On Fuzzy Isomorphism Theorem Of Hypernear-modules

M. Aliakbarnia. Omran¹, Y. Nasabi², E. Hendoukolaie³

¹Amol Institution of higher education, Amol, Iran, Mehdimran@gmail.com
²Young Researchers Club, Islamic Azad University, Ayatollah Amoli Branch, Amol, Iran, Yaser.nasabi@yahoo.com
³Young Researchers Club, Islamic Azad University, Ayatollah Amoli Branch, Amol, Iran, Edrishendoii@gmail.com

Abstract
In this paper, introduce the concept of normal fuzzy subhypernear-modules of hypernear-modules and establish three isomorphism theorems of hypernear-modules by using normal fuzzy subhypernear-modules.

Keywords: Near-module, Hypernear-module, Normal fuzzy subhypernear-module, Isomorphism theorems

1 Introduction
Hyperstructures, in particular hypergroups, were introduced in 1934 by a French mathematician, Marty, at the VIIIth Congress of Scandinavian Mathematicians ([20]). Since then, hundreds of papers and several books have been written on this topic. Nowadays, hyperstructures have a lot of applications to several domains of mathematics and computer science see [1, 2, 4, 6, 7, 9, 13], and they are studied in many countries of Europe, America and Asia. In 1971, Rosenfeld [23] introduced fuzzy sets in the context of group theory and formulated the concept of a fuzzy subgroup of a group. Since then, many researchers are engaged in extending the concepts of abstract algebra to the framework of the fuzzy setting. In 1990 Dasic [10] has introduced the notation of hypernear-rings in a particular case. The hypernear-rings generalize the concept of near-ring. More recently, Sen, Ameri and Chowdhury introduced and analyzed fuzzy semihypergroups in [24]. The fuzzy hyperring notion is defined and studied in [17]. Ameri and Hendoukolaie introduced and analyzed fuzzy hypernear-ring and a fuzzy hypernear-module on a hypernear-ring in [2, 3]. In [14] Hendoukolaie analyzed the fuzzy homomorphism between Hypernear-rings and in [15] Hendoukolaie, Ghasemi, Ghasemi introduced and analyzed the fuzzy isomorphism theorem of \( \Gamma \)-hypernear-rings by \( \Gamma \)-hyperideals. J. Zhan, B. Davvaz, K.P. Shum, introduced the concept of normal fuzzy
subhypermodules of hypermodules and analized three isomorphism theorems of hypermodules by using normal fuzzy subhypermodules in [29]. In this paper, introduce the concept of normal fuzzy subhypernear-modules of hypernear-modules and establish three isomorphism theorems of hypernear-modules by using normal fuzzy subhypernear-modules.

2 Preliminaries

First of all, recalled some notions and results that used in the following paragraphs. (see [1],[5],[6],[20]). A nonempty set \( R \) with two binary hyperoperations " \( \cdot \) and " \( + \) is called a \textit{Near-ring} if:

\begin{itemize}
  \item [(i)] \((R, +)\) is a group;
  \item [(ii)] \((R, \cdot)\) is a semigroup;
  \item [(iii)] \(x \cdot (y + z) = x \cdot y + x \cdot z, \quad \forall x, y, z \in R\).
\end{itemize}

\textbf{Definition 2.1} A right \textit{\( R \)-nearmodule} \( M \) over a \textit{Near-ring} \( R \) consists of an group \((M,+)\) and an operation \( M \times R \rightarrow M \) such that for all \( x, y \) of \( M \) and \( r, s \) of \( R \), we have:

\begin{itemize}
  \item [(i)] \((x + y).r = x.r + y.r\);
  \item [(ii)] \(x.(r + s) = x.r + x.s\);
  \item [(iii)] \(x.(r.s) = (x.r).s\);
  \item [(iv)] \(x.1_R = x\) if \( R \) has multiplicative identity \( 1_R \).
\end{itemize}

\textbf{Example 2.2} every module \( M \) over a ring \( R \) is a near-module.

\textbf{Example 2.3} If \( K \) is a field, then the concepts \( K \)-\textit{vector space} (a vector space over \( K \)) and \( K \)-\textit{nearmodule} are identical.

Let \( H \) be a nonempty set and let \( P^\star(H) \) be the set of all nonempty subsets of \( H \). A \textit{hyperoperation} on \( H \) is a map \( \circ : H \times H \rightarrow P^\star(H) \) and the couple \((H, \circ)\) is called a \textit{hypergroupoid}.

If \( A \) and \( B \) are nonempty subsets of \( H \), then we denote

\[ A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\}, \quad x \circ B = \{x\} \circ B. \]

A hypergroupoid \((H, \circ)\) is called a \textit{semihypergroup} if for all \( x, y, z \) of \( H \) we have

\[(x \circ y) \circ z = x \circ (y \circ z), \quad \text{which means that} \quad \bigcup_{a \in x, b \in y, c \in z} a \circ b = \bigcup_{a \in x, b \in y} a \circ b.\]

An element \( e \) of \( H \) is called an \textit{identity} (scalar identity) of \((H, \circ)\) if for all \( a \in H \), we have \( a \in (e \circ a) \cap (e \circ a), \quad ([a] = (e \circ a) \cap (e \circ a)).\)

A \textit{hypergroup} is a semihypergroup such that for all \( x \in H \), we have \( x \circ H = H = H \circ x.\)

A \textit{subhypergroup} \((K, \circ)\) of \((H, \circ)\) is a nonempty set \( K \), such that for all \( k \in K \), we have \( k \circ K = K = k \circ K.\)

\textbf{Definition 2.4} The triple \((R, +, \cdot)\) is a Hypernear\textit{ring} if:
(1) \((R, +)\) is a quasicanonical hypergroup, i.e. the following axioms hold for \((R, +)\):

(i) \((x + y) + z = x + (y + z), \forall x, y, z \in R;\)

(ii) \(\exists 0 \in R\) such that \(x + 0 = 0 + x, \forall x \in R;\)

(iii) \(\forall x \in H, \exists x' \in H\) such that \(0 \in (x + x') \cap (x' + x);\)

(iv) \(\forall x, y, z \in R\) and \(z \in x + y \implies x \in z + (-y), \quad y \in (-x) + z.\)

(2) \((R, \cdot)\) is a semihypergroup having \(0\) as a right absorbing element, i.e. \(0 \cdot x = 0, \forall x \in R;\)

(3) \((x + y) \cdot z = x \cdot z + y \cdot z, \forall x, y, z \in R.\)

Let \((R, +, \cdot)\) be a hypernear-ring. A non-empty subset \(A\) of \(R\) is called a subhypernear-ring of \(R\) if \((A, +, \cdot)\) itself a hypernear-ring. A subhypernear-ring \(A \subseteq R\) is called normal if for all \(x \in R\) holds:

\[ x + A - x \subseteq A. \]

Since \(A \subseteq x + A - x, \) it follows \(A = x + A - x, \) for all \(x \in R.\)

**Definition 2.5** Let \((R, +, \cdot)\) be a hypernear-ring. A nonempty set \(M\), endowed with two hyperoperations \(\oplus, \cdot\) is called a right hypernear-module over \((R, +, \cdot)\) if the following conditions hold:

1. \((M, \oplus)\) is a hypergroup (not necessarily commutative).
2. \(\cdot: M \times R \to P^*(M)\) is such that for all \(a, b\) of \(M\) and \(r, s\) of \(R\), we have:
   
   - (i) \((a \oplus b) \otimes r = (a \otimes r) \oplus (b \otimes r);\)
   - (ii) \(a \otimes (r + s) = (a \otimes r) \oplus (a \otimes s);\)
   - (iii) \(a \otimes (r \cdot s) = (a \otimes r) \otimes s;\)
   - (iv) \(a \otimes 0 = 0\) and \(0 \cdot r = 0.\)

Let \((M, \oplus, \cdot)\) be a hypernear-module. A non-empty subset \(A\) of \(M\) is called a subhypernear-module of \((M, \oplus, \cdot)\) if \((A, \oplus, \cdot)\) itself a hypernear-module.

A subhypernear-module \(A\) of \(M\) is called normal if the relation \(x + A - x \subseteq A\) holds for all \(x \in M.\)

**Example 2.6** Every right hypermodule \(M\) over a hyperring \(R\) is a right hypernear-module.

**Example 2.7** Let \((R, +)\) be a hypergroup (not necessarily commutative) and let \((M_0(R), +, \cdot)\) be a hypernear-ring of mapping from \(R\) into itself (see[8]). Then \((R, \oplus, \cdot)\) be a hypernear-ring over \((M_0(R), +, \cdot)\), Where the action \(\mu: R \times M_0(R) \to R\) is given by \((a, f) \to (a)f\) , for all \(a \in R\) and \(f \in M_0(R).\)

Let \(A\) be a subhypernear-module of an \(R\)-hypernear-module \(M.\) Then the hyperquotient group \(M/A = \{m + A | m \in M\}\) endowed with the following external composition \(M/A \times R \to M/A, (m + A, r) \to mr + A,\) is an \(R\)-hypernear-module, and \(M/A\) is called the quotient \(R\)-hypernear-module of \(M\) by \(A.\)

In what follows, all the hypernear-modules are right hypernear-modules.

**Definition 2.8** A fuzzy subset \(\mu\) of a hypernear-module \(M\) over a hypernear-ring \(R\) is called a
fuzzy subhypernear-module of $M$ if the following conditions hold:

(i) $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$, for all $x, y \in M$;

(ii) $\mu(x) \leq \mu(-x)$, for all $x \in M$;

(iii) $\mu(x) \leq \mu(x + y)$, for all $r \in R$ and $x \in M$.

A fuzzy subhypernear-module $\mu$ of $M$ is called normal if $\mu(y) \leq \inf_{z \in x+y} \mu(z)$, for all $x, y \in M$.

If $\mu$ be a fuzzy subhypernear-module of $M$, then it is clear that $\mu(-x) = \mu(x)$, $\min\{\mu(x), \mu(y)\} \leq \inf_{z \in x+y} \mu(z)$, for all $x, y \in M$.

Let $M$ be an $R$-hypernear-module. Then, for a fuzzy subset $\mu$ of $M$, the level subset $\mu_t$ and the strong level subset $\mu^>_t$ are defined by

$$\mu_t = \{x \in M \mid \mu(x) \geq t\}, t \in [0,1]$$

and

$$\mu^>_t = \{x \in M \mid \mu(x) > t\}, t \in [0,1].$$

A fuzzy subhypernear-module can be characterized by using its level subsets and strong level subsets. The following proposition is obvious.

**Proposition 2.9** Let $\mu$ be a fuzzy subset of an $R$-hypernear-module $M$. Then the following statements are equivalent:

1. $\mu$ is a fuzzy subhypernear-module of $M$,
2. each non-empty strong level subset of $\mu$ is a subhypernear-module of $M$,
3. each non-empty level subset of $\mu$ is a subhypernear-module of $M$.

**Definition 2.10** A mapping $f : M \to M'$ is called a homomorphism if for all $a, b \in M$ and $r \in R$, we have:

$$f(a + b) = f(a) + f(b), \quad f(ar) = f(a)r \quad \text{and} \quad f(0) = 0$$

It is clear that a homomorphism $f$ is an isomorphism if $f$ is both injective and surjective and write $M \cong M'$ if $M$ is isomorphic to $M'$.

### 3 The isomorphism theorem

In what follows, $M$ is always a hypernear-module over a hypernear-ring $R$ unless stated otherwise.

**Definition 3.1** Let $\mu$ be a normal fuzzy subhypernear-module of $M$. Define the following relation on $M$.

$$x \equiv y \ (\text{mod} \mu) \quad \text{if and only if there exists } \alpha \in (x - y) \text{ such that } \mu(\alpha) = \mu(0).$$

now denote the above relation by $x \mu^* y$. Then, for this relation, we have the following lemma.

**Lemma 3.2** The relation $\mu^*$ is an equivalence relation.

**Proof.** For all $x, y, z \in M$, we have

(i) $0 \in x - x$ implies $x \mu^* x$, i.e., $\mu^*$ is reflexive;

(ii) if $x \mu^* y$ then there exist $\alpha \in (x - y)$ such that $\mu(\alpha) = \mu(0)$. Since
\( \mu(\alpha) = \mu(-\alpha) \) and \(-\alpha \in (y-x), \ y\mu^* x \). Thus, \( \mu^* \) is symmetric.

(iii) To prove that \( \mu^* \) is transitive, let \( x\mu^* y \) and \( y\mu^* z \). Then there exist then there exist \( \alpha \in (x-y) \) and \( \beta \in (y-z) \) such that \( \mu(\alpha) = \mu(\beta) = \mu(0) \). Therefore, \( x \in \alpha + y \) and \( -z \in y + \beta \). Hence, we have \(-z+x\subseteq -y+\beta+\alpha+y \), and so for every \( a \in -z+x \), there exists \( b \in \beta+\alpha \) such that \( a \in -y+b+y \). Since \( \mu \) is normal, \( \mu(b) \leq \mu(a) \) and \( \mu(0) = \min\{\mu(\alpha), \mu(\beta)\} \leq \mu(b) \). These imply that \( \mu(b) = \mu(0) \). Consequently, we have \( a \in -z+x \) and \( \mu(a) = \mu(0) \), and so \((-z)\mu'(-x)\), that is, \( x\mu^* z \). This completes the proof.

**Lemma 3.3** If \( x\mu^* y \), then \( \mu(x) = \mu(y) \).

**Proof.** If \( x\mu^* y \) then there exist \( \alpha \in x-y \) such that \( \mu(\alpha) = \mu(0) \). Since \( \alpha \in x-y \) implies \( x \in \alpha + y \) and so \( \min\{\mu(\alpha), \mu(y)\} \leq \mu(x) \), that is, \( \mu(y) \leq \mu(x) \). Similarly, we have \( \mu(x) \leq \mu(y) \). Hence \( \mu(x) = \mu(y) \).

Let \( \nu \) be an equivalence relation on \( M \). If \( A, B \) are non-empty subsets of \( M \), then we write \( A \nu B \) to denote that

\[
\forall a \in A, \exists b \in B \quad \text{such that} \quad a \nu b \quad \text{and} \quad \forall b \in B, \exists a \in A \quad \text{such that} \quad a \nu b.
\]

An equivalence relation \( \nu \) on \( M \) is called regular if for every \( x, y \in M \),

\[
x \nu y \Rightarrow x + z \nu y + z, \quad \text{for all} \quad z \in M.
\]

**Lemma 3.4** \( \mu^* \) is a regular relation.

**Proof.** Suppose that \( x\mu^* y \). Then there exists \( \alpha \in x-y \) such that \( \mu(\alpha) = \mu(0) \). Now, for every \( z \in M \) and \( a \in x+z \), we have \( x \in a-z \) which implies that \( x-y \subseteq a-z-y \) or \( x-y \subseteq a-(y+z) \). Hence \( \alpha \in a-(y+z) \) and so there exists \( b \in y+z \) such that \( \alpha \in a-b \). Thus, \( a\mu^* b \) and so \( (x+z)\mu^*(y+z) \).

Let \( \mu^*[x] \) be the equivalence class containing the element \( x \). Then we denote \( M/\mu \) the set of all equivalence classes, i.e., \( M/\mu = \{\mu^*[x] | x \in M\} \). Define the following two operations on \( M/\mu \):

\[
\mu^*[x]\mu^*[y] = \{\mu^*[z] | z \in \mu^*[x] + \mu^*[y]\};
\]

\[
\mu^*[x]^r = \mu^*[x^r].
\]

Since \( \mu^* \) is regular, we can easily deduce the following theorem:

**Theorem 3.5** \( M/\mu, (\cdot)^* \) is a hypernear-module.

Let \( f : M \rightarrow M' \) be a map and \( \mu, \lambda \) be the fuzzy subsets of \( M \). \( M' \) respectively. Then the image \( f(\mu) \) of \( \mu \) is the fuzzy subset of \( M \) defined by

\[
f(\mu)(y) = \{\ell \sup_x f^{\cdot-1}(y) \{x\} \} \quad \text{if} \quad f^{\cdot-1}(y) \neq 0 \quad \text{otherwise}.
\]
for all \( y \in M' \). The inverse image \( f^{-1}(\lambda) \) of \( \lambda \) is the fuzzy subset of \( M \) defined by
\[
f^{-1}(\lambda)(x) = \lambda(f(x)) \quad \text{for all } x \in M.
\]
The following two lemmas can be easily proved and hence, we omit the details.

**Lemma 3.6** Let \( f : M \to M' \) be a homomorphism of hypernear-modules and \( \mu \) a (normal) fuzzy subhypernear-module of \( M \). Then \( f(\mu) \) is a (normal) fuzzy subhypernear-module of \( M' \).

**Lemma 3.7** Let \( f : M \to M' \) be a homomorphism of hypernear-modules and \( \mu, \lambda \) a normal fuzzy subhypernear-module of \( M, M' \), respectively. Then, the following statements hold:

1. If \( f \) is an epimorphism, then \( f(f^{-1}(\lambda)) = \lambda \);
2. If \( \mu \) is a constant on \( \text{Ker} f \), then \( f^{-1}(f(\mu)) = \mu \).

Let \( \mu \) be a normal subhypernear-module of \( M \). We now denote \( M_{\mu} = \{ x \in M \mid \mu(x) = \mu(0) \} \). Clearly, \( M_{\mu} \) is a normal subhypernear-module of \( M \). We now use the normal subhypernear-module of \( M \) to establish the isomorphism theorems.

**Theorem 3.8** (First fuzzy isomorphism theorem) Let \( f : M \to M' \) be an epimorphism of hypernear-modules and \( \mu \) a normal fuzzy subhypernear-module of \( M \) with \( M_{\mu} \supseteq \text{Ker} f \). Then \( M/\mu \cong M'/f(\mu) \).

**Proof.** First note that \( M/\mu \) and \( M'/f(\mu) \) are hypernear-modules. Now, Define \( \varphi : M/\mu \to M'/f(\mu) \) by \( \varphi(\mu^*[x]) = f(\mu)^*[f(x)] \), for all \( x \in M \). Then \( \varphi \) is clearly well-defined. In fact, if \( \mu^*[x] = \mu^*[y] \), then \( \mu(x) = \mu(y) \) by Lemma 3.3. Since \( M_{\mu} \supseteq \text{Ker} f \), \( \mu \) is a constant on \( \text{Ker} f \). By Lemma 3.7(ii) we have \( f^{-1}(f(\mu)) = \mu \). Thus, \( f^{-1}(f(\mu))(x) = f^{-1}(f(\mu))(y) \). It follows from above the definition that \( f(\mu)(f(x)) = f(\mu)(f(y)) \). Hence we \( f(\mu)^*[f(x)] = f(\mu)^*[f(y)] \). Moreover, we have
\[
\varphi(\mu^*[x][\mu^*[y]]) = \varphi(\mu^*[z]) \quad \text{for all } x, y, z \in M.
\]
(i) \( f(\mu)^*[f(x)] + f(\mu)^*[f(y)] = \varphi(\mu^*[x]) + \varphi(\mu^*[y]) \);
(ii) \( f(\mu^*[x]) = \varphi(\mu^*[x]) = f(\mu)^*[f(x)] \);\( f(\mu)^*[f(x)] = f(\mu)^*[f(x)] \);\( f(\mu)^*[f(x)] = f(\mu)^*[f(x)] \);
(iii) \( \varphi(\mu^*[0]) = f(\mu)^*[0] = f(\mu)^*[0] \).

Hence, we have shown that \( \varphi \) is a homomorphism. Clearly \( \varphi \) is an epimorphism. To show that \( \varphi \) is a monomorphism, let \( f(\mu)^*[f(x)] = f(\mu)^*[f(y)] \). Then \( f(\mu)(f(x)) = f(\mu)(f(y)) \), that is \( f^{-1}(f(\mu))(x) = f^{-1}(f(\mu))(y) \). Hence \( \mu(x) = \mu(y) \), and so \( \mu^*[x] = \mu^*[y] \), therefore, \( M/\mu \cong M'/f(\mu) \).

**Lemma 3.9** Let \( f : M \to M' \) be an epimorphism of hypernear-modules. If \( \lambda \) be a (normal) fuzzy subhypernear-module of \( M' \), then \( f^{-1}(\lambda) \) is a (normal) fuzzy subhypernear-module of \( M \).

**Corollary 3.10** Let \( f : M \to M' \) be an epimorphism of hypernear-modules. If \( \lambda \) be a normal fuzzy subhypernear-module of \( M' \), then \( Mf^{-1}(\lambda) \cong M/\lambda \).
Proof. First we observe that $Mf^{-1}(\lambda)$ and $M/\lambda$ are hypernear-modules by Lemma 3.9. In order to prove that $Mf^{-1}(\lambda) \cong ker f$, we consider $x \in Ker f$. Then we have $f(x) = f(0)$, and hence $\lambda(f(x)) = \lambda(f(0))$, i.e., $f^{-1}(\lambda)(x) = f^{-1}(\lambda)(0)$, This leads to $x \in Mf^{-1}(\lambda)$, and so $Mf^{-1}(\lambda) \cong ker f$. By Theorem 3.8, we have $Mf^{-1}(\lambda) \cong M/\lambda$.

Now, we proceed to establish the Second and Third Fuzzy Isomorphism Theorems. The following two lemmas are obvious.

**Lemma 3.11** Let $A$ be a normal subhypernear-module of $M$ and $\mu$ a normal fuzzy subhypernear-module of $M$. Then the following statements hold:

(i) If $\mu$ is restricted to $A$, then $\mu$ is a normal fuzzy subhypernear-module of $A$;

(ii) $A/\mu$ is a normal subhypernear-module of $M/\mu$.

**Lemma 3.12** If $\mu$ and $\lambda$ are any two normal fuzzy subhypernear-modules of $M$, then so is $\mu \cap \lambda$.

We now prove our second fuzzy isomorphism theorem:

**Theorem 3.13** (Second fuzzy isomorphism theorem) If $\mu$ and $\lambda$ are any two normal fuzzy subhypernear-modules of $M$ with $\mu(0) = \lambda(0)$, then,

$$M/\mu(\mu \cap \lambda) \cong (M/\mu + M/\lambda)/\lambda.$$ 

**Proof.** By Lemmas 3.11 and 3.12, $\lambda$ and $\mu \cap \lambda$ are two normal fuzzy subhypernear-modules of $M/\mu + M/\lambda$ and $M/\mu$, respectively. Now, it is clear that $(M/\mu + M/\lambda)/\lambda$ and $M/\mu(\mu \cap \lambda)$ are both hypernear-modules. Define $\psi : M/\mu \rightarrow (M/\mu + M/\lambda)/\lambda$ by $\psi(x) = \lambda^\delta[x]$, for all $x \in M/\mu$. Then, it is easy to check that $\psi$ is an epimorphism. To show that $Ker \psi = M_{\mu \cap \lambda}$, we consider the following equalities:

$$Ker \psi = \{x \in M/\mu \mid \psi(x) = \lambda^\delta[0]\} = \{x \in M/\mu \mid \lambda^\delta[x] = \lambda^\delta[0]\} = \{x \in M/\mu \mid \lambda(x) = \lambda(0)\} = \{x \in M/\mu \mid \mu(x) = \mu(0) = \lambda(0) = \lambda(x)\} = \{x \in M/\mu \mid x \in M/\lambda\} = M_{\mu \cap \lambda}.$$ 

Therefore, $M/\mu(\mu \cap \lambda) \cong (M/\mu + M/\lambda)/\lambda$.

**Theorem 3.14** (Third fuzzy isomorphism theorem) Let $\mu$ and $\lambda$ are any two normal fuzzy subhypernear-modules of $M$ with $\mu \geq \lambda$ and $\mu(0) = \lambda(0)$, then,

$$(M/\mu)(M/\lambda) \cong M/\mu.$$ 

**Proof.** By Lemma 3.11(ii), it is known that $M/\lambda$ is a normal subhypernear-module of $M/\lambda$. Define $f : M/\lambda \rightarrow M/\mu$ by $f(\lambda^\delta[x]) = \mu^\delta[x]$, for all $x \in M$. If $\lambda^\delta[x] = \lambda^\delta[y]$, for all $x, y \in M$, then there exists $\alpha \in x - y$ such that $\lambda(\alpha) = \lambda(0)$. Since $\mu \geq \lambda$ and $\mu(0) = \lambda(0)$,
we have $\mu(\alpha) \geq \lambda(\alpha) = \lambda(0) = \mu(0)$. This implies that $\mu(\alpha) = \mu(0)$, and so $\mu^\lambda[x] = \mu^\lambda[y]$. Hence, $f$ is well-defined. Moreover, we have

(i) \[ f(\lambda^\mu[x])(\lambda^\mu[y]) = f(\{\lambda^\mu[z] \in \lambda^\mu[x] + \lambda^\mu[y]\}) = \{\mu^\lambda[z] \in \lambda^\mu[x] + \lambda^\mu[y]\} = \mu^\lambda[\lambda^\mu[x]](\mu^\lambda[\lambda^\mu[y]]) = \mu^\lambda[x](\mu^\lambda[y]) = f(\lambda^\mu[x])f(\lambda^\mu[y]) \]

(ii) $f(\lambda^\mu[x] * r) = f(\lambda^\mu[x.r]) = \mu^\lambda[x] * r = f(\lambda^\mu[x]) * r$, $f(\lambda^\mu[0]) = \mu^\lambda[0] = 0$.

Hence, $f$ is a homomorphism. Clearly, $f$ is an epimorphism. Now we show that $\text{Ker} f = M_\mu/\lambda$.

In fact

$$ker f = \{\lambda^\mu[x] \in M/\lambda \mid f(\lambda^\mu[x]) = \mu^\lambda[0]\}$$

$$= \{\lambda^\mu[x] \in M/\lambda \mid \mu^\lambda[x] = \mu^\lambda[0]\}$$

$$= \{\lambda^\mu[x] \in M/\lambda \mid \mu(x) = \mu[0]\}$$

$$= \{\lambda^\mu[x] \in M/\lambda \mid x \in M_\mu\}$$

$$= M_\mu/\lambda.$$ 

Therefore, $(M/\lambda)/(M_\mu/\lambda) \cong M/\mu$.

References