Basic Unary Transformations and Functions
operating in Fuzzy Plane

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Abstract
In this paper first a series of basic transformation such integral, Rising and Falling has been defined. Then the integrals have been proved. So falling and rising planes have been studied and a theorem about it has been proved. At the end, operations on fuzzy time planes is shown and related proposition to it is proved.

Keywords: fuzzy plane, Y-function, operations on fuzzy time planes, Extend, Shift, Exp, Integrate.

Introduction
[1] Basic transformation about fuzzy interval time has been studied. First, basic concepts in fuzzy plane time have been studied and we argue a series of operations on fuzzy time planes by using [1], [20], [4, 7 and 18]. We define summary of formula of basic unary transformation such as integral, Rising and Falling. Then we continue to argue about integrals and we prove some theorems. Time planes usually don’t appear from nowhere, but they are constructed from other time planes. Plane operators are more general construction functions. They take one or more fuzzy time planes and construct a new one out of them.

We distinguish two ways of constructing new fuzzy time planes, first by means of Y-functions and then by means of plane operators. Y-functions map fuzzy values to fuzzy values. They can therefore be used to construct a new plane from a given one by applying the y-function point by point to the membership function values. Plane operators are more general construction functions.

In fact, our gold for presenting of this paper is that there are fuzzy planes which can be defined 2-dimension basic transformation for them, be defined some theorems for them.

Basic Unary Transformations

Definition (Basic Unary Transformations)
Let \( p \in F_{R^2} \) be a fuzzy plane. We define the following (parameterized) plane operators:

\[ \hat{S} = \text{sup}(p(x, y)) \] [1]

\[ f_m = \text{first maximom} \] [1]

\[ l_m = \text{last maximom} \] [1]

\[ \text{idemtity}(p) = p \]
\[
\text{Integrate} \\
\text{This operator integrates over the membership function and normalizes the integral to values } \leq 1. \text{ The two integration operators } \text{integrate}^+ \text{ and } \text{integrate}^- \text{ can be simplified for finite fuzzy time planes.}
\]

**Proposition (Integration for Finite planes)**
If the fuzzy plane \( p \) is finite then
\[
\text{integrate}^+(p)(f(x)) \equiv \lim_{a \to -\infty} \lim_{b \to +\infty} \int_a^b \int_y^z p(f(y_1, y_2)) \, dy_1 \, dy_2
\]
and
\[
\text{integrate}^-(p)(f(x)) \equiv \lim_{a \to -\infty} \lim_{b \to +\infty} \int_a^b \int_y^z p(f(y_1, y_2)) \, dy_1 \, dy_2
\]

**Proposition (Integration for planes with Finite Kernel)**
If the infinite fuzzy plane \( p \) has a finite kernel with \( p \equiv p(-\infty, -\infty) \) and \( p(+\infty, +\infty) \) then \( \text{integrate}^+(p)(f(x)) = \frac{p_1}{p_1 + p_2} \) and \( \text{integrate}^-(p)(f(x)) = \frac{p_2}{p_1 + p_2} \).

**Proof:** by using [2]

\[
\text{integrate}^+(p)(f(x)) = \lim_{a \to -\infty} \lim_{b \to +\infty} \int_a^b \int_y^z p(f(y_1, y_2)) \, dy_1 \, dy_2
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(|p|_{x}^{b} + |p|_{x}^{a} + |p|_{y}^{b} + |p|_{y}^{a})}{|p|_{a} + |p|_{b}^{a} + |p|_{b}^{b} + |p|_{y}^{b}}
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(|f|^{a} + |f|^{b})/i_1}{(a+b)/i_1}
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(a+b)/i_1}{i_1}
\]

\[
= \frac{i_1}{i_1 + i_2}
\]

\[
\text{integrate}^-(p)(f(x)) \equiv \lim_{a \to -\infty} \lim_{b \to +\infty} \int_a^b \int_y^z p(f(y_1, y_2)) \, dy_1 \, dy_2
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(|p|_{x}^{b} + |p|_{x}^{a} + |p|_{y}^{b} + |p|_{y}^{a})}{|p|_{a} + |p|_{b}^{a} + |p|_{b}^{b} + |p|_{y}^{b}}
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(|f|^{a} + |f|^{b})/i_2}{(a+b)/i_2}
\]

\[
= \lim_{a \to -\infty} \lim_{b \to +\infty} \frac{(a+b)/i_2}{i_2}
\]

\[
= \frac{i_2}{i_1 + i_2}
\]

Rising and Falling Fuzzy planes

**Definition (Rising and Falling Fuzzy planes and plane Operators)**
A fuzzy set \( p \) is rising \( \mathbf{iff} \) for its membership function \( p(x, y) = (1, 1) \) for all \( (x, y) > p^{fl} \). \( p \) is falling \( \mathbf{iff} \) for its membership function \( p(x, y) = (1, 1) \) for all \( (x, y) < p^{fl} \).

**Proposition**
The basic unary transformations \( \text{extend}^+ \) and \( \text{int}^+ \) are rising plane operators and the unary transformations \( \text{extend}^- \) and \( \text{int}^- \) are falling plane operators.

**Proof:** Any composition \( f_1 \circ \ldots \circ f_n \circ f \) where \( f \) is a rising (falling) plane operator is again a Rising (falling) plane operator. The proofs are straightforward [1].

**Linear Y-Functions**
A small, but important class of \( y \)-functions are linear \( y \)-functions. They are important firstly because very natural operators, like standard complement, intersection and union of fuzzy time planes can be described with linear \( y \)-
functions. Secondly they are important because they allow us to transform planes represented by polygons in a very efficient way: only the vertices of the polygons need to be transformed.

The main characterization of linear y-functions is therefore that they map non intersecting straight plane segments to straight plane segments.

**Definition (Y-Functions)**

\[ Y = FCT^n \equiv \{ f : [(0,0), (1,1)] \rightarrow [(0,0), (1,1)] \} \] is the set of n-place y-functions.

They map fuzzy values to fuzzy values.

\[ Y = FCT \subseteq \bigcup_{n=1}^{\infty} Y = FCT^n. \]

**Definition (plane Operators)**

\[ S = OP\mu^\infty = \{ f : F^n \rightarrow F^n \} \]

Is the set of n-place plane operators.

They map fuzzy planes to fuzzy planes.

\[ S = OP\mu^\infty \subseteq \bigcup_{n=2}^{\infty} S = OP\mu^n. \]

Every y-function can be used to construct a new fuzzy time plane from given ones by applying the y-function to the fuzzy values.

**Definition (Associated plane Operators)**

If \( f \in Y = FCT^n \) is a y-function then \( g_f \in S = OP\mu^\infty \) defined \( g_f(s_1, s_2, ..., s_n)(x, y) = f(s_1(x, y), ..., s_n(x, y)) \) is the associated plane operator.

**Definition (Linear Y-Function)**

A y-function \( f \in Y = FCT^n \) is linear if and only if the following condition holds:

\[ f((x_1, y_1), ..., (x_n, y_n)) = f(x_1, y_1, ..., x_n, y_n) \]

Maps non-intersecting plane segments

\[ (x_1, y_1, x_1), ..., (x_n, y_n), y_n) - (x_2, y_2, x_2), ..., (x_n, y_n, y_n) \]

To a line segment

\[ (z_1, f(x_1, y_1), ..., (x_n, y_n)) - (z_2, f(x_2, y_2), ..., (x_n, y_n)) \].

**One-place linear y-functions can be characterized in the following way**

**Proposition (Characterization of One-Place Linear y-Functions)**

A one-place linear y-function \( f \) is linear if and only if \( f(x, y) = f(0,0) + f(1,1) - f(0,0) \).

**Proof:** Suppose \( f \) is linear. We consider the case \( f((0,0)) \neq (f(1,1) - f(0,0)) \).

Then \( f((0,0)) = f(0,0) + f(1,1) - f(0,0) \).

Other direction: clearly.

An example for one-place linear y-function is the standard negation \( n(x, y) = 1 - (x, y) \).

**The characterization of two-place linear y-functions**

**Proposition (Characterization of Two-Place Linear y-Functions)**

A two-place y-function \( f \) is linear if and only if the following condition holds:

\[ f(x_1, y_1), (x_2, y_2) = f((0,0), (0,0)) + f((x_1, y_1), (x_2, y_2)) - f((0,0), (0,0)) \]

\[ = f(x_1, y_1), (x_2, y_2) \]

**Proof:** Suppose \( f \) is linear. We consider the case \( f(x_1, y_1) \leq (x_2, y_2) \) first. To this end we take the straight plane segment between \( ((0,0), (0,0)) \) and \( ((1,1), (1,1)) \). The line equation for this curve is just \( y = x \). Now take an arbitrary \((x_2, y_2) \in ((0,0), (1,1)) \) and an arbitrary \((x_1, y_1) \leq (x_2, y_2) \). The line equation for the plane segment starting at \((0,0), (0,0)\) and crossing \((x_2, y_2), (x_1, y_1)\) is \((x, y) = \frac{(x_1, y_1) - (0,0)}{(x_2, y_2) - (0,0)} \cdot (w_1, w_2) \).

For \((w_1, w_2) = (1,1)\) we get \((x_2, y_2) = \frac{x_1, y_1}{x_2, y_2} \).

Since \( f \) is linear we have

\[ f(x_1, y_1), (x_2, y_2) = f((0,0), (0,0)) + f((x_1, y_1), (x_2, y_2)) - f((0,0), (0,0)) \]

\[ = f((0,0), (0,0)) + f((x_1, y_1), (x_2, y_2)) - 1, 1 \]

Now consider the case \((x_1, y_1) \geq (x_2, y_2) \).
The plane starting at \((1,1),(1,1)\) and crossing \((x_2,y_2),(x_1,y_1)\) crosses the y-axis at \((x_v,y_v)\) if
\[
(x_v,y_v) = \left( (x_2,y_2) - (x_1,y_1) \right) + (x_1,y_1). 
\]
Since \(f\) is linear we have
\[
f((0,0),(x_1,y_1)) + \left( f((1,1),(1,1)) - f((0,0),(x_1,y_1)) \right) \cdot (x_2,y_2)
\]
\[
= f\left( (0,0),(x_2,y_2) - (x_1,y_1) \right) + \left( f((1,1),(1,1)) - f\left( (0,0),(x_2,y_2) - (x_1,y_1) \right) \right) \cdot (x_2,y_2)
\]

The other direction, showing that the two conditions imply linearity, is again straightforward.

Simple examples for linear two-place \(y\)-functions are the minimum and maximum function. The minimum function is used to realize standard intersection of two fuzzy time planes, and the maximum function is used to realize standard union of two fuzzy time planes.

References