On the Korobov and Changhee mixed-type polynomials and numbers

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Abstract

By using the Bosonic \( p \)-adic integral, Kim et al. [D. S. Kim, T. Kim, H.-I. Kwon, J.-J. Seo, Adv. Stud. Theor. Phys., 8 (2014), 745–754] studied some identities of the Korobov and Daehee mixed-type polynomials. In this paper, by using the fermionic \( p \)-adic integral, we define the Korobov and Changhee mixed-type polynomials and give some interesting identities of those polynomials. ©2017 All rights reserved.

Keywords: Korobov polynomials, Changhee polynomials, Korobov and Changhee mixed-type polynomials.

2010 MSC: 11B68, 11S40.

1. Introduction

Let \( p \) be an odd prime number. Throughout this paper, \( \mathbb{Z}_p \), \( \mathbb{Q}_p \), and \( \mathbb{C}_p \) denote the ring of \( p \)-adic integers, the field of \( p \)-adic rational numbers, and the completion of algebraic closure of \( \mathbb{Q}_p \), respectively.

The \( p \)-adic norm \( | \cdot |_p \) is normalized as \( |p|_p = \frac{1}{p} \). Let \( C(\mathbb{Z}_p) \) be the space of continuous functions on \( \mathbb{Z}_p \).

For \( f \in C(\mathbb{Z}_p) \), the fermionic \( p \)-adic integrals on \( \mathbb{Z}_p \) is defined by Kim to be

\[
I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{pN-1} f(x)(-1)^x, \quad \text{(see [1–21])}. \tag{1.1}
\]

From (1.1), it is well-known that

\[
I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.2}
\]

where \( f_1(x) = f(x + 1) \). By using (1.2), we get

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doi:10.22436/jmcs.017.03.05

Received 2017-02-08
\[
\int_{Z_p} (1-t)^q y^x \, d\mu_{-1}(y) = \frac{2}{(1-t)^q + 1} (1-t)^x. \tag{1.3}
\]

Recall that the \(q\)-Chaghee polynomials are defined by the generating function
\[
\frac{2}{(1+t)^q + 1} (1+t)^x = \sum_{x=0}^{\infty} \text{Ch}_{n,q}(x) \frac{t^n}{n!}, \quad \text{see} \ [20]. \tag{1.4}
\]

From (1.3) and (1.4), we have
\[
\frac{2}{(1-t)^q + 1} (1-t)^x = \sum_{n=0}^{\infty} \text{Ch}_{n,q}(x) (-1)^n \frac{t^n}{n!}. \tag{1.5}
\]

By replacing \(t\) by \(1 - e^t\) in (1.5), we have
\[
\text{LHS of (1.5)} = \frac{2}{1 + e^t} e^{xt} = \frac{2}{1 + e^t} e^{\frac{x}{q}t} = \sum_{n=0}^{\infty} \text{E}_n \left( \frac{x}{q} \right) q^n \frac{t^n}{n!}, \tag{1.6}
\]

and
\[
\text{RHS of (1.5)} = \sum_{n=0}^{\infty} \text{Ch}_{n,q}(x) (-1)^n \frac{(1 - e^t)^n}{n!}
= \sum_{n=0}^{\infty} \text{Ch}_{n,q}(x) \frac{1}{n!} \sum_{m=n}^{\infty} S_2(m,n) \frac{t^m}{m!}
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \text{Ch}_{n,q}(x) S_2(m,n) \right) \frac{t^m}{m!}. \tag{1.7}
\]

From (1.6) and (1.7), we obtain the following theorem.

**Theorem 1.1.** For \(m \in \mathbb{N} \cup \{0\}\), we have
\[
\text{E}_m \left( \frac{x}{q} \right) q^m = \sum_{n=0}^{m} \text{Ch}_{n,q}(x) S_2(m,n).
\]

We observe that
\[
\int_{Z_p} (1-t)^q y^x \, d\mu_{-1}(y) = \sum_{m=0}^{\infty} \int_{Z_p} (qy + x) \frac{t^m}{m!} \, d\mu_{-1}(y) = \sum_{m=0}^{\infty} \int_{Z_p} (qy + x)_m \, d\mu_{-1}(y)(-1)^m \frac{t^m}{m!}. \tag{1.8}
\]

From (1.3), (1.5), and (1.8), we obtain the following theorem.

**Theorem 1.2.** For \(m \in \mathbb{N} \cup \{0\}\), we have
\[
\int_{Z_p} (qy + x)_m \, d\mu_{-1}(y) = \text{Ch}_{m,q}(x). \tag{1.9}
\]
The Stirling numbers of the first kind are defined by
\[
(x)_n = \sum_{l=0}^{n} S_1(n, l)x^l, \quad (n \geq 0).
\]
The Stirling numbers of the second kind are defined by
\[
x^n = \sum_{l=0}^{n} S_2(n, l)x^l, \quad (n \geq 0).
\]

By using the Bosonic $p$-adic integral, Kim et al. ([11, 12, 14]) studied some identities of the Korobov and Dahee mixed-type polynomials. In this paper, we observe the Korobov and Dahee mixed-type polynomials in a slightly different way and use the Fermionic $p$-adic integral in stead of the Bosonic $p$-adic integral. From the Fermionic $p$-adic integral, we define the Korobov and Changhee mixed-type polynomials and give some interesting identities of those polynomials.

2. The Korobov and Changhee mixed-type polynomials

Let us define Korobov and Changhee mixed-type polynomials $\text{KCh}_{n,q}(x)$ of the first kind as follows:
\[
\text{KCh}_{n,q}(x) = (-1)^n \int_{\mathbb{Z}_p} (qy + x)_n d\mu_{-1}(y), \quad (n \geq 0).
\] (2.1)

Then, by (1.9) and (2.1), we have
\[
\text{KCh}_{n,q}(x) = \text{Ch}_{l,q}(x)(-1)^n.
\]

By (2.1), we derive the generating function of $\text{KCh}_{n,q}(x)$ as follows:
\[
\sum_{n=0}^{\infty} \text{KCh}_{n,q}(x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{(qy + x)_n}{n!} d\mu_{-1}(y)
\]
\[
= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\frac{qy + x}{n}\right)(-t)^n d\mu_{-1}(y)
\]
\[
= \int_{\mathbb{Z}_p} (1-t)^{qy+x} d\mu_{-1}(y)
\]
\[
= \frac{2}{(1-t)^{q+1}} (1-t)^x.
\]

Note that the generating function of the Stirling number is given by
\[
(e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \quad (n \geq 0).
\]

Recall that the Euler polynomials was defined by the generating function as follows:
\[
\int_{\mathbb{Z}_p} e^{(y+x)t} d\mu_{-1}(y) = \frac{2}{e^{xt} + 1}e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.
\]

Replacing $t$ by $1 - e^t$, we have
\[
\sum_{m=0}^{\infty} \text{KCh}_{m,q}(x)(-1)^m \frac{(e^t - 1)^m}{m!} = \frac{2}{1 + e^t}e^{\frac{x}{q}t}e^{\frac{x}{q}t}
\]
\[
= \sum_{n=0}^{\infty} E_n \left(\frac{x}{q}\right) q^n \frac{t^n}{n!},
\] (2.2)
and
\[
\sum_{m=0}^{\infty} \text{KCh}_{m,q}(x)(-1)^m \frac{(e^t - 1)^m}{m!} = \sum_{m=0}^{\infty} \text{KCh}_{m,q}(x)(-1)^m \sum_{n=m}^{\infty} S_2(n, m) \frac{t^n}{n!}
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} (-1)^m \text{KCh}_{m,q}(x) S_2(n, m) \right) \frac{t^n}{n!}.
\]
(2.3)

Thus by (2.2) and (2.3), we obtain the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have
\[
E_n \left( \frac{x}{q} \right) q^n = \sum_{l=0}^{n} (-1)^l \text{KCh}_{l,q}(x) S_2(l, n).
\]

In view of (2.1), we define the Korobov and Changhee mixed-type polynomials of the second kind as following:
\[
\tilde{\text{KCh}}_{n,q}(x) = (-1)^n \int_{\mathbb{Z}_p} (-qy + x)_n \, d\mu_{-1}(y), \quad (n \geq 0).
\]
(2.4)

From (2.4), we get
\[
\sum_{n=0}^{\infty} \tilde{\text{KCh}}_{n,q} \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \int_{\mathbb{Z}_p} (-qy + x)_n \, d\mu_{-1}(y) \frac{t^n}{n!}
= \int_{\mathbb{Z}_p} (1-t)^{-qy+x} \, d\mu_{-1}(y).
\]
(2.5)

From (1.2), we have
\[
\int_{\mathbb{Z}_p} (1-t)^{-qy+x} \, d\mu_{-1}(y) = \frac{2}{(1-t)^{-q+1}}(1-t)^x.
\]
(2.6)

By (2.5) and (2.6), we derive the generating function of \( \tilde{\text{KCh}}_{n,q}(x) \) as follows:
\[
\sum_{n=0}^{\infty} \tilde{\text{KCh}}_{n,q} \frac{t^n}{n!} = \frac{2}{(1-t)^{-q+1}}(1-t)^x.
\]
(2.7)

From (2.4), we have
\[
\tilde{\text{KCh}}_{n,q}(x) = (-1)^n \int_{\mathbb{Z}_p} (-qy + x)_n \, d\mu_{-1}(y)
= \sum_{l=0}^{n} S_1(n, l) \int_{\mathbb{Z}_p} (-qy + x)^l \, d\mu_{-1}(y)(-1)^n
= \sum_{l=0}^{n} S_1(n, l)(-1)^{l+n} q^l \int_{\mathbb{Z}_p} \left( y - \frac{x}{q} \right)^l \, d\mu_{-1}(y)
= \sum_{l=0}^{n} S_1(n, l)(-1)^{n+l} q^l E_l \left( \frac{x}{q} \right).
\]
(2.8)

Thus, by (2.8), we obtain the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have
\[
\tilde{\text{KCh}}_{n,q}(x) = \sum_{l=0}^{n} S_1(n, l)(-1)^{n+l} q^l E_l \left( \frac{x}{q} \right).
\]
(2.9)
We observe that
\[
\sum_{n=0}^{\infty} E_n(-x) \frac{t^n}{n!} = \frac{2}{e^t+1} e^{-xt} \\
= \frac{2}{1+e^{-t}} e^{-(x+1)t} \\
= \sum_{n=0}^{\infty} E_n(1+x)(-1)^n \frac{t^n}{n!}.
\] (2.10)

By (2.10), we get
\[
E_n,q(-x) = E_n(1+x)(-1)^n. 
\] (2.11)

By (2.8) and (2.11), we obtain the following theorem.

**Theorem 2.3.** For \( n \geq 0 \),
\[
\hat{K}\text{Ch}_{n,q}(x) = \sum_{l=0}^{n} S_1(n,1)(-1)^n q^l E_1 \left( 1 + \frac{x}{q} \right).
\]

By replacing \( t \) by \( 1 - e^t \) in (2.7)
\[
\sum_{n=0}^{\infty} \hat{K}\text{Ch}_{n,q}(x) \frac{(1-e^t)}{n!} = \frac{2}{1+e^{qt}} e^{qt} e^{tx} \\
= \frac{2}{1+e^{qt}} e^{(q+x)t} \\
= \frac{2}{1+e^{qt}} e^{(1+\frac{x}{q})t} \\
= \sum_{n=0}^{\infty} E_n \left( 1 + \frac{x}{q} \right) q^n \frac{t^n}{n!},
\] (2.12)

and
\[
\sum_{m=0}^{\infty} \hat{K}\text{Ch}_{m,q}(x) \frac{1}{m!}(1-e^t)^m = \sum_{m=0}^{\infty} \hat{K}\text{Ch}_{m,q}(x) \frac{1}{m!} m!(-1)^m \sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \hat{K}\text{Ch}_{m,q}(x)(-1)^m S_2(n,m) \right) \frac{t^n}{n!}.
\] (2.13)

Thus, by (2.12) and (2.13), we obtain the following theorem.

**Theorem 2.4.** For \( n \geq 0 \), we have
\[
E_n \left( 1 + \frac{x}{q} \right) q^n = \sum_{m=0}^{n} \hat{K}\text{Ch}_{m,q}(x)(-1)^m S_2(n,m).
\]

We observe that
\[
\frac{\hat{K}\text{Ch}_{n,q}}{n!} = \frac{(-1)^n}{n!} \int_{Z_p} (-qy)_n d\mu_{-1}(y) \\
= \left[ \frac{qy + n - 1}{n} \right]_{Z_p} d\mu_{-1}(y) \\
= \sum_{l=0}^{n} \left( \begin{array}{c} n-1 \cr l \end{array} \right) \int_{Z_p} \left( \frac{qy}{l} \right) d\mu_{l-1}(y) \\
= \sum_{l=0}^{n} \left( \begin{array}{c} n-1 \cr l \end{array} \right) \frac{1}{l!} \int_{Z_p} \left( \frac{qy}{l} \right) d\mu_{l-1}(y) \\
= \sum_{l=0}^{n} \left( \begin{array}{c} n-1 \cr l-1 \end{array} \right) \frac{\hat{K}\text{Ch}_{l,q}}{l!}.
\] (2.14)
Thus, by (2.14), we obtain the following theorem.

**Theorem 2.5.** For \( n \geq 1 \), we have

\[
\frac{KCh_{n,q}}{n!} = \sum_{l=0}^{n} \frac{(n-1)}{(l-1)} \frac{KCh_{l,q}}{l!}.
\]

3. The Korobov and Changhee mixed-type polynomials of order \( r \)

For \( r \in \mathbb{N} \), let us consider Korobov and Changhee mixed-type polynomials of order \( r \) as follows:

\[
KCh_{n,q}(x) = (-1)^{n} \int_{Z_p} \cdots \int_{Z_p} (qx_1 + \cdots + qx_r + x)_{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r). \tag{3.1}
\]

Then we have

\[
KCh_{n,q}(x) = (-1)^{n} \int_{Z_p} \cdots \int_{Z_p} (qx_1 + \cdots + qx_r + x)_{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
= \sum_{l=0}^{n} S_1(n, l)(-1)^{n} \int_{Z_p} \cdots \int_{Z_p} (qx_1 + \cdots + qx_r + x)^{r} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \tag{3.2}
= \sum_{l=0}^{n} S_1(n, l)q^{r}(1)^{n} \int_{Z_p} \cdots \int_{Z_p} (x_1 + \cdots + x + \frac{x}{q})^{r} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).
\]

Recall that the Euler polynomials of order \( r \) were defined by the generating function as follows:

\[
\int_{Z_p} \cdots \int_{Z_p} e^{x_1 + \cdots + x_r + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) = \left(\frac{2}{e^t + 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x) \frac{t^n}{n!} \tag{3.3}
\]

From (3.2) and (3.3), we obtain the following theorem.

**Theorem 3.1.** For \( n \geq 0 \), we have

\[
KCh_{n,q}(x) = \sum_{l=0}^{n} S_1(n, l)q^{r}(1)^{n} E_n^{(r)}\left(\frac{x}{q}\right).
\]

From (3.1), we can derive the generating function of \( KCh_{n,q}(x) \) as follows:

\[
\sum_{n=0}^{\infty} KCh_{n,q}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (-1)^{n} \int_{Z_p} \cdots \int_{Z_p} (qx_1 + \cdots + qx_r + x)_{n} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \frac{t^n}{n!}
= \int_{Z_p} \cdots \int_{Z_p} \left(\frac{q x_1 + \cdots + q x_r + x}{n}\right)(-t)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \tag{3.4}
= \int_{Z_p} \cdots \int_{Z_p} (1-t)^{q x_1 + \cdots + q x_r + x} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
= \left(\frac{2}{(1-t)^{q+1}}\right)^r (1-t)^x.
\]

As is known, the \( q \)-Changhee polynomials of order \( r \) are defined by the generating function to be

\[
\left(\frac{2}{(1+t)^{q+1}}\right)^r (1+t)^x = \sum_{n=0}^{\infty} Ch_{n,q}(x) \frac{t^n}{n!}.
\]
By replacing $t$ by $1 - e^t$ in (3.4)

$$
\sum_{n=0}^{\infty} \text{KCh}_{n,q}^{(r)}(x) \frac{(1 - e^t)^n}{n!} = \sum_{n=0}^{\infty} \text{KCh}_{n,q}^{(r)}(x) \frac{1}{n!} n! \sum_{l=1}^{\infty} S_2(l,n)(-1)^n \frac{t^l}{l!}
$$

and

$$
\sum_{n=0}^{\infty} \text{KCh}_{n,q}^{(r)}(x) \frac{(1 - e^t)^n}{n!} = \left(\frac{2}{e^{qt} + 1}\right)^r e^{tx}
$$

From (3.5) and (3.6), we obtain the following theorem.

**Theorem 3.2.** For $n \geq 0$,

$$
q^n \text{E}_n^{(r)}(x) = \sum_{l=0}^{n} \text{KCh}_{l,q}^{(r)}(x)(-1)^l S_2(n,l).
$$

Let us consider Korobov and Changhee mixed-type polynomials of second kind with order $r$ as follows:

$$
\text{KCh}_{n,q}^{(r)}(x) = (-1)^n \int_{Z_p} \cdots \int_{Z_p} (-qx_1 - qx_2 - \cdots - qx_r + x)_n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r),
$$

where $n \geq 0$.

Thus, by (3.7), we get

$$
\text{KCh}_{n,q}^{(r)}(x) = \sum_{l=0}^{n} S_1(n,l)(-1)^n \int_{Z_p} \cdots \int_{Z_p} (-qx_1 - qx_2 - \cdots - qx_r + x)^l d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)
$$

By (3.8), we obtain the following theorem.

**Theorem 3.3.** For $n \geq 0$,

$$
\text{KCh}_{n,q}^{(r)}(x) = \sum_{l=0}^{n} S_1(n,l) q^l (-1)^{n+l} \text{E}_l^{(r)}(-\frac{x}{q}).
$$

We observe that

$$
\sum_{n=0}^{\infty} \text{E}_n^{(r)}(-x) \frac{t^n}{n!} = \left(\frac{2}{e^t + 1}\right)^r e^{-xt}
$$

$$
= \left(\frac{2}{1+e^{-t}}\right)^r e^{-rt} e^{-xt}
$$

$$
= \left(\frac{2}{1+e^{-t}}\right)^r e^{-(x+r)t}
$$

$$
\sum_{n=0}^{\infty} \text{E}_n^{(r)}(x+r)(-1)^n \frac{t^n}{n!}
$$
By (3.10), we have
\[ E_n^{(r)}(-x) = E_n^{(r)}(x + r)(-1)^n. \] (3.11)

By (3.9) and (3.11), we obtain the following theorem.

**Theorem 3.4.** For \( n \geq 0 \),
\[ \hat{KCH}_{n,q}(r, x) = \sum_{l=0}^{n} S_1(n, l) q^l (-1)^n E_l^{(r)}(\frac{x}{q} + r). \]

References


