consisting of functions f that map $\mathbb{U}$ onto starlike and convex domain, respectively.

In 1984 Clunie and Sheil-Small [2] investigated the class $\mathcal{H}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $\mathcal{H}$ and its subclasses such that Silverman [6], Silverman and Silvia [7], and Jahangiri [3] studied the harmonic univalent functions.

2. Preliminaries

A continuous complex-valued mapping F in $\mathcal{D}$ is *biharmonic* if the Laplacian of F is harmonic, i.e., F satisfies the equation $\Delta(\Delta F) = 0$. It can be shown that in a simply connected domain $\mathcal{D}$, every biharmonic mapping has the representation

$$F(z) = G_1(z) + |z|^2 G_2(z), \tag{2.1}$$

where both G_1 and G_2 are harmonic in $\mathcal{D}$.

More generally, a complex-valued mapping F of a domain $\mathcal{D}$ is called *polyharmonic* (or *p-harmonic*) if F satisfies the equation $\Delta^p F = \Delta(\Delta^{p-1} F) = 0$ for $p \in \mathbb{N}^+$.

In a simply connected domain, a mapping F is polyharmonic if and only if F has the following representation:

$$F(z) = H(z) + \overline{G(z)} = \sum_{k=1}^{p} |z|^{2(k-1)} J_{p-k+1}(z), \tag{2.2}$$

where $\Delta J_{p-k+1}(z) = 0$ and for each $J_{p-k+1} = h_{p-k+1} + \overline{g}_{p-k+1}$, $(k \in \{1, \ldots, p\})$ is harmonic in $\mathcal{D}$, where

$$h_{p-k+1}(z) = \sum_{n=1}^{\infty} a_{n,p-k+1} z^n, \qquad g_{p-k+1}(z) = \sum_{n=1}^{\infty} b_{n,p-k+1} z^n, \qquad (a_{1,p} = 1, |b_{1,p}| < 1).$$

Denote by $\mathcal{H}_p^0$ $(b_{1,p} = 0, a_{1,p-k+1} = b_{1,p-k+1} = 0)$ the subclass of $\mathcal{H}_p$ the class of function F of the form (2.1) that are harmonic, univalent, and sense-preserving in the unit disk. Obviously, if $p = 1$ and $p = 2$, F is harmonic and biharmonic, respectively. Biharmonic mappings arise in a lot of physical situations, particularly in fluid dynamics and elasticity problems, and have many important applications in engineering and biology.

In [5], Qiao and Wang introduced the class $\mathcal{HS}_p$ of polyharmonic mappings F given by (2.1) satisfying the condition

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n](|a_{n,p-k+1}| + |b_{n,p-k+1}|) \leqslant 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|), \tag{2.3}$$

where $0 \leqslant |b_{1,1}| + \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|) < 1$, and the subclass $\mathcal{HC}_p$ of $\mathcal{HS}_p$, where

$$\sum_{k=1}^{p} \sum_{n=2}^{\infty} [2(k-1) + n^2](|a_{n,p-k+1}| + |b_{n,p-k+1}|) \leqslant 1 - |b_{1,1}| - \sum_{k=2}^{p} (2k-1)(|a_{1,p-k+1}| + |b_{1,p-k+1}|). \tag{2.4}$$

The classes of all mappings F in $\mathcal{HS}_p^0$ which are of the form (2.1), and subject the conditions (2.3) and (2.4) are denoted by $\mathcal{HS}_p^0, \mathcal{HC}_p^0$, respectively.

Now we introduce new classes of polyharmonic mappings, denoted by $\mathcal{HS}_p(\alpha)$ and $\mathcal{HC}_p(\alpha)$ as follows:

Denote by $\mathcal{HS}_p(\alpha)$ the class of all functions of the form (2.1) that satisfy the condition

$$\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big) \geqslant \alpha, \qquad (0 \leqslant \alpha < 1,\ |z| = r < 1). \tag{2.5}$$

Also, denote by $\mathcal{HT}_p(\alpha)$ the subclass of $\mathcal{HS}_p(\alpha)$ such that the functions H and G in $F = H + \overline{G}$ are of the form:

$$\begin{aligned} H(z) &= z - \sum_{n=2}^{\infty} |a_{n,1}|z^n - \sum_{k=2}^{p}\sum_{n=2}^{\infty} |z|^{2(k-1)}|a_{n,p-k+1}|z^n, \\ G(z) &= \sum_{n=2}^{\infty} |b_{n,1}|\overline{z^n} + \sum_{k=2}^{p}\sum_{n=1}^{\infty} |z|^{2(k-1)}|b_{n,p-k+1}|\overline{z^n}. \end{aligned} \tag{2.6}$$

Also denote by $\mathcal{HC}_p(\alpha)$ the class of all functions of the form (2.1) that satisfy the condition

$$\frac{\partial}{\partial\theta}\Big[\arg\big(\frac{\partial}{\partial\theta}F(re^{i\theta})\big)\Big] \geqslant \alpha, \qquad (0 \leqslant \alpha < 1,\ |z| = r < 1).$$

Note that:

(i) if $\alpha = 0$, then the classes $\mathcal{HS}_p(\alpha)$ and $\mathcal{HC}_p(\alpha)$ reduce to the classes $\mathcal{HS}_p$ and $\mathcal{HC}_p$ introduced and studied by Qiao and Wang [5];

(ii) if $p = 1$, then the classes $\mathcal{HS}_p(\alpha)$ and $\mathcal{HC}_p(\alpha)$ reduce to the classes $\mathcal{HS}(\alpha)$ and $\mathcal{HC}(\alpha)$ studied by Jahangiri [4];

(iii) if $\alpha = 0$, $p = 1$, then the classes $\mathcal{HS}_p(\alpha)$ and $\mathcal{HC}_p(\alpha)$ reduce to the classes $\mathcal{HS}$ and $\mathcal{HC}$ studied by Avci and Zlotiewicz [1].

3. Main results

Theorem 3.1. *Let F be given by (2.1) and*

$$\begin{aligned} &\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{\left[\frac{2(k-1)+n-\alpha}{1-\alpha}\right]|a_{n,p-k+1}| + \left[\frac{2(k-1)+n+\alpha}{1-\alpha}\right]|b_{n,p-k+1}|\right\} \\ &\qquad \leqslant 1 - \frac{1+\alpha}{1-\alpha}|b_{1,1}| - \sum_{k=2}^{p}\left\{\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|\right\}, \end{aligned} \tag{3.1}$$

where $0 \leqslant \frac{1+\alpha}{1-\alpha}|b_{1,1}| + \sum_{k=2}^{p}\left\{\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|\right\} < 1$. Then F is univalent and sense preserving in $\mathbb{U}$ and $F \in \mathcal{HS}_p(\alpha)$.

Proof. First, we note that F is locally univalent and sense-preserving in $\mathbb{U}$. This is because

$$\begin{aligned} |H'(z)| &> 1 - \sum_{k=2}^{p}((2k-1)|a_{1,p-k+1}|r^{2(k-1)} - \sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n]|a_{n,p-k+1}|r^{2(k-1)+n-1} \\ &> 1 - \sum_{k=2}^{p}(2k-1)|a_{1,p-k+1}| - \sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n]|a_{n,p-k+1}| \end{aligned}$$

$$\geqslant 1-\sum_{k=2}^{p}\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}|-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\frac{2(k-1)+n+1-\alpha}{1-\alpha}|a_{n,p-k+1}|$$
$$\geqslant \sum_{k=1}^{p}\sum_{n=1}^{\infty}\Big[\frac{2(k-1)+n+\alpha}{1-\alpha}\Big]|b_{n,p-k+1}|$$
$$\geqslant \sum_{k=1}^{p}\sum_{n=1}^{\infty}[2(k-1)+n]|b_{n,p-k+1}|$$
$$> \sum_{k=1}^{p}\sum_{n=1}^{\infty}2(k-1)+n|b_{n,p-k+1}|r^{2(k-1)+n-1}\geqslant |G'(z)|.$$

To show that F is univalent in $\mathbb{U}$ we notice that for $|z_1|\leqslant|z_2|<1$, and by (3.1), we have

$$\begin{aligned}
|F(z_1)-F(z_2)| &\geqslant |H(z_1)-H(z_2)|-|\overline{G(z_1)-G(z_2)}|\\
&=\left|(z_1-z_2)+\sum_{k=1}^{p}\sum_{n=2}^{\infty}a_{n,p-k+1}(z_1^n-z_2^n)\right|-\left|\sum_{k=1}^{p}\sum_{n=1}^{\infty}\overline{b_{n,p-k+1}(z_1^n-z_2^n)}\right|\\
&\geqslant |z_1-z_2|\Bigg\{1-\Bigg|\sum_{n=2}^{\infty}a_{n,p}\frac{z_1^n-z_2^n}{z_1-z_2}+\sum_{n=1}^{\infty}\overline{b_{n,p}\frac{z_1^n-z_2^n}{z_1-z_2}}\Bigg|\\
&\quad -\Bigg|\sum_{k=2}^{p}\Bigg(\sum_{n=1}^{\infty}a_{n,p-k+1}\frac{|z_1|^{2(k-1)}(z_1^n-z_2^n)}{z_1-z_2}+\sum_{n=1}^{\infty}\overline{b_{n,p-k+1}}\frac{|z_1|^{2(k-1)}(\overline{z_1^n}-\overline{z_2^n})}{z_1-z_2}\Bigg)\Bigg|\Bigg\}\\
&\geqslant |z_1-z_2|\Bigg(1-|b_{1,1}|-|z_2|\sum_{n=2}^{\infty}n(|a_{n,p}|+|b_{n,p}|)\\
&\quad -|z_2|\sum_{k=2}^{p}\sum_{n=1}^{\infty}(2(k-1)+n)(|a_{n,p-k+1}|+|b_{n,p-k+1}|)\Bigg)\\
&\geqslant |z_1-z_2|\Bigg(1-|b_{1,1}|-|z_2|\sum_{n=2}^{\infty}\Big\{\frac{n-\alpha}{1-\alpha}|a_{n,p}|+\frac{n+\alpha}{1-\alpha}|b_{n,p}|\Big\}\\
&\quad -|z_2|\sum_{k=2}^{p}\sum_{n=1}^{\infty}\Big\{\Big(\frac{2(k-1)+n-\alpha}{1-\alpha}\Big)|a_{n,p-k+1}|+\Big(\frac{2(k-1)+n+\alpha}{1-\alpha}\Big)|b_{n,p-k+1}|\Big\}\Bigg)\\
&\geqslant |z_1-z_2|(1-|b_{1,1}|)(1-|z_2|)>0.
\end{aligned}$$

Consequently, F is univalent in $\mathbb{U}$.

Now we show that $F\in\mathcal{HS}_p(\alpha)$. According to the condition (2.4) we only need to show that if (3.1) holds, then

$$\frac{\partial}{\partial\theta}\big(\arg F(re^{i\theta})\big)=\Im\Big(\frac{\partial}{\partial\theta}\log F(re^{i\theta})\Big)=\Re\Big(\frac{zH'(z)-\overline{zG'(z)}}{H(z)+\overline{G(z)}}\Big)\geqslant\alpha,$$

where $z=re^{i\theta}$, $0\leqslant\theta<2\pi$, $0\leqslant r<1$, and $0\leqslant\alpha<1$.

Using the fact that $\Re w\geqslant\alpha$ if and only if $|1-\alpha+w|\geqslant|1+\alpha-w|$, it suffices to show that

$$\Big|A(z)+(1-\alpha)B(z)\Big|-\Big|A(z)-(1+\alpha)B(z)\Big|\geqslant 0, \tag{3.2}$$

where $B(z)=H(z)+\overline{G(z)}$ and $zH'(z)-\overline{zG'(z)}$.

Substituting for $B(z)$ and $A(z)$ in (3.2),

$$
\begin{aligned}
&\left|A(z)+(1-\alpha)B(z)\right|-\left|A(z)-(1+\alpha)B(z)\right|\\
&\quad=\left|(1-\alpha)H(z)+zH'(z)+\overline{(1-\alpha)G(z)-zG'(z)}\right|\\
&\qquad-\left|(1+\alpha)H(z)-zH'(z)+\overline{(1+\alpha)G(z)+zG'(z)}\right|\\
&\quad\geqslant\sum_{k=1}^{p}\sum_{n=1}^{\infty}[2(k-1)+n+1-\alpha]|a_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad-\sum_{k=1}^{p}\sum_{n=1}^{\infty}[2(k-1)+n-1+\alpha]|b_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad+\sum_{k=1}^{p}\sum_{n=1}^{\infty}[2(k-1)+n-1-\alpha]|a_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad-\sum_{k=1}^{p}\sum_{n=1}^{\infty}[2(k-1)+n+1+\alpha]|b_{n,p-k+1}||z|^{2(k-1)+n}\\
&\quad\geqslant(2-\alpha)|z|+\sum_{k=2}^{p}[2(k-1)+2-\alpha]|a_{1,p-k+1}|+\sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n+1-\alpha]|a_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad-\alpha|b_{1,1}||z|-\sum_{k=2}^{p}[2(k-1)+\alpha]|b_{1,p-k+1}||z|^{2(k-1)+1}\\
&\qquad-\sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n-1+\alpha]|b_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad-\alpha|z|+\sum_{k=2}^{p}[2(k-1)-\alpha]|a_{1,p-k+1}|+\sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n-1-\alpha]|a_{n,p-k+1}||z|^{2(k-1)+n}\\
&\qquad-(2+\alpha)|b_{1,1}||z|-\sum_{k=2}^{p}[2(k-1)+2+\alpha]|b_{1,p-k+1}||z|^{2(k-1)+1}\\
&\qquad-\sum_{k=1}^{p}\sum_{n=2}^{\infty}[2(k-1)+n+1+\alpha]|b_{n,p-k+1}||z|^{2(k-1)+n}\\
&\quad\geqslant 2(1-\alpha)|z|\left\{1-\frac{1+\alpha}{1-\alpha}|b_{1,1}|-\sum_{k=2}^{p}\frac{(2k-1-\alpha)}{1-\alpha}|a_{1,p-k+1}||z|^{2k+n-3}\right.\\
&\qquad-\sum_{k=2}^{p}\frac{(2k-1+\alpha)}{1-\alpha}|b_{1,p-k+1}||z|^{2k+n-3}-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\frac{[2(k-1)+n-\alpha]}{1-\alpha}|a_{n,p-k+1}||z|^{2k+n-3}\\
&\qquad\left.-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\frac{[2(k-1)+n+\alpha]}{1-\alpha}|b_{n,p-k+1}||z|^{2k+n-3}\right\}\\
&\quad\geqslant 2(1-\alpha)|z|\left\{1-\frac{1+\alpha}{1-\alpha}|b_{1,1}|-\sum_{k=2}^{p}\frac{(2k-1-\alpha)}{1-\alpha}|a_{1,p-k+1}|-\sum_{k=2}^{p}\frac{(2k-1+\alpha)}{1-\alpha}|b_{1,p-k+1}|\right.\\
&\qquad\left.-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\frac{[2(k-1)+n-\alpha]}{1-\alpha}|a_{n,p-k+1}|-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\frac{[2(k-1)+n+\alpha]}{1-\alpha}|b_{n,p-k+1}|\right\}\geqslant 0,\ \text{by (3.1)}.
\end{aligned}
$$

The starlike polyharmonic mappings

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}\Big\{\sum_{n=1}^{\infty}\Big\{\frac{1-\alpha}{[2(k-1)+n-\alpha]}x_{n,p-k+1}z^{n}+\frac{1-\alpha}{[2(k-1)+n+\alpha]}\overline{y_{n,p-k+1}z^{n}}\Big\}\Big\}, \tag{3.3}$$

where $\sum_{k=1}^{p}\Big\{\sum_{n=1}^{\infty}\Big\{|x_{n,p-k+1}|+|y_{n,p-k+1}|\Big\}\Big\}=1$, show that the coefficient bound given by (3.1) is sharp. The functions of form (3.3) are in $\mathcal{HS}_p(\alpha)$ because

$$\begin{aligned}&\sum_{k=1}^{p}\sum_{n=1}^{\infty}\left\{\Big[\frac{2(k-1)+n-\alpha}{1-\alpha}\Big]|a_{n,p-k+1}|+\Big[\frac{2(k-1)+n+\alpha}{1-\alpha}\Big]|b_{n,p-k+1}|\right\}\\&=1+\sum_{k=1}^{p}\Big\{\sum_{n=1}^{\infty}\Big\{|x_{n,p-k+1}|+|y_{n,p-k+1}|\Big\}\Big\}\\&=1-\frac{1+\alpha}{1-\alpha}|b_{1,1}|-\sum_{k=2}^{p}\Big\{\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}|+\frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|\Big\}.\end{aligned}$$

□

The restriction placed in Theorem 3.1 on the moduli of the coefficients of $F=H+\overline{G}$ enables us to conclude for arbitrary rotation of the coefficients of F that the resulting functions would still be harmonic univalent and $F\in\mathcal{HS}_p(\alpha)$

Next, we discuss the geometric properties of mappings belonging to $\mathcal{HS}_p(\alpha)$.

Theorem 3.2. *Each mapping in $\mathcal{HS}_p(\alpha)$ maps $\mathbb{U}$ onto a starlike domain with respect to the origin.*

Proof. Let $r\in(0,1)$ be a fixed number and

$$F_r(z)=z+\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{p}r^{2(k-1)}a_{n,p-k+1}\Big)z^{n}+\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{p}r^{2(k-1)}\overline{b_{n,p-k+1}}\Big)\overline{z^{n}}.$$

Obviously, F_r is a harmonic mapping. Since

$$\begin{aligned}F_r(z)&=\sum_{n=2}^{\infty}n\Big|\sum_{k=1}^{p}r^{2(k-1)}a_{n,p-k+1}\Big|+\sum_{n=2}^{\infty}n\Big|\sum_{k=1}^{p}r^{2(k-1)}b_{n,p-k+1}\Big|\\&\leqslant\sum_{n=2}^{\infty}\sum_{k=1}^{p}(2(k-1)+n)\Big(|a_{n,p-k+1}|+|b_{n,p-k+1}|\Big)\\&\leqslant\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left(\frac{2(k-1)+n-\alpha}{1-\alpha}|a_{n,p-k+1}|+\frac{2(k-1)+n-\alpha}{1-\alpha}|b_{n,p-k+1}|\right)\leqslant 1,\end{aligned}$$

it follows that $F_r\in\mathcal{HS}_p(0)$. By (2.5), we know that F_r maps $\mathbb{U}$ onto a starlike domain with respect to the origin for each $r\in(0,1)$, we show that F is starlike with respect to the origin. □

Example 3.3. Let $F_1(z)=z+\frac{1}{4}z^2+\frac{1}{8}\overline{z^2}$. Then $F_1\in\mathcal{HS}_1(\frac{2}{7})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_1 maps $\mathbb{U}$ onto a starlike domain with respect to the origin (see Figure 1).

Example 3.4. Let $F_2(z)=z+\frac{1}{103}z^2+\frac{50}{103}\overline{z^2}$. Then $F_2\in\mathcal{HS}_1(\frac{1}{52})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_1 maps $\mathbb{U}$ onto a starlike domain with respect to the origin (see Figure 1).

Figure 1: The images of $\mathbb{U}$ under the mappings $F_1(z) = z + \frac{1}{4}z^2 + \frac{1}{8}\overline{z^2}$ (left) and $F_2(z) = z + \frac{1}{103}z^2 + \frac{50}{103}\overline{z^2}$ (right).

We next show that the condition (3.1) is also necessary for functions in $\mathcal{HT}_p(\alpha)$.

Theorem 3.5. *Let* $F = H + \overline{G}$ *with* H *and* G *are given by* (2.6). *Then* $F \in \mathcal{HT}_p(\alpha)$ *if and only if*

$$\begin{aligned}\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{\left[\frac{2(k-1)+n-\alpha}{1-\alpha}\right]|a_{n,p-k+1}| + \left[\frac{2(k-1)+n+\alpha}{1-\alpha}\right]|b_{n,p-k+1}|\right\}\\ \leqslant 1 - \frac{1+\alpha}{1-\alpha}|b_{1,1}| - \sum_{k=2}^{p}\left\{\frac{2k-1-\alpha}{1-\alpha}|a_{n,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{n,p-k+1}|\right\},\end{aligned} \tag{3.4}$$

where $0 \leqslant \frac{1+\alpha}{1-\alpha}|b_{1,1}| + \sum\limits_{k=2}^{p}\left\{\frac{2k-1-\alpha}{1-\alpha}|a_{n,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{n,p-k+1}|\right\} < 1.$

Proof. We first suppose that $F \in \mathcal{HT}_p(\alpha)$, then by (2.5) we have

$$\Re\left\{\frac{zH'(z) - \overline{G'(z)}}{H(z) + G(z)}\right\} - \alpha \geqslant 0.$$

The above condition must hold for all values of z, $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leqslant z = r < 1$ we must have

$$\begin{aligned}&\Bigg(\left[(1-\alpha) - ((1+\alpha)|b_{1,p}|)\right] - \sum_{k=2}^{p}\left\{(2k-1-\alpha)|a_{n,p-k+1}| + (2k-1+\alpha)|b_{n,p-k+1}|r^{2(k-1)}\right\}\\ &\quad - \sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{[2(k-1)+n-\alpha]|a_{n,p-k+1}| + [2(k-1)+n+\alpha]|b_{n,p-k+1}|r^{2k+n-3}\right\}r^{2k+n-3}\Bigg)\\ &\quad \Big/ \left(1 - \sum_{k=2}^{p}\left\{|a_{1,p-k+1}| + |b_{1,p-k+1}|r^{2(k-1)}\right\} - \sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{|a_{n,p-k+1}| + |b_{n,p-k+1}|\right\}r^{2k+n-3}\right) \geqslant 0.\end{aligned} \tag{3.5}$$

If the condition (3.4) does not hold then the numerator in (3.5) is negative for r sufficiently close to 1. Thus there exits a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (3.5) is negative. This contradicts the required condition for $F \in \mathcal{HT}_p(\alpha)$ and so the proof is complete. □

Example 3.6. Let $F_1(z) = z - \frac{1}{4}z^2 + \frac{1}{8}\overline{z^2}$. Then $F_1 \in \mathcal{HT}_1(\frac{10}{11})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_1 maps $\mathbb{U}$ onto a starlike domain with respect to the origin (see Figure 2).

Example 3.7. Let $F_2(z) = z - \frac{1}{103}z^2 + \frac{50}{103}\overline{z^2}$. Then $F_2 \in \mathcal{HT}_1(\frac{5}{151})$ is a univalent, sense preserving polyharmonic mapping. In particular, F_1 maps $\mathbb{U}$ onto a starlike domain with respect to the origin (see Figure 2).

Figure 2: The images of $\mathbb{U}$ under the mappings $F_1(z) = z - \frac{1}{4}z^2 + \frac{1}{8}\overline{z^2}$ (left) and $F_2(z) = z - \frac{1}{103}z^2 + \frac{50}{103}\overline{z^2}$ (right).

Next, we shall obtain distortion bounds for functions in $\mathcal{HT}_p(\alpha)$ and also provide extreme points for the class $\mathcal{HT}_p(\alpha)$.

Theorem 3.8. $F \in \mathcal{HT}_p(\alpha)$ *if and only if* F *can be expressed as*

$$F(z) = \sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(X_{n,p-k+1}H_{n,p-k+1}(z) + Y_{n,p-k+1}G_{n,p-k+1}(z)\big),$$

where

$$\begin{aligned} H_{1,1}(z) &= z,\ H_{n,1}(z) = z - \frac{1-\alpha}{n-\alpha}z^n \quad (n = 2, 3, \ldots),\\ H_{n,p-k+1}(z) &= z - |z|^{2(k-1)}\frac{1-\alpha}{2(k-2)+n-\alpha}z^n \quad (n = 1, 2, \ldots, 2 \leqslant k \leqslant p),\\ G_{n,1}(z) &= z + \frac{1-\alpha}{n+\alpha}\overline{z^n} \quad (n = 1, 2, \ldots),\\ G_{n,p-k+1}(z) &= z + |z|^{2(k-1)}\frac{1-\alpha}{2(k-2)+n+\alpha}\overline{z^n} \quad (n = 1, 2, \ldots, 2 \leqslant k \leqslant p), \end{aligned}$$

and

$$\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(X_{n,p-k+1} + Y_{n,p-k+1}\big) = 1,\ (X_{n,p-k+1} \geqslant 0,\ Y_{n,p-k+1} \geqslant 0).$$

In particular, the extreme points of $\mathcal{HT}_p(\alpha)$ are $\{H_{n,p-k+1}\}$ and $\{G_{n,p-k+1}\}$.

Proof. Note that for F we may write

$$\begin{aligned} F(z) &= \sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(X_{n,p-k+1}H_{n,p-k+1}(z) + Y_{n,p-k+1}G_{n,p-k+1}(z)\big)\\ &= \sum_{n=1}^{\infty}X_{n,1}H_{n,1}(z) + Y_{n,1}G_{n,1}(z) + \sum_{k=2}^{p}\sum_{n=1}^{\infty}X_{n,p-k+1}H_{n,p-k+1}(z) + Y_{n,p-k+1}\\ &= z - \sum_{k=2}^{p}\sum_{n=2}^{\infty}|z|^{2(k-1)}\frac{1-\alpha}{2(k-1)+n-\alpha}X_{n,p-k+1}z^n\\ &\quad + \sum_{k=2}^{p}\sum_{n=1}^{\infty}|z|^{2(k-1)}\frac{1-\alpha}{2(k-1)+n+\alpha}Y_{n,p-k+1}\overline{z^n} - \sum_{n=2}^{\infty}\frac{1-\alpha}{n-\alpha}z^n + \sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha}\overline{z^n}. \end{aligned}$$

Then, by Theorem 3.5 we have

$$\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{\frac{1-\alpha}{2(k-1)+n-\alpha}\Big(\frac{2(k-1)+n-\alpha}{1-\alpha}X_{n,p-k+1}\Big)+\frac{1-\alpha}{2(k-1)+n+\alpha}\Big(\frac{2(k-1)+n+\alpha}{1-\alpha}Y_{n,p-k+1}\Big)\right\}$$
$$+Y_{1,1}+\sum_{k=2}^{p}\Big(\frac{2k-1-\alpha}{1-\alpha}\frac{1-\alpha}{2k-1-\alpha}X_{1,p-k+1}+\frac{2k-1+\alpha}{1-\alpha}\frac{1-\alpha}{2k-1+\alpha}Y_{1,p-k+1}\Big)$$
$$\leqslant\sum_{k=1}^{p}\sum_{n=2}^{\infty}(X_{n,p-k+1}+Y_{n,p-k+1})+\sum_{k=1}^{p}(X_{1,p-k+1}+Y_{1,p-k+1})+Y_{1,1}\leqslant 1-Y_{1,1}\leqslant 1,$$

so $F\in\mathcal{HT}_p(\alpha)$. Conversely, suppose that $F\in\mathcal{HT}_p(\alpha)$. Then

$$\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{\Big[\frac{2(k-1)+n-\alpha}{1-\alpha}\Big]|a_{n,p-k+1}|+\Big[\frac{2(k-1)+n+\alpha}{1-\alpha}\Big]|b_{n,p-k+1}|\right\}$$
$$\leqslant 1-\frac{1+\alpha}{1-\alpha}|b_{1,1}|-\sum_{k=2}^{p}\left\{\frac{2k-1-\alpha}{1-\alpha}|a_{n,p-k+1}|+\frac{2k-1+\alpha}{1-\alpha}|b_{n,p-k+1}|\right\}.$$

Setting

$$X_{n,p-k+1}=\Big(\frac{2(k-1)+n-\alpha}{1-\alpha}\Big)|a_{n,p-k+1}|\quad(2\leqslant k\leqslant p,\ n=1,2,\ldots),$$
$$X_{n,1}=\Big(\frac{n-\alpha}{1-\alpha}\Big)|a_{n,1}|\quad(n=2,3,\ldots),$$
$$Y_{n,p-k+1}=\Big(\frac{2(k-1)+n+\alpha}{1-\alpha}\Big)|b_{n,p-k+1}|\quad(1\leqslant k\leqslant p,\ n=1,2,\ldots),$$

and

$$X_{1,1}=1-\sum_{k=1}^{p}\sum_{n=2}^{\infty}\big(X_{n,p-k+1}+Y_{n,p-k+1}\big)-\sum_{k=2}^{p}\big(X_{1,p-k+1}+Y_{1,p-k+1}\big)-Y_{1,1},$$

we obtain

$$F(z)=\sum_{k=1}^{p}\sum_{n=1}^{\infty}\big(X_{n,p-k+1}H_{n,p-k+1}(z)+Y_{n,p-k+1}G_{n,p-k+1}(z)\big),$$

as required. □

Finally, we give the distortion bounds for functions in $\mathcal{HT}_p(\alpha)$, which yields a covering result for $\mathcal{HT}_p(\alpha)$.

Theorem 3.9. *If* $F\in\mathcal{HT}_p(\alpha)$, *then*

$$|f(z)|\leqslant(1+|b_{1,1}|)r+\Big(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_{1,1}|\Big)r^2,\quad |z|=r<1,$$

and

$$|f(z)|\geqslant(1-|b_{1,1}|)r-\Big(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_{1,1}|\Big)r^2,\quad |z|=r<1.$$

Proof. We only prove the first inequality. The argument for second inequality is similar and will be omitted. Let $F \in \mathcal{HT}_p(\alpha)$. Taking the absolute value of F, we obtain

$$\begin{aligned}
|F(z)| &\leqslant (1+|b_{1,1}|)|z| + \Big(\sum_{k=1}^{p}\sum_{n=2}^{\infty}(|a_{n,p-k+1}|+|b_{n,p-k+1}|) + \sum_{k=1}^{p}(|a_{1,p-k+1}|+|b_{1,p-k+1}|)\Big)|z|^n \\
&\leqslant (1+|b_{1,1}|)r + \Big(\sum_{k=1}^{p}\sum_{n=2}^{\infty}(|a_{n,p-k+1}|+|b_{n,p-k+1}|) + \sum_{k=1}^{p}(|a_{1,p-k+1}|+|b_{1,p-k+1}|)\Big)r^2 \\
&= (1+|b_{1,1}|)r + \frac{1-\alpha}{2-\alpha}\Big(\sum_{k=1}^{p}\sum_{n=2}^{\infty}(\frac{1-\alpha}{2-\alpha}|a_{n,p-k+1}| + \frac{1-\alpha}{2-\alpha}|b_{n,p-k+1}|) \\
&\quad + \sum_{k=1}^{p}(\frac{1-\alpha}{2-\alpha}|a_{1,p-k+1}| + \frac{1-\alpha}{2-\alpha}|b_{1,p-k+1}|)\Big)r^2 \\
&\leqslant (1+|b_{1,1}|)r + \frac{1-\alpha}{2-\alpha}\Big(\sum_{k=1}^{p}\sum_{n=2}^{\infty}(\frac{2(k-1)+n-\alpha}{1-\alpha}|a_{n,p-k+1}| + \frac{2(k-1)+n+\alpha}{1-\alpha}|b_{n,p-k+1}|) \\
&\quad + \sum_{k=1}^{p}(\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|)\Big)r^2 \\
&\leqslant (1+|b_{1,1}|)r + \frac{1-\alpha}{2-\alpha}\Big(1-\frac{1+\alpha}{1-\alpha}|b_{1,1}|\Big)r^2 \quad \text{(by (3.4))} \\
&= (1+|b_{1,1}|)r + \Big(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_{1,1}|\Big)r^2.
\end{aligned}$$

The bounds given in Theorem 3.5 for the functions $F = H + \overline{G}$ of the form (2.6) also hold for functions of the form (2.2) if the coefficient condition (3.1) is satisfied. The functions F given by

$$F(z) = z + |b_{1,1}|\overline{z} + \Big(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_{1,1}|\Big)\overline{z^2} \quad \text{and} \quad F(z) = z - |b_{1,1}|z - \Big(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_{1,1}|\Big)z^2$$

for $|b_{1,1}| \leqslant (1-\alpha)/(1+\alpha)$ show that the bounds given in Theorem 3.5 are sharp. □

The following covering result follows from the second inequality in Theorem 3.5.

Corollary 3.10. *If* $F \in \mathcal{HT}_p(\alpha)$, *then*

$$\left\{w : |w| < \frac{1}{2-\alpha}(1-|b_1|)\Big[1+(2\alpha-1)|b_{1,1}|\Big]\right\} \subset F(\mathbb{U}).$$

The corresponding definition for polyharmonic convex function of order α leads to the following corollary.

Corollary 3.11. *Let* F *be given by* (2.1) *and*

$$\sum_{k=1}^{p}\sum_{n=2}^{\infty}\left\{\Big[\frac{2(k-1)+n(n-\alpha)}{1-\alpha}\Big]|a_{n,p-k+1}| + \Big[\frac{2(k-1)+n(n+\alpha)}{1-\alpha}\Big]|b_{n,p-k+1}|\right\}$$
$$\leqslant 1 - \frac{1+\alpha}{1-\alpha}|b_{1,1}| - \sum_{k=2}^{p}\Big\{\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|\Big\},$$

where $0 \leqslant \frac{1+\alpha}{1-\alpha}|b_{1,1}| + \sum_{k=2}^{p}\Big\{\frac{2k-1-\alpha}{1-\alpha}|a_{1,p-k+1}| + \frac{2k-1+\alpha}{1-\alpha}|b_{1,p-k+1}|\Big\} < 1$. *Then* F *is univalent and sense preserving in* $\mathbb{U}$ *and* $F \in \mathcal{HC}_p(\alpha)$.

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