A numerical method for space fractional diffusion equations using a semi-discrete scheme and Chebyshev collocation method

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Abstract
In the present paper, a numerical approach to efficiently calculate the solution of space fractional diffusion equations is investigated. The finite difference scheme and Chebyshev collocation method is applied to solve this problems. Also, the matrix form of the proposed method is obtained. The numerical examples and comparison with other methods shows that the present method is effective.

Keywords: Fractional diffusion equation, Finite difference, Collocation, Chebyshev polynomials

1. Introduction

The use of fractional partial differential equations (FPDEs) in mathematics, physics, engineering and chemistry has become increasingly popular in recent years [4, 8, 14].

To obtain an analytical solution of this problems is extremely difficult thus many authors are seeking ways to numerically solve these problems.


Meerschaert, Tadjeran and et al. suggested three kinds of finite difference approximations which are the implicit Euler method, the explicit Euler method and the fractional Cranck-Nicholson method
for FPDEs based on shifted Grünwald formula. Also they derived some detailed stability and convergence analysis [10, 11, 12, 18, 19].

The space fractional diffusion equations are a type of fractional partial differential equations which by many authors are solved numerically. For example Khader [6] used Chebyshev collocation method to discretize space fractional diffusion equations to obtain a linear system of ordinary differential equations and used the finite difference method for solving the resulting system. Saadatmandi and Dehghan [15] used tau approach and Sousa [16] applied splines to solve space fractional diffusion equations.

In this paper, we proposed a different approach to obtain the solution of space fractional diffusion equation

\[
\frac{\partial u(x,t)}{\partial t} = p(x) \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + v(x,t), \quad 0 < x < 1, \quad 0 \leq t \leq T, \quad 1 < \alpha \leq 2, \tag{1}
\]

with initial condition

\[ u(x,0) = g(x), \quad 0 < x < 1, \tag{2} \]

and boundary conditions

\[ u(0,t) = h_0(t), \quad 0 < t \leq T, \tag{3} \]

\[ u(1,t) = h_1(t), \quad 0 < t \leq T. \tag{4} \]

Where the function \( v(x,t) \) is a source term and the fractional derivative is the Caputo derivative can be defined by [16]:

\[
\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-s)^{m-\alpha-1} \frac{\partial^m u(s,t)}{\partial s^m} \, ds, \quad m-1 < \alpha \leq m.
\]

Note that for \( \alpha = 2 \), Eq. (1) is the classical diffusion equation

\[
\frac{\partial u(x,t)}{\partial t} = p(x) \frac{\partial^2 u(x,t)}{\partial x^2} + v(x,t).
\]

This work presents a numerical method to solve this kind of problems using finite difference scheme and collocation method via Chebyshev polynomials.

2. Description of the method

In this section, the process of solving the space fractional diffusion equations is described as in 1-4.

Let \( t_n = n \Delta t \), \( n = 0,1,...,M \) where \( \Delta t = \frac{T}{M} \), \( t_0 = 0, t_M = T \).

First, we use a finite difference technique and \( \theta \)-weighted scheme [2] to discretize the time derivative.

\[
\frac{u(x,t + \Delta t) - u(x,t)}{\Delta t} = \frac{p(x)}{\Gamma(2-\alpha)} \int_0^1 (x-s)^{1-\alpha} \left[ \theta \frac{d^2 u(s,t + \Delta t)}{ds^2} + (1-\theta) \frac{d^2 u(s,t)}{ds^2} \right] ds + v(x,t), \tag{5}
\]

where \( \theta \in [0,1] \). Using the notation \( u_n(x) = u(x,t_n) \) in (5), where \( t_n = t_{n-1} + \Delta t \), we obtain

\[
\begin{align*}
\frac{u_{n+1}(x) - u_n(x)}{\Delta t} & = \frac{p(x)}{\Gamma(2-\alpha)} \int_0^1 (x-s)^{1-\alpha} \left[ \theta \frac{d^2 u_{n+1}(s)}{ds^2} + (1-\theta) \frac{d^2 u_n(s)}{ds^2} \right] ds + \Delta t v_n(x), 
\end{align*}
\]

where \( v_n(x) = v(x,t_n) \).
Definition 1. [9] The well-known Chebyshev polynomials of the first kind of degree \( n \) are defined on the interval \([-1,1]\) as

\[
T_n(x) = \cos(n \arccos(x)),
\]

where \( T_0(x) = 1, \ T_1(x) = x \) and they satisfy the recurrence relations:

\[
T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad n = 1,2,\ldots.
\]

In order to use these polynomials on the interval \([0,1]\) we define the so called shifted Chebyshev polynomials by introducing the change of variable \( z = 2x - 1 \). The shifted Chebyshev polynomials are defined as: \( T_n^*(x) = T_n(2x-1) \).

Now we expand \( u_n(x) \) by shifted Chebyshev polynomials:

\[
 u_n(x) = \sum_{i=0}^{N} r_i^n T_i^*(x), \quad n = 1,\ldots,M,
\]

where \( r_0^n, r_1^n, \ldots, r_N^n \) are unknown coefficients.

From equation (6) and (7) we obtain:

\[
\sum_{i=0}^{N} r_i^{n+1} T_i^*(x) - \sum_{i=0}^{N} r_i^n T_i^*(x) = p(x) \frac{\Delta t}{\Gamma(2-\alpha)} \int_{0}^{1} (x - s)^{1-\alpha} \left[ \theta \sum_{i=0}^{N} r_i^{n+1} T_i^*(s) + (1 - \theta) \sum_{i=0}^{N} r_i^n T_i^*(s) \right] ds + (1 - \theta) \sum_{i=0}^{N} r_i^n v^n_i(x) ds + \Delta t v_n(x), \quad n = 0,1,\ldots,M-1,\ k = 1,\ldots, N-1.
\]

Also boundary conditions (3) and (4) for \( n = 0,1,\ldots, M-1 \) are used to obtain:

\[
u_{n+1}(x_N) = \sum_{i=0}^{N} r_i^{n+1}(x_N) = h_0(t_{n+1}),
\]

\[
u_{n+1}(x_0) = \sum_{i=0}^{N} r_i^{n+1}(x_0) = h_k(t_{n+1}).
\]

Therefor equations (9)-(11) generate a set of \((N+1)\) algebraic equations, which can be solved for unknown coefficients.

Clearly \( u_0(x) \) can be obtained from the initial condition (2) as follows:

\[
u_0(x) = u(x,t_0) = g(x)
\]

3. The matrix form of the proposed method

In order to find the matrix form of suggested method, first by using (9) for \( k = 1,2,\ldots, N-1 \) we obtain the preliminary matrices, \( T^* \) and \( Q \). Finally by using (10) and (11) the matrix form for this method is

Achieved.

Let \([u]^n = [u_1^n, u_2^n, \ldots, u_{N-1}^n]^T\) and \([r]^n = [r_0^n, r_1^n, \ldots, r_N^n]^T\) where \( u_k^n = u_n(x_k), \ k = 1,2,\ldots, N-1 \).
From (7) we obtain
\[ [u]^n = T^*[r]^n \]
where \( T^* \) is a \((N-1) \times (N+1)\) matrix that given by
\[
T^* = \begin{bmatrix}
T_0^*(x_1) & T_1^*(x_1) & T_N^*(x_1) \\
T_0^*(x_2) & T_1^*(x_2) & T_N^*(x_2) \\
& & \\
T_0^*(x_{N-1}) & T_1^*(x_{N-1}) & T_N^*(x_{N-1})
\end{bmatrix}
\]

Also from the right side of equations (6) and (9) for \( k = 1,2,...,N - 1 \) we obtain
\[
\begin{align*}
p(x_k)\Delta t & \frac{p(x_k)\Delta t}{\Gamma(2-\alpha)} \int_0^1 (x_k - s)^{1-\alpha} \left[ \theta \frac{d^2 u_n}{ds^2} + (1-\theta) \frac{d^2 u_n}{ds^2} \right] ds + \Delta t x_n(x_k) \\
& = \frac{p(x_k)\Delta t}{\Gamma(2-\alpha)} \int_0^1 (x_k - s)^{1-\alpha} \left[ \theta \sum_{j=0}^N r^{\alpha}_{i,j} T_i^*(s) + (1-\theta) \sum_{j=0}^N r^{\alpha}_{i,j} T_i^*(s) \right] ds + \Delta t x_n(x_k)
\end{align*}
\]
\[
= \theta \Delta t \sum_{j=0}^N r^{\alpha}_{i,j} \frac{p(x_k)}{\Gamma(2-\alpha)} \int_0^1 (x_k - s)^{1-\alpha} T_i^*(s) ds + (1-\theta) \Delta t \sum_{j=0}^N r^{\alpha}_{i,j} \left( \frac{p(x_k)}{\Gamma(2-\alpha)} \int_0^1 (x_k - s)^{1-\alpha} T_i^*(s) ds \right)
\]
\[
+ \Delta t v_n(x_k).
\]

Therefor the matrix form for this equations is as follows:
\[
\theta \Delta t Q\{r\}_{n+1} + (1-\theta) \Delta t Q\{r\}^n + \Delta t v_n^*,
\]
where
\[
v_n = [v_n(x_1), v_n(x_2), ..., v_n(x_{N-1})]^T.
\]
and the matrix element of \( Q \) are
\[
q_{ij} = \frac{p(x_i)}{\Gamma(2-\alpha)} \int_0^1 (x_i - s)^{1-\alpha} T_j^*(s) ds,
\]
for \( i = 1,...,N - 1, j = 1,...,N + 1 \).

Since
\[
D_i^\alpha T_j^*(x) = \frac{1}{\Gamma(2-\alpha)} \int_0^1 (x-s)^{1-\alpha} T_j^*(s) ds, 1 < \alpha \leq 2,
\]
and from [6] we have:
\[
D_i^\alpha T_j^*(x) = \sum_{k=0}^{[\alpha]} (-1)^k 2^{2(j-k-1)} j!(2j-k-1)!(j-k-1)! k!(2j-2k-2)! \Gamma(j-k+1-\alpha) x^{j-k-\alpha},
\]
thus the matrix elements of \( Q \) can be demonstrated as follows:
\[
q_{ij} = \begin{cases} 
0, & 1 \leq i \leq N - 1, j = 1,2, \\
p(x_i) \sum_{k=0}^{[\alpha]-1} (-1)^k 2^{2(j-k-1)} j!(2j-k-3)!(j-k-1)! k!(2j-2k-2)! \Gamma(j-k+1-\alpha) x^{j-k-\alpha}, & 1 \leq i \leq N - 1, 3 \leq j \leq N + 1.
\end{cases}
\]

Note that \( D_i^\alpha T_j^*(x) = 0 \) for \( j = 0,1,...,[\alpha]-1 \) thus \( q_{ij} = 0 \) for \( j=1,2 \).

Therefore above matrices and equations (9) for \( k = 1,2,...,N - 1 \) leads to the following matrix form:
\[
(T^* - \theta \Delta t Q)[r]^{n+1} = ((1-\theta)\Delta t Q + T^*)[r]^n + \Delta t v_n^*,
\]
(12)

Now by using boundary conditions (10), (11) and relation (12) we obtain the matrix form of our method as follows:
\[ A[r]^{n+1} = B[r]^n + d_n, \quad (13) \]

where the matrix elements of \( A \) are:
\[
\begin{align*}
T_{j-1}^*(x_0), & \quad i = 1, 1 \leq j \leq N + 1, \\
T_{j-1}^*(x_N), & \quad i = N + 1, 1 \leq j \leq N + 1, \\
T_{j-1}^*(x_{-1}), & \quad 2 \leq i \leq N, j = 1, 2, \\
\end{align*}
\]

\[
a_g = \begin{cases} 
T_{j-1}^*(x_{i-1}), & \quad 2 \leq i \leq N, 3 \leq j \leq N + 1,
\end{cases}
\]

matrix \( B \) is obtained by adding two zero rows to the first and last row of matrix \((1-\theta)\Delta t \mathbf{Q} + \mathbf{T}^d\)
and
\[
d_n = [h_i(t_{n+1}), \Delta t v_n^T, h_j(t_{n+1})]^T.
\]

4. Numerical examples

In this section, we consider some examples of proposed scheme for the space fractional diffusion equations and for showing the accuracy, efficiency and validity of the method, we compare the our numerical results with other methods.

**Remark:** In all of examples the time step is taken as \( \Delta t = 0.001 \) and \( \theta = \frac{1}{2} \).

**Example 4.1.** [15] Consider the following space fractional diffusion equation
\[
\frac{\partial u(x,t)}{\partial t} = p(x) \frac{\partial^{1.5} u(x,t)}{\partial x^{1.5}} + v(x,t),
\]
on a finite domain \( 0 < x < 1 \), with the diffusion coefficient
\[
p(x) = \Gamma(1.5)x^{0.5},
\]
the source function
\[
v(x,t) = (x^2 + 1)\cos(t+1) - 2x\sin(t+1),
\]
with the initial condition
\[
u(x,0) = (x^2 + 1)\sin(1),
\]
and the boundary conditions
\[
u(0,t) = \sin(t+1), \quad u(1,t) = 2\sin(t+1), \quad \text{for} \quad t > 0.
\]

The exact solution of this problem is \( u(x,t) = (x^2 + 1)\sin(t+1) \).

We applied the present method with \( N = 2 \) and compared absolute error function \( |u(x,1) - u_{approx}(x,1)| \) of our scheme with the method in [15] which are shown in Table 1. In Table 2 the maximum absolute errors for \( t = 0, 0.1, 0.2, ..., 0.9, 1 \) and \( 0 < x < 1 \) are reported.

Also, figure 1 shows the exact solution and approximate solution for \( u(x, 1) \) with \( N = 2 \). Note that the matrices \( A \) and \( B \) for this example are as follows:
\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1.002 \\ 1 & -1 & 1 \end{bmatrix},
\]
\[
B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -0.998 \\ 0 & 0 & 0 \end{bmatrix}.
\]
Easily seen that $A$ is non-singular matrix and the spectral radius of matrix $A^{-1}B$ is less than one therefore the proposed method has unique solution and is unconditionally stable [17].

**Table 1:** Comparison of present method for $u(x,1)$ with the tau method [15] for Exa. 4.1.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Method[15] with $m = 7$</th>
<th>present method with $N = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$4.66 \times 10^{-5}$</td>
<td>$9.34 \times 10^{-7}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$7.74 \times 10^{-5}$</td>
<td>$3.23 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.3</td>
<td>$5.00 \times 10^{-5}$</td>
<td>$6.29 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.30 \times 10^{-5}$</td>
<td>$9.54 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.5</td>
<td>$2.74 \times 10^{-5}$</td>
<td>$1.45 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$4.38 \times 10^{-5}$</td>
<td>$3.27 \times 10^{-5}$</td>
</tr>
<tr>
<td>0.7</td>
<td>$3.87 \times 10^{-5}$</td>
<td>$1.26 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$1.01 \times 10^{-5}$</td>
<td>$1.26 \times 10^{-6}$</td>
</tr>
<tr>
<td>0.9</td>
<td>$3.35 \times 10^{-6}$</td>
<td>$1.43 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

**Table 2:** Absolute errors for example 4.1 with $N=2$ in domain $0 < x < 1$

| $t$  | $|u(x,t) - u_{approx}(x,t)|$ |
|------|-------------------------------|
| 0    | 0                             |
| 0.1  | $2.24 \times 10^{-8}$         |
| 0.2  | $3.20 \times 10^{-8}$         |
| 0.3  | $4.33 \times 10^{-7}$         |
| 0.4  | $7.79 \times 10^{-7}$         |
| 0.5  | $8.25 \times 10^{-7}$         |
| 0.6  | $2.40 \times 10^{-6}$         |
| 0.7  | $4.30 \times 10^{-6}$         |
| 0.8  | $5.44 \times 10^{-6}$         |
| 0.9  | $4.54 \times 10^{-5}$         |
| 1    | $6.21 \times 10^{-5}$         |

**Figure 1:** Approximate solution and exact solution for $u(x,1)$ with $N = 2$ for Ex. 4.1
Example 4.2. [15] In this example, we consider the following space fractional diffusion equation:

\[
\frac{\partial u(x,t)}{\partial t} = \Gamma(1.2) x^{1.8} \frac{\partial^{1.8} u(x,t)}{\partial x^{1.8}} + 3x^2(2x-1)e^{-t},
\]

with the initial condition

\[ u(x,0) = x^2 - x^3, \]

and the boundary conditions

\[ u(0,t) = 0, \quad u(1,t) = 0, \quad \text{for } t > 0. \]

The exact solution of this problem is \( u(x,t) = x^2(1-x)e^{-t} \). We solved this equation by using proposed method and in Table 3 we compared our results with results of obtained in [6, 15]. Also in Table 4 maximum absolute errors for \( t = 0.2,0.4,\ldots,1,1.2,\ldots,1.8,2 \) and \( 0 < x < 1 \) are reported.

Also, figure 2 shows the exact solution and approximate solution for \( u(x, 2) \) with \( N = 3 \). Note that the matrices \( A \) and \( B \) for this example are as follows:

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 0.5 & -0.5039 & -1.0081 \\
1 & -0.5 & -0.5003 & 1.0013 \\
1 & -1 & 1 & -1
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0.5 & -0.4961 & -0.9919 \\
1 & -0.5 & -0.4997 & 0.9987 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Easily seen that \( A \) is non-singular matrix and the spectral radius of matrix \( A^{-1}B \) is less than one therefore the proposed method has unique solution and is unconditionally stable.

<table>
<thead>
<tr>
<th>( x )</th>
<th>Method [6] with ( m=5 )</th>
<th>Method [15] with ( m=5 )</th>
<th>present method(N=3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.74×10^{-5}</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.1</td>
<td>4.20×10^{-5}</td>
<td>4.47×10^{-6}</td>
<td>3.15×10^{-7}</td>
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<td>0.2</td>
<td>3.76×10^{-5}</td>
<td>2.78×10^{-7}</td>
<td>4.23×10^{-7}</td>
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<tr>
<td>0.3</td>
<td>8.44×10^{-5}</td>
<td>5.81×10^{-6}</td>
<td>2.94×10^{-7}</td>
</tr>
<tr>
<td>0.4</td>
<td>3.27×10^{-5}</td>
<td>1.02×10^{-5}</td>
<td>5.70×10^{-6}</td>
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<td>6.28×10^{-5}</td>
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<tr>
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<td>1.94×10^{-5}</td>
<td>1.08×10^{-5}</td>
<td>1.05×10^{-5}</td>
</tr>
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<td>2.95×10^{-5}</td>
<td>8.54×10^{-6}</td>
<td>2.43×10^{-6}</td>
</tr>
<tr>
<td>0.8</td>
<td>4.92×10^{-5}</td>
<td>6.06×10^{-6}</td>
<td>8.50×10^{-6}</td>
</tr>
<tr>
<td>0.9</td>
<td>2.83×10^{-5}</td>
<td>3.67×10^{-6}</td>
<td>7.68×10^{-7}</td>
</tr>
<tr>
<td>1.0</td>
<td>7.73×10^{-5}</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
The matrices $A$ and $B$ for this example for $\alpha = 1.2$ are as follows:

$$A = \begin{pmatrix}
\alpha & 1 \\
1 & \alpha
\end{pmatrix}, \quad B = \begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix}$$
Easily seen that $A$ is non-singular matrix and the spectral radius of matrix $A^{-1}B$ is less than one therefore the proposed method has unique solution and is unconditionally stable.

Note that in [16] this problem has been solved by finite difference method and splines. The maximum errors for $\alpha = 1.2, \alpha = 1.4, \alpha = 1.5$ and $\alpha = 1.8$ with $\Delta x = \frac{1}{30}$ are $0.3566 \times 10^{-3}$, $0.24616 \times 10^{-3}$, $0.2067 \times 10^{-3}$ and $0.1150 \times 10^{-3}$ respectively.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$\alpha = 1.2$</th>
<th>$\alpha = 1.4$</th>
<th>$\alpha = 1.5$</th>
<th>$\alpha = 1.8$</th>
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<td>$2.43 \times 10^{-7}$</td>
<td>$1.04 \times 10^{-7}$</td>
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<td>$6.22 \times 10^{-7}$</td>
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<td>$3.73 \times 10^{-7}$</td>
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<tr>
<td>0.3</td>
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<td>$4.26 \times 10^{-7}$</td>
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<td>0.4</td>
<td>$8.00 \times 10^{-7}$</td>
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<td>1</td>
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<td>$8.44 \times 10^{-5}$</td>
<td>$6.34 \times 10^{-5}$</td>
<td>$2.43 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

**Figure 3:** Approximate solution and exact solution for $u(x, 1)$ with $N = 4$, $\alpha = 1.2$. 

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5. Conclusion

In this paper, finite difference scheme and Chebyshev collocation method have been successfully applied to find the solution of the space fractional diffusion equations. Since using matrix form of the method is more convenient for application of collocation method, thus the matrix form of the proposed method was obtained. The results and comparison of the our proposed method and other methods indicate that this scheme is accurate and efficient approach for the solution of this problems. Proposed method for the examples of this paper was unconditionally stable but, unconditionally stable for the general case is open problem.

References