ON SOME GENERALIZED FINSLER CONNECTION WITH TORSION

H. Wosoughi
Departement of Mathematics
Islamic Azad University, Babol Branch, Iran

Received: January 2010, Revised: March 2010
Online Publication: August 2010

Abstract
In the present article we generalize the connection of M. Hashiguchi and S. Hojo and study some uniqueness theorems and finally obtain the relationship between Cartan connection and the so-called Finsler connection.

Keywords: Generalized Finsler connection, Hashiguchi and Hojo connection, Cartan connection, Chern-Rund connection, Berwald connection.

AMS subject classification 53C60, 53B40.

1. Introduction.
There are four Finsler connections on a Finsler space: The connections named Berwald, Cartan, Chern-Rund and Hashiguchi connection, respectively. The theory of connections is an important field of differential geometry. It was initially developed to solve pure geometrical problems. The most important linear connections in Finsler geometry were studied in ([1], [2], [3], [4], [5], ... etc.). Recently N. L. Youssef et al. [6] studied the Finslerian connections globally. Historically in 1934 E. Cartan [7] established a connection which is metrical and torsion free. In 1970 M. Matsumbto [8] determined uniquely the Cartan connection by the following conditions:

1) Metrical,
2) Without torsion tensors and
3) With trivial deflection tensor i.e. with non-linear connection obtained from contracting $F^i_{jk}$ by $y^i$.

On the other hand M. Hashiguchi [9] established the connection with the given torsion tensor $T$, while S. Hojo [10] established the new connection known as $^{(p)}C \Gamma$ in which the torsion tensor $T$ vanished but it is not metrical with v-covariant derivative.

In the present article we have generalized the connection of M. Hashiguchi and S. Hojo. In process of determination of connection coefficient, we have taken the covariant derivatives of $g_{ij}$ to vanish just as covariant derivatives of $g_{ij}$ vanish in Cartan connection where $\Phi_{ij}$ is defined in the next sub-article.

2. The generalized Finsler Connection $^{(p)}C \Gamma T$

Let $p \neq 1$ be a real number, we define $^{(p)}\Phi(x, y)$ as

$$^{(p)}\Phi = \frac{1}{L} L^p \text{ if } p \neq 0, \quad ^{(p)}\Phi = \log L \text{ if } p = 0,$$

(2.1)

where $L(x, y)$ is fundamental function. We write,

$$\partial_1^{(p)}\Phi(\cdot) = \Phi_i, \quad \partial_j \partial_i^{(p)}\Phi = \Phi_{ij}$$

and so on.

Thus if $l_i$ denote the unit vector along element of support then,

a) $^{(p)}\Phi_i = L^{p-1}l_i$,  

b) $^{(p)}\Phi_{ij} = L^{p-2}\left[ g_{ij} + (P - 2)l_i l_j \right]$ (2.2)

where $g_{ij}(x, y)$ is metric tensor.

Let us assume that the matrix of $^{(p)}\Phi_{ij}$ is non-singular. Then its inverse $^{(p)}\Phi_{ij}$ is given by,

$$^{(p)}\Phi^{ij} = L^{-(p-2)}\left[ g^{ij} - \frac{p-2}{p-1} l^i l^j \right]$$

(2.3)

Differentiating (2.2)(b) by $y^k$ we get

$$^{(p)}\Phi_{ijk} = L^{p-2}\left[ 2g_{ijk} + (P - 2)L^{-1}\left( h_{ij}l_k + h_{ik}l_j + h_{jk}l_i + (P - 1)l_i l_j l_k \right) \right],$$

(2.4)

where $h_{ij}$ is the angular metric tensor defined as $h_{ij} = g_{ij} - l_i l_j$.

Definition 2.1. A Finsler connection $^{FT} = (F^i_{jk}, N^i_{ij}, C^i_{jk})$ on Finsler manifold $M^n$ defined as a triad consisting of a $v$-connection $F^i_{jk}$, a non-linear connection $N^i_{ij}$ and a vertical connection $C^i_{jk}$ and satisfying the following axioms is called the Cartan connection and is denoted by $^{(p)}C \Gamma$,

A1) $g_{ijk} = 0$  

A2) (h)h-torsion $T^i_{jk} = 0$  

A3) deflection tensor $D^i_{jk} = 0$
A3) $g_{ij} |k = 0$  
A3) $(\nu)$-torsion $S^i_{jk} = 0$,

where “$|$” and “$\llbracket\llbracket$” are the symbols for $h$-and $v$-covariant differentiation respectively.

To distinguish a general Finsler connection and Cartan’s connection we shall denote Cartan connection $\Gamma$ as $(\Gamma^i_{jk}, N^i_j, C^i_{jk})$, where

$$\Gamma^i_{jk} = \gamma^i_{jk} - C^i_m N^m_{j} - C^i_{jm} N^m_{k} + g^{hi} C^i_{km} N^m_h$$

and

$$N^i_j = \gamma^i_{jk} y^k - g^i_{jm} \gamma^m_{hi} y^h y^i,$$  \hspace{1cm} (2.6)

where $\gamma^h_{ijk} = g^{hi} \gamma^i_{jk}$ is the Christoffel’s symbol of second kind and $\gamma^i_{ij} = \frac{1}{2} \partial_i g_{ij} + \partial_j g_{ik} - \partial_j g_{ik}$ is the Christoffel’s symbol of first kind.

**Theorem 2.1.** The generalized Finsler connection $(F^i_{jk}, N^i_j, C^i_{jk})$ is uniquely determined by

$$g_{ij} |k = 0 \quad \text{(connection is } h\text{-metrical)},$$  \hspace{1cm} (2.7)

$$N^i_j = y^i F^i_{rj} \quad \text{(deflection tensor field vanishes)},$$  \hspace{1cm} (2.8)

$$\text{(h) } h\text{-torsion tensor } T_{jk}^i = F^i_{jk} - F^i_{kj}$$  \hspace{1cm} (2.9)

is a given skew symmetric and $(0)$ $p$ homogeneous tensor

$$\Phi^p_{ij} |k = 0, \quad C^i_{jk} = C^i_{kj},$$  \hspace{1cm} (2.10)

where $|k$ and $\llbracket k$ denote $h$ and $v$-covariant derivative with respect to the connection $(\Gamma)$.  

**Proof.** From (2.7) it follows that

$$g_{ij} |k = \partial_k g_{ij} - g_{ij} F^i_{rk} - g_{ir} F^r_{jk} = 0$$  \hspace{1cm} (2.7)'

Applying Christoffel process to (2.7)' we get

$$F^i_{jk} = y^i_{jk} - N^m_{j} g^{i}_{mk} - N^m_{k} g^{i}_{mj} + g^{ir} N^m_{r} g_{mjk} + \frac{1}{2} (T^i_{jk} + g^{ir} T^m_{rj} + g^{ir} T^m_{rk} - g^{ir} T^m_{rk}).$$  \hspace{1cm} (2.11)

Contracting (2.11) with $y^i$ and using (2.8) we get

$$N^i_k = \gamma^i_{ok} + N^m_{i0} g^{i}_{mk} - \frac{1}{2} (T^i_{ko} + g^{ir} T^m_{ro} + g^{ir} T^m_{rk} - g^{ir} T^m_{rk}),$$  \hspace{1cm} (2.12)

where $o$ denotes the contraction with $y^i$ i.e. $\gamma^i_{ok} = \gamma^i_{jk} y^j$.

Again contracting (2.12) with $y^k$ we get...
\( N' = \gamma'_{\alpha\beta} - g^{\alpha i} T'^{\alpha}_{\beta} \).

(2.13)

Substituting (2.13) in (2.12) we get

\[
N = \gamma^{\beta}_{\alpha\beta} - (\gamma^{\beta}_{\alpha\beta} - g^{\beta m} T'^{\beta}_{m}) g^i_{\alpha} = \frac{1}{2} \left( T'^{i}_{\alpha} + g_{\alpha\beta} g^{\beta i} T'^{\beta}_{m} + g^{\beta i} T'^{\beta}_{i} \right).
\]

(2.14)

From (2.11) and (2.14) we get unique \((p) F'_{ij}\) and \((p) N'_{ij}\) with given \((p) T'_{ijk}\).

Form (2.10) we get

\[
\Phi_{ij} = \Phi_{ijk} - C_{ijk} - C'_{ijk} = 0,
\]

(2.15)

where \( C'_{ijk} = C_{ijk} \). Applying Christoffel process to (2.15) and using the axiom

\[
C_{ijk} = C_{ikj}
\]

(2.16)

which gives

\[
C'_{ijk} = \frac{1}{2} \left( \Phi_{ijk} + \Phi_{jki} - \Phi_{kij} \right) = \frac{1}{2} \Phi_{ijk},
\]

(2.17)

where \( \sigma'_{ik} \) is given by

\[
\sigma'_{ik} = \frac{p-2}{2L} \left\{ \delta'^{i}_{l} \delta'^{j}_{l} + \delta'^{j}_{l} \delta'^{i}_{l} + \frac{h_{ik} l'^{p}}{(p-1)} - l'_{ik} l' \right\}.
\]

Thus from (2.17) we get unique \( C'_{ijk} \).

**Theorem 2.2.** Any connection \((F'_{ij}, N'_{ij}, C'_{ijk})\) determined by theorem (2.1) also satisfy \( \Phi_{ijk} = 0 \) for \( p\neq0 \).

Proof. Conditions (2.7) and (2.8) of theorem (2.1) are equivalent to \( g_{ij\parallel k} = 0, y'^{i}_{\parallel k} = 0 \),

which in view of relation \( L^2 = g_{ij} y'^{i} y'^{j} \) gives \( L\parallel k = 0 \). Hence \( l'_{ijk} = 0 \), which in view of (2.2) yields

\[
\Phi_{ijk} = 0.
\]

**Theorem 2.3.** The generalized Finsler connection \((F'_{ij}, N'_{ij}, C'_{ijk})\) is uniquely determined for \( p\neq0 \) by

\[
\Phi_{ijk} = 0, \quad \Phi_{ij\parallel k} = 0
\]

(2.18)

\[
N'_{ij} = y'^{l} F'_{ij}
\]

(2.19)

(h)h - torsion tensor \( T'_{ijk} \) is given skew symmetric and \((0)p\) homogeneous tensor (2.20)

\[
C'_{ijk} = C_{ijk},
\]

(2.21)
and is the same connection as that determined in theorem (2.1)

Proof. If we take the connection \( (F^i_{jk}, N^i_j, C^i_j) \) determined by theorem (2.1), then theorem (2.2) gives \( \Phi_i\|l = 0 \). If we take the connection determined by axioms (2.18) to (2.21), then from (2.19) we get \( y^i_l = 0 \). Also from homogeneity of \( \Phi_i \) we have

\[
\Phi_i y^i = (P - 1) \Phi_i,
\]

which gives \( \Phi_i\|l = 0 \). Again \( \Phi_i y^i = \Phi \) implies that \( \Phi\|l = 0 \). Hence from (2.1) we have \( L\|k = 0 \), from which we get \( l\|k = 0 \).

Thus equation (2.2) gives \( g_{ij}\|k = 0 \). The uniqueness follows from theorem (2.1).

3. relation between Cartan connection and the generalized Finsler Connection \( C \Gamma_T \).

We establish the relation between Cartan connection \( C\Gamma = (\Gamma^i_{jk}, N^i_j, C^i_j) \) and the generalized Finsler connection \( C\Gamma_T = (F^i_{jk}, N^i_j, C^i_j) \).

In view of (2.11), (2.14), (2.5) and (2.6) we have the following relations

\[
\begin{align*}
F^i_{jk} &= \Gamma^i_{jk} + A^i_{jk}, \\
N^i_j &= N^i_j + A^i_j,
\end{align*}
\]

where

\[
\begin{align*}
A^i_{jk} &= (g^i_{ms} g^m_{jk} - g^i_{mj} g^m_{jk}) T^s_{ko}^{(P)} + \frac{1}{2} (g^i_{jk} T^m_{ko} + g^i_{mk} T^m_{jo} - g^i_{mj} T^m_{ko} + \frac{1}{2} (g^i_{jk} T^m_{ko} + g^i_{mk} T^m_{jo} - g^i_{mj} T^m_{ko} \right) \\
+ \frac{1}{2} (g^i_{jk} T^m_{ko} + g^i_{mk} T^m_{jo} - g^i_{mj} T^m_{ko} - \frac{1}{2} g^i_{jk} T^m_{ko} + g^i_{mk} T^m_{jo} - g^i_{mj} T^m_{ko}) \right)
\end{align*}
\]

And

\[
T^s_{ko}^{(P)} = g^s_{kl} T^r_{ko}^{(P)}.
\]

Now if \( U^i_{\|k} \) and \( U^i_j\|k \) denote h-covariant derivative of a tensor field \( U^i_j \) with respect to the connection \( C\Gamma_T \) and Cartan connection \( C\Gamma \) respectively, then in view of (2.17)

\[
U^i_j\|k = U^i_{j\|k} = U^i_j \sigma^i_{jk} - U^i_r \sigma^i_{jk}.
\]

The (v)h–torsion tensor and (v)hv–torsion tensor of \( C\Gamma_T \) are given by

\[
R^p_{jk} = \delta^p_k N^i_j - \delta^p_j N^i_k,
\]

140
\[ \hat{P}_{jk}^{i} = \partial_{k}^{i} - N_{k}^{j} \partial_{j}^{i}, \quad (3.7) \]

where \( \partial_{k}^{i} = \partial_{k} - N_{k}^{i} \partial_{j} \).

In view of (3.1) and (3.2) we have

\[ R_{jk}^{i} = R_{jk}^{i} \quad (p) + A_{mjk} \quad (p) = A_{mjk} \quad (p) - F_{mjk}^{i} \quad (p) + F_{mjk}^{i} \quad (p) - (\hat{\partial}_{j} \quad A_{mjk}^{i} \quad (p) + (\hat{\partial}_{k} \quad A_{mjk}^{i} \quad (p) + (\hat{\partial}_{j} \quad A_{mjk}^{i} \quad (p), \quad (3.8) \]

\[ P_{jk}^{i} = P_{jk}^{i} \quad (p) + A_{mjk} \quad (p), \quad (3.9) \]

where \( R_{jk}^{i} \) and \( P_{jk}^{i} \) are (v)h-torsion tensor and (v)hv- torsion tensor of \( \mathbb{C} \Gamma \) respectively and

\[ F_{hjr}^{i} = \hat{\partial}_{r} \quad \Gamma_{hjr}^{i}, \quad A_{mjk}^{i} = \hat{\partial} \quad A_{mjk}^{i}. \quad (3.10) \]

Here \( F_{hjr}^{i} \) is the hv- curvature tensor of Chern-Rund connection.

**References**


E-mail : hmd_vosoghi@yahoo.com