A Unique Common Fixed Point Theorem For
Three Mappings in G –Cone metric spaces

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Abstract
In this paper we obtain a unique common fixed point theorem for three mappings in
G-cone metric spaces and obtain an extension and improvement of a theorem of I. Beg et. al. [ 1 ].

Keywords: G – cone metric space, common fixed points, symmetric space.

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1. Introduction and preliminaries

Based on cone metric spaces introduced by [2] and on G-metric spaces introduced by [4], I. Beg et. al. [1] introduced generalized cone metric spaces as follows:

Let E be a real Banach space and P be a subset of E. The subset P is called a Cone if it has the following properties:

(i) \( P \) is non empty, closed and \( P \neq \{0\} \);
(ii) \( 0 \leq a, b \in \mathbb{R} \) and \( x, y \in P \Rightarrow ax + by \in P \);
(iii) \( P \cap (-P) = \{0\} \).

For a given cone \( P \subseteq E \), we can define a partial ordering \( \leq \) on \( E \) with respect to \( P \) by \( x \leq y \) if and only if \( y - x \in P \). We will write \( x < y \) if \( x \leq y \) and \( x \neq y \), while \( x \ll y \) will stands for \( y - x \in P^0 \), where \( P^0 \) denotes the interior of \( P \).

**Proposition 1.1 ([5]).** Let \( P \) be a cone in a real Banach space \( E \). If \( a \in P \) and \( a \leq \lambda a \) for some \( \lambda \in (0, 1) \) then \( a = 0 \).

**Proposition 1.2 ([3], Cor.1.4).** Let \( P \) be a cone in a real Banach space \( E \).

(i) If \( a \leq b \) and \( b \ll c \), then \( a \ll c \)
(ii) If \( a \in E \) and \( a \ll c \) for all \( c \in P^0 \), then \( a = 0 \).

**Remark 1.3 ([3]).** \( \lambda P^0 \subseteq P^0 \) for \( \lambda > 0 \) and \( P^0 + P^0 \subseteq P^0 \).

**Definition 1.4 ([1]).** Let \( X \) be a nonempty set and let \( G : X \times X \times X \to E \) be a function satisfying the following properties :

(G1): \( G(x, y, z) = 0 \) if \( x = y = z \),
(G2): \( 0 < G(x, x, y) \) for all \( x, y \in X \) with \( x \neq y \),
(G3): \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \),
(G4): \( G(x, y, z) = G(x, z, y) = G(y, z, x) = ... \) (symmetry in three variables),
(G5): \( G(x, y, z) \leq G(x, a, a) + G(a, y, z) \) for all \( x, y, z, a \in X \).

Then the function \( G \) is called a generalized cone metric on \( X \) and \( X \) is called a generalized cone metric space or a \( G \)-cone metric space. It is clear that if \( G(x, y, z) = 0 \) then \( x = y = z \) for any \( x, y, z \in X \).

**Definition 1.5 ([1]).** A \( G \)-cone metric space \( X \) is called symmetric if \( G(x, x, y) = G(x, y, y) \) for all \( x, y \in X \).

**Definition 1.6 ([1]).** Let \( X \) be a \( G \)-cone metric space and \( \{x_n\} \) be a sequence in \( X \). The sequence \( \{x_n\} \) is said to converge to a point \( x \in X \) if for every \( c \in E \) with \( 0 \ll c \) there is \( N \) such that \( G(x_n, x_m, x) \ll c \) for all \( n, m > N \). In this case, we write \( x_n \to x \) as \( n \to \infty \).

The sequence \( \{x_n\} \) is said to be a \( G \)-Cauchy sequence in \( X \) if for every \( c \in E \) with \( 0 \ll c \) there is \( N \) such that \( G(x_n, x_m, x_l) \ll c \) for all \( n, m, l > N \).

\( X \) is said to be complete if every \( G \)-Cauchy sequence in \( X \) is convergent in \( X \).
Proposition 1.7 ([1], Lemma 2.8). Let $X$ be a $G$–cone metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent

(i) $\{x_n\}$ is $G$–convergent to $x$,
(ii) $G(x_n, x_{n}, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $n, m \to \infty$.

Proposition 1.8 ([1], Lemma 2.9). Let $X$ be a $G$–cone metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Remark 1.9 ([5]). If $c \in P_0$, $0 \leq a_n$ and $a_n \to 0$, then there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $a_n \ll c$.

Ismat Beg et al. [1] proved the following

Theorem 1.10 ([1], Theorem 3.1). Let $X$ be a complete symmetric $G$–cone metric space and $T : X \to X$ be a mapping satisfying one of the following conditions

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, Tx, Tx) + cG(y, Ty, Ty) + dG(z, Tz, Tz)$$
and

$$G(Tx, Ty, Tz) \leq aG(x, y, z) + bG(x, x, Tx) + cG(y, y, Ty) + dG(z, z, Tz)$$

for all $x, y, z \in X$, where $0 \leq a + b + c + d < 1$.

Then $T$ has a unique fixed point in $X$.

Now, we give a Lemma in $G$–cone metric spaces which is similar in cone metric spaces given by Jain et al. [6].

Lemma 1.11: Let $X$ be a $G$–cone metric space, $P$ be a cone in a real Banach space $E$ and $k_1, k_2, k_3, k_4 \geq 0$ such that $k_1 + k_2 + k_3 + k_4 > 0$ and $k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in $X$ and

(1.11.1) $ka \leq k_1G(x_n, x_m, x) + k_2G(y_n, y_m, y) + k_3G(z_n, z_m, z) + k_4G(p_n, p_m, p)$

then $a = 0$.

Proof. Since $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$, we have for $c \in P^0$, there exists a positive integer $N_c$ such that

$$\frac{c}{k_1 + k_2 + k_3 + k_4} \leq G(x_n, x_m, x), \quad \frac{c}{k_1 + k_2 + k_3 + k_4} \leq G(y_n, y_m, y),$$
$$\frac{c}{k_1 + k_2 + k_3 + k_4} \leq G(z_n, z_m, z), \quad \frac{c}{k_1 + k_2 + k_3 + k_4} \leq G(p_n, p_m, p) \in P^0 \forall n > N_c.$$

From Remark 1.3, we have

$$\frac{k_1 c}{k_1 + k_2 + k_3 + k_4} \leq k_1 G(x_n, x_m, x), \quad \frac{k_2 c}{k_1 + k_2 + k_3 + k_4} \leq k_2 G(y_n, y_m, y),$$
$$\frac{k_3 c}{k_1 + k_2 + k_3 + k_4} \leq k_3 G(z_n, z_m, z), \quad \frac{k_4 c}{k_1 + k_2 + k_3 + k_4} \leq k_4 G(p_n, p_m, p) \in P^0 \forall n > N_c.$$

Adding these four and by Remark 1.3, we have
c − [k_1G(x_n, x_m, x) + k_2G(y_n, y_m, y) + k_3G(z_n, z_m, z) + k_4G(p_n, p_m, p)] ∈ P^0 ∀ n > N_c.

Now from (1.11.1) and Proposition 1.2(i), we have ka << c for all c ∈ P^0.

By Proposition 1.2(ii), we have a = 0 as k > 0.

2. Main result

Theorem 2.1. Let (X,G) be a symmetric G–cone metric space and A,B,C : X → X be satisfying

\[ G(Ax, By, Cz) \leq k \max \left\{ G(x, y, z), G(x, Ax, By), \\
G(y, By, Cz), G(z, Cz, Ax), \\
G(x, Ax, Ax), G(y, By, By), G(z, Cz, Cz) \right\} \] (2.1.1)

for all x, y, z ∈ X, where 0 ≤ k < 1.

Then the mappings A, B and C have a unique common fixed point in X.

**Proof.** Choose x_0 ∈ X. Define x_{3n+1} = Ax_{3n}, x_{3n+2} = Bx_{3n+1}, x_{3n+3} = Cx_{3n+2}, n = 0, 1, 2, ....

Case(I) If x_{3n} = x_{3n+1} then x_{3n} is a fixed point of A. Denote x_{3n} = x. Then Ax = x.

Suppose Bx ≠ Cx. Then from (2.1.1)

\[ G(x, Bx, Cx) = G(x, Bx, Cx) \leq k \max \left\{ 0, G(x, x, Bx), G(x, Bx, Cx), G(x, Cx, x) \right\} \]

\[ = k \max \{ G(x, x, Bx), G(x, Bx, Cx), G(x, x, Cx) \} \] (1), as X is symmetric

It is a contradiction. Hence Bx = Cx.

Now from (1), G(x, Bx, Bx) ≤ k G(x, Bx, Bx).

Now from Proposition 1.1, Bx = x. Hence Cx = x.

Thus x is a common fixed point of A, B and C.

Suppose x^1 is another common fixed point of A,B and C. Then

\[ G(x, x^1) = G(Ax, Bx^1) \]

\[ \leq k \max \{ G(x, x, x^1), 0, G(x, x, x^1), G(x^1, x^1, x), 0, 0, 0 \} \]

\[ = k G(x, x, x^1) \] as X is symmetric

Hence x = x^1. Thus x is the unique common fixed point of A,B and C.

Similarly, if x_{3n+1} = x_{3n+2} or x_{3n+2} = x_{3n+3} then we can show that A, B and C have a unique common fixed point in X.

Case(II): Assume that x_n ≠ x_{n+1} for all n.

As X is symmetric and from (G_3), we have

\[ G(x_{3n+1}, x_{3n+2}, x_{3n+3}) = G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}) \]

\[ \leq k \max \left\{ G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n}, x_{3n+1}, x_{3n+2}), G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \right\} \]

Hence x = x^1. Thus x is the unique common fixed point of A,B and C.

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Thus, we have

\[ G(A_p, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

Thus we have

\[ G(A_p, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

\[ k \max \left\{ G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+1}, x_{3n+3}, x_{3n+2}) \right\} \]

\[ G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \]

\[ (1-k) G(A_p, p, p) \leq (2+k) G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

\[ G(A_p, p, p) \leq (2+k) G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

\[ (1-k) G(A_p, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

\[ (1-k) G(A_p, p, p) \leq 2G(x_{3n+2}, p, p) + G(p, x_{3n+3}, x_{3n+3}) \]

\[ k \max \left\{ G(x_{3n+1}, x_{3n+2}, x_{3n+3}), G(x_{3n+1}, x_{3n+3}, x_{3n+2}) \right\} \]

\[ G(x_{3n+1}, x_{3n+2}, x_{3n+3}) \]
\[ G(A_p, p, p) \leq (2+k) \ G(x_{3n+2}, p, p) + k \ G(p, x_{3n+1}, x_{3n+2}) \] \\
\[ G(A_p, p, p) \leq (2+k) \ G(x_{3n+2}, p, p) + (1+k) \ G(p, x_{3n+3}, x_{3n+3}). \]

Now from Proposition 1.7 and from Lemma 1.11, it follows that \( G(A_p, p, p) = 0 \) so that \( A_p = p \). The rest of the proof follows as in Case (I).

References.


