Convergence of Common Fixed Point Theorems in Fuzzy Metric Spaces

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Abstract

Fixed point theorems in a fuzzy metric space are proved by considering a contractive condition for a triplet of mappings and this is the totally a new approach for obtaining the fixed point.

Key words:- Fixed point, quasi-contraction, fuzzy metric space, Cauchy sequence.

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1. Introduction

The notion of fuzzy set was introduced by Zadeh [9]. It was developed extensively by many authors and used in various fields. In this paper we deal with the fuzzy metric space defined by Kramosil and Michalek [6] and modified by George and Veeramani [3]. The most interesting references in this direction are Chang [1], Cho [2], Grabiec [4], and Kaleva [5]. In the present paper, first we prove a common fixed point theorem. Then we study the relationship between the convergence of three sequences of mappings and the convergence of their common fixed points.
2. Preliminaries

**Definition 2.1**[8]. A binary operation $\ast : [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1], \ast)$ is an abelian topological monoid with the unit 1 such that $a \ast b \leq c \ast d$ and whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

**Definition 2.2**[6]. The 3-tuple $(X, M, \ast)$ is called a fuzzy metric space (shortly, FM-space) if $X$ is an arbitrary set, $\ast$ a continuous t-norm and $M$ is a fuzzy set in $X \times X \times [0,\infty)$ satisfying the following conditions:

- for all $x, y, z \in X$ and $s, t > 0$.
- (FM-1) $M(x, y, 0) = 0$,
- (FM-2) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
- (FM-3) $M(x, y, t) = M(y, x, t)$
- (FM-4) $M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)$,
- (FM-5) $M(x, y, \cdot) : [0, \infty] \rightarrow [0,1]$ is left continuous,

Note that $M(x, y, t)$ can be considered as the degree of nearness between $x$ and $y$ with respect to $t$. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

**Example 2.3.**[3]. Let $(X, d)$ be a metric space. Define $a \ast b = \min\{a, b\}$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and all $t > 0$. Then $(X, M, \ast)$ is a Fuzzy metric space. It is called the Fuzzy metric space induced by $d$.

**Lemma 2.4.**[4]. For all $x, y \in X$, $M(x, y, \cdot)$ is a non decreasing function.

**Definition 2.5**[4]. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, \ast)$ is said to be a Cauchy sequence if and only if for each $\epsilon > 0$, $t > 0$, there exists $n_0 \in \mathbb{N}$, such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$. The sequence $\{x_n\}$ is said to converge to a point $x$ in $X$ if and only if for each, $\epsilon > 0$, $t > 0$, $n_0 \geq N$ such that $M(x_n, x, t) > 1 - \epsilon$ for all $n \geq n_0$.

A fuzzy metric space $(X, M, \ast)$ is said to be complete if every Cauchy sequence in it converges to a point in it.

**Remark 2.6.** Since $\ast$ is continuous, it follows from (FM-4) that the limit of the sequence in FM-space is uniquely determined. Let $(X, M, \ast)$ be a fuzzy metric space with the following conditions (FM-6) $\lim_{t \rightarrow \infty}M(x, y, t) = 1$ for all $x, y \in X$.

**Lemma 2.7**[2]. Let $\{x_n\}$ be a sequence in a fuzzy metric space $(X, M, \ast)$ with $t \ast t > t$ for all $t \in [0,1]$ and condition (FM-6). If there exists a number $k \in (0,1)$ such that $M(x_{n+2}, x_{n+1}, qt) \geq M(x_{n+1}, x_n, t)$

for all $t > 0$ and $n = 1, 2, \ldots$ then $\{x_n\}$ is a Cauchy sequence in $X$.

**Lemma 2.8**[7]. If for all $x, y \in X$, $t > 0$ with positive number $k \in (0,1)$ and

$M(x, y, kt) \geq M(x, y, t),$

then $x = y$.

3. Main results
Theorem 3.1. Let \((X, M, \ast)\) be a complete fuzzy metric space. Suppose that \(P, Q\) and \(T\) are mappings from \(X\) to itself such that
(a) \(PT = TP, QT = TQ,\)
(b) \(P(X) \cup Q(X) \subseteq T(X),\)
(c) \(T\) is continuous, and
(d) The triplet \((P, Q, T)\) is a quasi-contraction, i.e.,
\[
M(Px, Qy, kt) \geq \min \{M(Tx, Ty, t), M(Px, Tx, t), M(Ty, Qy, t), M(Ty, Px, t), M(Tx, Qy, t)\},
\]
with \(k \in (0,1)\), then \(P, Q\) and \(T\) have a unique common fixed point.

Proof. Let \(x_0 \in X\) be any arbitrary point in \(X\). We define sequence \(\{y_n\}\) and \(\{x_n\}\) such that
\[
y_{2n} = Tx_{2n} = Qx_{2n+1} \quad \text{and} \quad y_{2n+1} = Tx_{2n+1} = Px_{2n}, \quad n = 1, 2, \ldots
\]
This is always possible because of the condition (b).

Now taking \(x = x_{2n}\) and \(y = x_{2n+1}\) in (1) we have
\[
M(y_{2n+1}, y_{2n+2}, kt) = M(Px_{2n}, Qx_{2n+1}, kt)
\]
\[
\geq \min \{M(Tx_{2n}, Tx_{2n+1}, t), M(Px_{2n}, Tx_{2n}, t), M(Tx_{2n+1}, Qx_{2n+1}, t),
M(Tx_{2n+1}, Px_{2n}, t), M(Tx_{2n}, Qx_{2n+1}, t)\}.\]
which implies
\[
M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t)
\]
In general
\[
M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t). \quad (2)
\]

To prove that \(\{y_n\}\) is a Cauchy sequence we prove by the method of induction that for all \(n \geq n_0\), and for every \(m \in \mathbb{N},\)
\[
M(y_n, y_{n+m}, t) > 1 - \lambda. \quad (3)
\]
From (2) we have
\[
M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, \frac{t}{k}) \geq M(y_{n-2}, y_{n-1}, \frac{t}{k^2}) \geq \ldots \geq M(y_0, y_1, \frac{t}{k^n}) \to 1 \text{ as } n \to \infty.
\]
For \(t > 0, \lambda \in (0,1),\) there exists \(n_0 \in \mathbb{N},\) such that
\[
M(y_n, y_{n+1}, t) > 1 - \lambda. \quad (4)
\]
Thus (3) is true for \(m = 1.\) Suppose (3) is true for all \(m\) then we will show that it is also true for \(m+1.\)

Using the definition of fuzzy metric space, (2) and (3), we have
\[
M(y_n, y_{n+m+1}, t) \geq \min \{M(y_n, y_{n+m}, \frac{t}{k}), M(y_{n+m}, y_{n+m+1}, \frac{t}{k})\} > 1 - \lambda.
\]
Hence (3) is true for \(m+1.\)

Thus \(\{y_n\}\) is a Cauchy sequence. By completeness of \((X, M, \ast)\), \(\{y_n\}\) convergence to some point \(z\) in \(X.\)

Therefore \(\{Tx_{2n}\}, \{Px_{2n}\}\) and \(\{Qx_{2n+1}\}\) also converge to \(z.\)

By continuity of \(T\) and the fact that \(PT = TP,\) it follows that
\[
TTx_{2n} \to Tz \quad \text{and} \quad PTx_{2n} = TPx_{2n} \to Tz.
\]

Taking \(x = Tx_{2n}\) and \(y = z\) in (1) we have
\[
M(PTx_{2n}, Qz, kt) \geq \min \{M(TTx_{2n}, Tz, t), M(PTx_{2n}, TTx_{2n}, t), M(Tz, Qz, t), M(Tz, PTx_{2n}, t),
M(TTx_{2n}, Qz, t)\}.
\]
Taking limit \(n \to \infty\) we have
\[
M(Tz, Qz, kt) = \min \{1, 1, M(Tz, Qz, t), 1, M(Tz, Qz, t)\}
\]
Therefore,
\[ M(Tz, Qz, kt) \geq M(Tz, Qz, t), \]
which give \( Tz = Qz \). Similarly \( Tz = Pz \).
Again taking \( x = x_{2n} \) and \( y = z \) in (1) we can show that
\[ z = Qz = Tz = Pz. \]

Uniqueness:
Let \( w \) be another common fixed point of \( P, Q \) and \( T \). Then we have
\[ M(Pz, Qw, kt) \geq \min \{ M(Tz, Tw, t), M(Pz, Tz, t), M(Tw, Qw, t), M(Tw, Pz, t), M(Tz, Qw, t) \}. \]

or
\[ M(z, w, kt) \geq M(z, w, t). \]
Thus \( z = w \) and hence \( z \) is a unique common fixed point of \( P, Q \) and \( T \).

Next, we have the convergence theorem.

**Theorem 3.2.** Let \((X, M, \ast)\) be a complete fuzzy metric space. Let \( \{P_n\}, \{Q_n\} \) and \( \{T_n\} \) be sequences of mappings from \( X \) to itself such that the triplet \( \{P_n, Q_n, T_n\} \) is a quasi-contraction. If \( P, Q \) and \( T: X \to X \) are point wise limit of \( P_n, Q_n \) and \( T_n \) respectively, and if \( k_n \to k < 1 \), then \( (P, Q, T) \) is a quasi-contraction. Furthermore, the sequence of the unique common fixed point \( u_n \) of \( P_n, Q_n \) and \( T_n \), converges to the unique common fixed point \( u \) of \( P, Q \) and \( T \).

**Proof:** For any \( x, y \in X \), we have, for \( x \neq y \),
\[ M(Px, Qy, kt) = M(Px, P_n x, t) \ast M(P_n x, Q_n y, t) \ast M(Q_n y, Qy, t) \]
\[ \geq \min \{ M(T_n x, T_n y, t), M(T_n x, Q_n y, t), M(T_n y, P_n x, t), M(T_n x, Q_n y, t) \} \ast M(Q_n y, Qy, t) \]
\[ = \min \{ M(T_n x, T_n y, t), M(T_n x, Q_n y, t), M(T_n y, P_n x, t) \} \ast M(Q_n y, Qy, t). \]
Since \( P_n x \to Px, Q_n y \to Qy, T_n x \to Tx \) and \( T_n y \to Ty \) for \( x \neq y \), as \( n \to \infty \), and also \( k_n \to k < 1 \), we have,
\[ M(Px, Qy, kt) \geq \min \{ M(Tx, Ty, t), M(Tx, Ty, t) \} \ast M(Qy, Qy, t) \]
\[ = \min \{ M(Tx, Ty, t), M(Tx, Ty, t) \} \ast 1 \ast 1. \]
\[ = \min \{ M(Tx, Ty, t), M(Tx, Ty, t) \} \ast M(Qy, Qy, t). \]
Hence we get \((P, Q, T)\) is a quasi-contraction, and \( P, Q \) and \( T \) have the unique common fixed point \( u \) since \( X \) is complete.

Now suppose that \( u_n \) be the common fixed point of \( P_n, Q_n \) and \( T_n \) for each \( n \). Then we have
\[ M(u_n, u, kt) \geq M(u_n, Q_n u, u, t) \ast M(Q_n u, u, t) \]
\[ = M(u_n, Q_n u, u, t) \ast M(Q_n u, u, t) \]
\[ \geq \min \{ M(u_n, Q_n u, u, t), M(u_n, Q_n u, u, t), M(u_n, Q_n u, u, t), M(u_n, Q_n u, u, t) \} \ast M(Q_n u, u, t) \]
\[ = \min \{ M(u_n, Q_n u, u, t), M(u_n, Q_n u, u, t), M(u_n, Q_n u, u, t) \} \ast M(Q_n u, u, t). \]
Now if \( M(T_n u, Q_n u, t) \) is minimum then
\[ M(u_n, u, kt) \geq M(T_n u, Q_n u, t) \ast M(Q_n u, u, t) \]
\[ = M(T_n u, u, t) \ast M(u, u, t). \]
Since \( T_n \to T \) and \( Q_n \to Q \), point wise it follows that, as \( n \to \infty \),
\[ M(u_n, u, t) \to 1, \]
and therefore \( u_n \to u \).
Now if \( M(u_n, T_n u, t) \) is minimum, then
\[ M(u_n, u, kt) \geq M(u_n, T_n u, t) \ast M(Q_n u, u, t) \]

so that, as \( n \to \infty \), again we can get \( u_n \to u \).

Similarly, for the last term in the bracket, we can prove that \( u_n \to u \).

References


