COMMON FIXED POINT THEOREMS FOR A PAIR OF MAPPINGS IN COMPLEX-VALUED METRIC SPACES

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Abstract
The purpose of this paper is to prove common fixed point theorems for a pair of mappings satisfying a quasi-contraction condition in a complex-valued metric space \((X, d)\). For this, we have defined the \(\text{max}\) function for the partial order \(\leq\) in complex-valued metric \(d\).

Keywords: Common fixed point, contraction mapping, contractive condition, Banach contraction condition, Complex-valued metric space.

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1. Introduction.

An ordinary metric $d$ is a real-valued function from a set $X \times X$ into $\mathbb{R}$, where $X$ is a nonempty set. That is, $d: X \times X \rightarrow \mathbb{R}$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers, whose first co-ordinate is called $Re(z)$ and second coordinate is called $Im(z)$. Thus a complex-valued metric $d$ would be a function from a set $X \times X$ into $\mathbb{C}$, where $X$ is a nonempty set and $\mathbb{C}$ is the set of complex number. That is, $d: X \times X \rightarrow \mathbb{R}$.

Define a partial order $\leq$ on $\mathbb{C}$ as follows; let $z_1, z_2 \in \mathbb{C}$.

$$z_1 \leq z_2 \text{ if and only if } Re(z_1) \leq Re(z_2), \text{ } Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), \text{ } Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2), \text{ } Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2), \text{ } Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2), \text{ } Im(z_1) = Im(z_2)$.

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So that, $|z_1| \leq |z_2|$. In particular, $z_1 \not\leq z_2$ if $z_1 \neq z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1| < |z_2|$. Also $z_1 \leq z_2$ if only (iii) satisfy. Further,

$$0 \leq z_1 \not\leq z_2 \text{ implies } |z_1| < |z_2|,$$

$$z_1 \leq z_2, \text{ } z_2 < z_3 \text{ implies } z_1 < z_3.$$

From this definition of complex-valued metric $d$, Azam et. al. [1] defined the complex-valued metric space $(X, d)$ in the following way:

Definition 1.1. Let $X$ be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

(C1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(C2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(C3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a complex-valued metric in $X$, and $(X, d)$ is called a complex-valued metric space.
A point $x \in X$ is called an **interior point** of $A \subseteq X$ if there exists $r \in C$, where $0 < r$, such that

$$B(x, r) = \{y \in X: d(x, y) < r\} \subseteq A.$$

A point $x \in X$ is called a **limit point** of $A \subseteq X$, if for every $0 < r \in C$,

$$B(x, r) \cap (AX) \neq \emptyset.$$

The set $A$ is called **open** whenever each element of $A$ is an interior point of $A$. A subset $B$ is called **closed** whenever each limit point of $B$ belongs to $B$.

The family $F := \{B(x, r): x \in X, 0 < r\}$ is a sub-basis for a Hausdorff topology $\tau$ on $X$. Let $\{x_n\}$ be a sequence in $X$ and $x \in X$. If for every $c \in C$ with $0 < c$, there exists $n_0 \in N$ such that for all $n > n_0$, $d(x_n, x) < c$, then $\{x_n\}$ is called **convergent**. Also, sequence $\{x_n\}$ converges to $x$ (written as, $x_n \to x$ or $\lim_{n \to \infty} x_n = x$); and $x$ is the **limit point** of $\{x_n\}$. The sequence $\{x_n\}$ converges to $x$ if and only if

$$\lim_{n \to \infty} |d(x_n, x)| = 0.$$  
If for every $c \in C$ with $0 < c$, there exists $n_0 \in N$ such that for all $n > n_0$,

$$d(x_n, x_{n+m}) < c,$$

then $\{x_n\}$ is called **Cauchy sequence** in $(X, d)$. If every Cauchy sequence converges in $X$, then $X$ is called a complete complex-valued metric space. The sequence $\{x_n\}$ is called **Cauchy** if and only if $\lim_{n \to \infty} |d(x_n, x_{n+m})| = 0$.

**Definition 1.2.** We define the ‘max’ function for the partial order relation $\leq$ by:

1. $\max\{z_1, z_2\} = z_2$ if and only if $z_1 \leq z_2$,
2. $z_1 \leq \max\{z_2, z_3\}$ implies $z_1 \leq z_2$, or $z_1 \leq z_3$.

Using Definition 1.2 we have the following Lemma:
**Lemma 1.3.** Let \( z_1, z_2, z_3, \ldots \in C \) and the partial order relation \( \leq \) is defined on \( C \). Then

(i) If \( z_1 \leq \max\{z_2, z_3\} \) then \( z_1 \leq z_2 \) if \( z_3 \leq z_2 \);

(ii) If \( z_1 \leq \max\{z_2, z_3, z_4\} \) then \( z_1 \leq z_2 \) if \( \max\{z_3, z_4\} \leq z_2 \);

(iii) If \( z_1 \leq \max\{z_2, z_3, z_4, z_5\} \) then \( z_1 \leq z_2 \) if \( \max\{z_3, z_4, z_5\} \leq z_2 \), and so on.

Since \((X, d)\) is a complex-valued metric space, the ‘usual metric’ in \( \mathbb{R} \) is not definable; as shown in Example 7 [1]. Keeping this in view, we need to generalize the Banach contraction principal [2] in complex-valued metric space, as follows:

**Theorem 1.4.** Let \((X, d)\) be a complete, complex-valued metric space and \( T \) be a mapping of \( X \) into itself, satisfying:

\[
d(Tx, Ty) \leq k d(x, y), \quad \text{for all } x, y \in X; \tag{1.1}
\]

where \( k \) is a constant in \((0,1)\). Then \( T \) has a unique common fixed point in \( X \).

**Proof.** For an arbitrary \( x_0 \) in \( X \), we have \( T^nx_0 = x_n \). The sequence \( \{x_n\} \) is Cauchy. For, we have since

\[
d(x_1, x_2) = d(Tx_0, Tx_1) \leq k d(x_0, x_1),
\]

\[
d(x_2, x_3) = d(Tx_1, Tx_2) \leq k d(x_1, x_2) \leq k^2 d(x_0, x_1),
\]

\[
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\]

\[
d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq k d(x_{n-1}, x_n) \leq k^n d(x_0, x_1) \quad \text{(A)}
\]

Hence for any \( m > n, \quad m, n \in \mathbb{N} \)

\[
d(x_m, x_{n+m}) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m})
\]

\[
\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \ldots + k^{n+m-1} d(x_0, x_1)
\]
\begin{align*}
\leq & k^n d(x_0, x_1)/(1-k) \leq d(x_0, x_1), \quad \text{as } 0 < k < 1.
\end{align*}

Therefore \(|d(x_n, x_{n+m})| \leq \frac{k^n}{(1-k)^2} \rightarrow 0\); as \(m, n \rightarrow \infty\). Thus \(\{x_n\}\) is a Cauchy sequence. The completeness of \(X\) implies that sequence \(\{x_n\}\) converges to some \(x \in X\). We claim that \(x = Tx\), otherwise \(|d(x, Tx)| = |z| > 0\), and we would then have

\[|d(x, Tx)| = |z| \leq |d(x, x_n) + d(x_n, Tx)| = |d(x, x_n) + d(Tx_{n-1}, Tx)|\]

\[\leq |d(x, x_n)| + |d(Tx_{n-1}, Tx)| = |d(x, x_n)| + k|d(x_{n-1}, x)| \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Thus \(x = Tx\). The uniqueness of \(x\) follows easily. For, if \(x'\) be another fixed point then

\[d(x, x') \leq d(x, Tx) + d(Tx, x') = d(Tx, Tx') \leq k d(x, x'), \quad \text{by (1.1)}.
\]

Taking modulus in above, we have

\[|d(x, x')| \leq k|d(x, x')| < |d(x, x')|,
\]

a contradiction. Thus \(x\) is unique fixed point in \(X\). This completes the proof.

2. Main Results

Theorem 2.1. Let \((X, d)\) be a complete complex-valued metric space and mappings \(S, T : X \rightarrow X\) satisfying:

\[d(Sx, Ty) \leq h \max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\} \quad (2.1)
\]
for all $x, y \in X$; where $0 < h < \frac{1}{2}$. Then $S$ and $T$ have a unique common fixed point in $X$.

**Proof.** Choose an arbitrary point $x_0$ in $X$. Sequence $\{x_n\}$ can be formed in $X$ such that $Sx_0 = x_1$, $Tx_1 = x_2$, $Sx_2 = x_3$, $Tx_3 = x_4$, ....

\[ Sx_{2n} = x_{2n+1}, \quad Tx_{2n+1} = x_{2n+2}. \]  

(2.2)

We show that the sequence $\{x_n\}$ is Cauchy. For, putting $x = x_{2k}$ and $y = x_{2k+1}$ in (2.1), we have

\[
d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1}) \]

\[
\leq h \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, Sx_{2k}), d(x_{2k+1}, Tx_{2k+1}), d(x_{2k}, Tx_{2k+1}), d(x_{2k+1}, Sx_{2k})\} \]

\[
= h \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2}), 0\}, \text{ by (2.2)} \]

(2.3)

whence,

\[
d(x_{2k+1}, x_{2k+2}) \leq h [d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})], \text{ as other co-ordinates are less}\]

i.e.,

\[
d(x_{2k+1}, x_{2k+2}) \leq \frac{h}{1-h}.d(x_{2k}, x_{2k+1}). \]

Similarly, by putting $x = x_{2k+2}$ and $y = x_{2k+1}$ in (2.1), we have

\[
d(x_{2k+2}, x_{2k+3}) \leq \frac{h}{1-h}.d(x_{2k+1}, x_{2k+2}). \]

Hence for each $n = 1, 2, 3, \ldots$ we have

\[
d(x_n, x_{n+1}) \leq H.d(x_{n-1}, x_n), \quad (C) \]
where $0 < H = h/(1 - h) < 1$. From this we have, inductively

$$d(x_n, x_{n+1}) \leq H \cdot d(x_{n-1}, x_n) \leq H^2 \cdot d(x_{n-2}, x_{n-1}) \leq \ldots \leq H^n \cdot d(x_0, x_1) \quad (2.3)$$

Thus for any $m > n, m, n \in \mathbb{N}$, we have

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \ldots + d(x_{m-1}, x_m)$$

$$\leq H^n + H^{n+1} + H^{n+2} + \ldots + H^{m-1} \cdot d(x_0, x_1), \quad \text{by (2.3)}$$

$$\leq \frac{H^n}{1-H} \cdot d(x_0, x_1),$$

So that $|d(x_n, x_m)| \leq \frac{|H^n/(1-H)| \cdot d(x_0, x_1)|}{0}$ as $n \to \infty$.

Thus $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, therefore $\{x_n\}$ converges to some point $u$ (say) in $X$. We claim that $u$ is a fixed point of $S$. Otherwise $u \neq Su$ and $|d(u, Su)| = |z| > 0$. From triangle inequality and using (2.1), we have successively

$$d(u, Su) \leq d(u, x_{2k+2}) + d(x_{2k+2}, Su)$$

$$\leq d(u, x_{2k+2}) + d(Tx_{2k+1}, Su)$$

$$\leq d(u, x_{2k+2}) + h \max\{d(u, x_{2k+1}), d(u, Su), d(x_{2k+1}, Tx_{2k+1}), d(u, Tx_{2k+1}), d(x_{2k+1}, Su)\}. $$

Taking magnitude in above, and using $|a+b| \leq |a|+|b|$, for all $a, b \in \mathbb{C}$, we have
\[ |d(u, Su)| \leq d(u, x_{2k+2}) + h \max\{ |d(u, x_{2k+1})|, |d(u, Su)|, |d(x_{2k+1}, Tx_{2k+1})|, \]
\[ |d(u, Tx_{2k+1})|, |d(x_{2k+1}, Su)| \}. \]

Letting \( n \to \infty \) we have

\[ |z| = |d(u, Su)| \leq 0 + h \max\{0, |z|, 0, 0, |z|\} = h |z| < |z|, \]
a contradiction. Thus \( |z| = |d(u, Su)| = 0 \), yielding \( u = Su \).

Further, since \( X \) is complete, there exist some \( v \) in \( X \) such that \( v = Tu \). We claim that \( u = v \). If not, then from (2.1), we have

\[ d(u, v) = d(Su, Tu) \leq h \max\{d(u, u), d(u, Su), d(u, Tu), d(u, Tu), d(u, Su)\} \]
\[ \leq h \max\{0, 0, d(u, v), d(u, v), 0\} = h d(u, v). \]

Whence, on taking magnitude, \( |d(u, v)| \leq |h d(u, v)| < |d(u, v)| \), a contradiction.

Thus \( u = v = Tu = Su \), and \( u \) is the common fixed point of \( S \) and \( T \). For uniqueness of common fixed point, let \( u_0 \) be another common fixed point of \( S \) and \( T \). Then from (2.1), we have

\[ d(u, u_0) = d(Su, Tu_0) \leq h \max\{d(u, u_0), d(u, Su), d(u_0, Tu_0), d(u, Tu_0), d(u_0, Su)\}, \]

whence,

\[ |d(u, u_0)| \leq h \max\{ |d(u, u_0)|, 0, 0, |d(u, u_0)|, |d(u_0, u)|\} = h |d(u, u_0)| < |d(u, u_0)|, \]
a contradiction. Thus \( S \) and \( T \) have unique common fixed point. This completes the proof.
If the function ‘max’ has only three variables, as shown in (2.4) below, then we have the following theorem:

**Corollary 2.2.** Let \((X, d)\) be a complete complex-valued metric space and mappings \(S, T:X \to X\) satisfying:

\[
d(Sx, Ty) \leq h \max\{d(x, y), d(x, Sx), d(y, Ty)\}
\]  

(2.4)

for all \(x, y \in X\); where \(0 < h < 1\). Then \(S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** In this case, eq.(B) reduces to:

\[
d(x_{2k+1}, x_{2k+2}) \leq h \max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k}, x_{2k+1})
\]

so, eq.(C) reduces to:

\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n), \text{ where } 0 < h < 1.
\]

This is eq.(A). Further proof runs smoothly as Theorem 1.4 and Theorem 2.1.

**Remark.** By putting \(S = T\) in above corollary, we obtain Theorem 1.4. Thus, Corollary 2.2 is a generalization of Theorem 1.4.

**References.**
